The ideal structure of semigroups of linear transformations with lower bounds on their nullity or defect

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Abstract

Suppose V is an infinite-dimensional vector space and let $T(V)$ denote the semigroup (under composition) of all linear transformations of V. In this paper, we study the semigroup $OM(p,q)$ consisting of all $\alpha \in T(V)$ for which $\dim\ker\alpha \geq q$ and the semigroup $OE(p,q)$ of all $\alpha \in T(V)$ for which $\operatorname{codim}\operatorname{ran}\alpha \geq q$, where $\dim V = p \geq q \geq \aleph_0$. It is not difficult to see that $OM(p,q)$ and $OE(p,q)$ are a right and a left ideal of $T(V)$, respectively, and using these facts we show that they belong to the class of all semigroups whose sets of bi-ideals and quasi-ideals coincide. Also, we describe the Green's relations and the two-sided ideals of each semigroup, and we determine its maximal regular subsemigroup. Finally, we determine some maximal right cancellative subsemigroups of $OE(p,q)$.

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1. Introduction

Suppose V is a vector space over a field F with dimension $p \geq \aleph_0$ and let $T(V)$ denote the semigroup (under composition) of all linear transformations from V into itself. Given $\alpha \in T(V)$, we write $\ker \alpha$ and $\operatorname{ran} \alpha$ for the *kernel* and the *range* of α, respectively, and put

$$n(\alpha) = \dim \ker \alpha,\ r(\alpha) = \dim \operatorname{ran} \alpha,\ d(\alpha) = \operatorname{codim} \operatorname{ran} \alpha.$$

As usual, these cardinals are called the *nullity*, the *rank* and the *defect* of α, respectively.

In [7], the authors considered the semigroups $AM(p,q) = \{\alpha \in T(V) : n(\alpha) < q\}$ and $AE(p,q) = \{\alpha \in T(V) : d(\alpha) < q\}$, where $p \geq q \geq \aleph_0$, and they showed that they do not belong to **BQ**, the class of all semigroups whose sets of bi-ideals and quasi-ideals coincide. For each semigroup, they described its maximal regular subsemigroup and characterised its Green's relations and ideals. Also, they determined all the maximal right simple subsemigroups of $AM(p,q)$.

In this paper, we study related semigroups defined as follows. For each cardinal q such that $\aleph_0 \leq q \leq p$, we write

$$\begin{aligned} OM(p,q) &= \{\alpha \in T(V) : n(\alpha) \geq q\} \ \text{ and} \\ OE(p,q) &= \{\alpha \in T(V) : d(\alpha) \geq q\}. \end{aligned}$$

Clearly, $0 \in OM(p,q) \cap OE(p,q)$, where 0 denotes the zero map on V. In [5] Theorem 3.3, Kemprasit and Namnak showed that $OE(p, \aleph_0)$ is in **BQ** and in [8] Theorem 3.4, they proved that $OM(p, \aleph_0) \in \mathbf{BQ}$. In section 2, we generalise these results: we show that $OM(p,q)$ and $OE(p,q)$ are a right and a left ideal of $T(V)$, respectively, and using this, we conclude that $OM(p,q)$ and $OE(p,q)$ are always in **BQ**. Also, we characterise the regular elements of each semigroup and determine its unique maximal regular subsemigroup. In section 3, we describe the Green's relations and ideals in $OM(p,q)$ and $OE(p,q)$.

In [6] Mendes-Gonçalves considered the semigroup $KN(p,q)$ of all injective elements of $OE(p,q)$. In section 4, we prove that $KN(p,q)$ is a maximal right cancellative subsemigroup of $OE(p,q)$. Moreover, we show that $OE(p,q)$ admits other maximal right cancellative subsemigroups.

2. Basic properties

In what follows, if Y is a *disjoint* union of A and B, we write $Y = A \dot{\cup} B$, and id_Y denotes the identity transformation on Y.

As an abbreviation, we write $\{e_i\}$ to denote a subset $\{e_i : i \in I\}$ of V, taking as understood that the subscript i belongs to some (unmentioned) index set I. The subspace A of V generated by a linearly independent subset $\{e_i\}$ of V is denoted by $\langle e_i \rangle$, and we write $\dim A = |I|$.

We adopt the convention introduced in [10]. That is, often it is necessary to define some $\alpha \in T(V)$ by first choosing a basis $\{e_i\}$ for V and some $\{a_i\} \subseteq V$, and then letting $e_i\alpha = a_i$ for each i and extending this action by linearity to the whole of V. To abbreviate matters, we simply say, given $\{e_i\}$ and $\{a_i\}$ within context, that $\alpha \in T(V)$ is defined by letting

$$\alpha = \begin{pmatrix} e_i \\ a_i \end{pmatrix}.$$

It is easily verified that if $\alpha, \beta \in T(V)$, then $\ker \alpha \subseteq \ker(\alpha\beta)$ and $\operatorname{ran}(\alpha\beta) \subseteq \operatorname{ran} \beta$. Thus, $n(\alpha) \le n(\alpha\beta)$ and $d(\beta) \le d(\alpha\beta)$, and these imply that the sets $OM(p,q)$ and $OE(p,q)$, as defined above, are subsemigroups of $T(V)$. In fact, we may conclude that $OM(p,q)$ is a right ideal of $T(V)$ and $OE(p,q)$ is a left ideal of $T(V)$. In passing, we observe that $OM(p,q)$ and $OE(p,q)$ are semigroups even if q is finite.

A subsemigroup Q of a semigroup S is called a *quasi-ideal* of S if $SQ \cap QS \subseteq Q$. A subsemigroup B of S is a *bi-ideal* of S if $BSB \subseteq B$. Clearly, every right and every left ideal of S is a quasi-ideal, and every quasi-ideal Q of a semigroup S is a bi-ideal of S (since $QSQ \subseteq SQ \cap QS$). We denote the quasi-ideal and the bi-ideal generated by a non-empty subset X of S by $(X)_Q$ and $(X)_B$, respectively. If $X = \{x_1, x_2, \ldots, x_n\}$ then we write $(x_1, x_2, \ldots, x_n)_Q$ and $(x_1, x_2, \ldots, x_n)_B$ instead of $(\{x_1, x_2, \ldots, x_n\})_Q$ and $(\{x_1, x_2, \ldots, x_n\})_B$, respectively. By [1] Vol. 1, pp. 84-85, Exercises 15 and 17, if X is a non-empty subset of a semigroup S, then

$$\begin{aligned} &(X)_Q = S^1X \cap XS^1 = (SX \cap XS) \cup X, \quad \text{and} \\ &(X)_B = (XS^1X) \cup X = XSX \cup X \cup X^2. \end{aligned}$$

It is known that regular semigroups, right [left] simple semigroups and right [left] 0-simple semigroups are in the class **BQ** of all semigroups whose sets of bi-ideals and quasi-ideals coincide (see [4] Propositions 1.2 and 1.3). The following result and its dual extend this remark: it can be used to simplify some of the arguments in [4], [5] and [8].

Lemma 1. If S is a regular semigroup, then any right ideal R of S belongs to **BQ**.

Proof. Suppose S is a regular semigroup and let R be a right ideal of S. Let X be a non-empty subset of R. We know that $(X)_B \subseteq (X)_Q$ always. We assert that $(X)_Q \subseteq (X)_B$. Let $a \in RX \cap XR$. Then, there exist $b, c \in R$ and $s, t \in X$ such that $a = bs = tc$. Since S is regular, $s = sxs$ for some $x \in S$. Since R is a right ideal of S, $cx \in R$. Therefore, $a = bs = b(sxs) = (bs)xs = (tc)xs = t(cx)s \in XRX$. Hence, $RX \cap XR \subseteq XRX$ and so $(X)_Q = R^1X \cap XR^1 \subseteq XRX \cup X \cup X^2 = (X)_B$. Thus, $(X)_B = (X)_Q$ for every non-empty subset X of R and so $R \in$ **BQ**. □

As mentioned before, $OM(p,q)$ and $OE(p,q)$ are a right and a left ideal of $T(V)$, respectively. Moreover, by [1] Vol. 1, p. 57, Exercise 6, the semigroup $T(V)$ is regular. Hence, by the above result and its dual, $OM(p,q)$ and $OE(p,q)$ are always in **BQ**. We shall see that $OM(p,q)$ and $OE(p,q)$ are not regular semigroups and neither right 0-simple nor left 0-simple.

In [9], Namnak and Kemprasit considered the semigroup $OM(p,\aleph_0) \cap OE(p,\aleph_0)$, and they showed that this is a regular subsemigroup of $T(V)$ and hence belongs to **BQ**. The next result extends their work by determining all the regular elements of $OM(p,q)$.

Theorem 1. Let $\alpha \in OM(p,q)$. Then, α is regular if and only if $\alpha \in OE(p,q)$. Consequently, $OM(p,q) \cap OE(p,q)$ is the largest regular subsemigroup of $OM(p,q)$.

Proof. Suppose $\alpha \in OM(p,q) \cap OE(p,q)$ and let $\{e_j\}$ be a basis for $\ker \alpha$ with $|J| = n(\alpha) \geq q$. Expand $\{e_j\}$ to a basis $\{e_j\} \dot\cup \{e_i\}$ for V and write $e_i\alpha = a_i$ for each i. Then, $\{a_i\}$ is a basis for $\operatorname{ran} \alpha$ and it can be expanded to a basis for V, say $\{a_i\} \dot\cup \{a_k\}$, where $|K| = d(\alpha) \geq q$. Define $\beta \in T(V)$ by

$$\beta = \begin{pmatrix} a_i & a_k \\ e_i & 0 \end{pmatrix}.$$

Then, $n(\beta) = \dim\langle a_k\rangle = d(\alpha) \geq q$ and $d(\beta) = \dim\langle e_j\rangle = n(\alpha) \geq q$, and hence $\beta \in OM(p,q) \cap OE(p,q)$. Also, $\alpha\beta\alpha = \alpha$ and so α is regular in $OM(p,q)$.

Conversely, suppose $\alpha \in OM(p,q)$ and $\alpha = \alpha\beta\alpha$ for some $\beta \in OM(p,q)$. Then $d(\alpha) = d(\alpha(\beta\alpha)) \geq d(\beta\alpha)$. Since $\beta\alpha$ is idempotent, it follows that $d(\beta\alpha) = n(\beta\alpha) \geq n(\beta) \geq q$. Hence $\alpha \in OE(p,q)$ as required. Also, if S is a regular subsemigroup of $OM(p,q)$, then it is contained in $OE(p,q)$. Therefore, $S \subseteq OM(p,q) \cap OE(p,q)$ and the latter is the largest regular subsemigroup of $OM(p,q)$. □

We now determine all regular elements of $OE(p,q)$.

Theorem 2. Let α in $OE(p,q)$. Then, α is regular if and only if $\alpha \in OM(p,q)$. Consequently, $OM(p,q) \cap OE(p,q)$ is the largest regular subsemigroup of $OE(p,q)$.

Proof. By Theorem 1, if $\alpha \in OM(p,q) \cap OE(p,q)$ then there exists some $\beta \in OM(p,q)$ such that $\alpha = \alpha\beta\alpha$ and $\beta = \beta\alpha\beta$, and this implies $\beta \in OE(p,q)$ (again, by Theorem 1). In other words, every $\alpha \in OM(p,q) \cap OE(p,q)$ is a regular element of $OE(p,q)$. Conversely, suppose $\alpha \in OE(p,q)$ and $\alpha = \alpha\beta\alpha$ for some $\beta \in OE(p,q)$. Then, $n(\alpha) = n((\alpha\beta)\alpha) \geq n(\alpha\beta)$. Also $\alpha\beta$ is an idempotent in $T(V)$, hence $V = \ker(\alpha\beta) \oplus \operatorname{ran}(\alpha\beta)$ and, since $OE(p,q)$ is closed, it follows that $n(\alpha) \geq n(\alpha\beta) = d(\alpha\beta) \geq d(\beta) \geq q$. Therefore, $\alpha \in OM(p,q)$ as required. Clearly, every regular subsemigroup of $OE(p,q)$ is contained in $OM(p,q) \cap OE(p,q)$, hence this semigroup is the largest regular subsemigroup of $OE(p,q)$. □

3. Green's relations and ideals

It is well-known that if $\alpha, \beta \in T(V)$, then $\alpha \mathcal{L} \beta$ if and only if $\operatorname{ran} \alpha = \operatorname{ran} \beta$, $\alpha \mathcal{R} \beta$ if and only if $\ker \alpha = \ker \beta$, and $\mathcal{D} = \mathcal{J}$ (see [1] Vol. 1, Exercise 2.2.6.). In this section, we characterise Green's relations on the semigroups $OM(p,q)$ and $OE(p,q)$: although the $\mathcal{L}$ and $\mathcal{J}$ relations on $OM(p,q)$ and the $\mathcal{R}$ and $\mathcal{J}$ relations on $OE(p,q)$ can be described just like the corresponding ones on $T(V)$, the other Green's relations differ substantially from the corresponding ones on $T(V)$.

We begin with analogues of [7] Lemmas 2 and 3, respectively: the proofs of those Lemmas hold verbatim for the present situation but, for completeness, we include proofs for the next two results.

Lemma 2. Let $\alpha, \beta \in OM(p,q)$. Then $\alpha \mathcal{L} \beta$ if and only if $\operatorname{ran} \alpha = \operatorname{ran} \beta$.

Proof. Let $\alpha, \beta \in OM(p,q)$ and suppose $\alpha \mathcal{L} \beta$. Then, $\beta = \lambda\alpha$ and $\alpha = \mu\beta$, for some $\lambda, \mu \in OM(p,q)^1$. Therefore, $\operatorname{ran} \alpha = \operatorname{ran}(\mu\beta) \subseteq \operatorname{ran} \beta$ and $\operatorname{ran} \beta = \operatorname{ran}(\lambda\alpha) \subseteq \operatorname{ran} \alpha$, hence $\operatorname{ran} \alpha = \operatorname{ran} \beta$.

Conversely suppose $\alpha \neq \beta$ and $\operatorname{ran}\alpha = \operatorname{ran}\beta$, and let $\{e_j\}$ be a basis for $\ker\beta$. Expand $\{e_j\}$ to a basis $\{e_j\} \dot\cup \{e_i\}$ for V and write $e_i\beta = b_i$ for each i. Then, $\{b_i\}$ is a basis for $\operatorname{ran}\beta = \operatorname{ran}\alpha$. For every i, choose $f_i \in b_i\alpha^{-1}$. Clearly, $\{f_i\}$ is linearly independent. Now define $\lambda \in T(V)$ by

$$\lambda = \begin{pmatrix} e_j & e_i \\ 0 & f_i \end{pmatrix}.$$

Since $\ker\lambda = \ker\beta$, it follows that $\lambda \in OM(p,q)$. Also, $\beta = \lambda\alpha$. Similarly, we conclude that there exists $\mu \in OM(p,q)$ such that $\alpha = \mu\beta$, and so $\alpha \mathcal{L} \beta$. □

Lemma 3. Let $\alpha, \beta \in OE(p,q)$. Then $\alpha \mathcal{R} \beta$ if and only if $\ker\alpha = \ker\beta$.

Proof. Suppose $\alpha, \beta \in OE(p,q)$ are such that $\alpha \mathcal{R} \beta$. Then, $\alpha = \beta\lambda$ and $\beta = \alpha\mu$, for some $\lambda, \mu \in OE(p,q)^1$. Thus, $\ker\alpha \subseteq \ker(\alpha\mu) = \ker\beta$ and $\ker\beta \subseteq \ker(\beta\lambda) = \ker\alpha$, and so $\ker\alpha = \ker\beta$.

Conversely, suppose $\alpha \neq \beta$ and $\ker\alpha = \ker\beta$. Let $\{e_j\}$ be a basis for $\ker\alpha$ and expand it to a basis $\{e_j\} \dot\cup \{e_i\}$ for V. For each i, write $e_i\alpha = a_i$ and $e_i\beta = b_i$. Clearly, $\{a_i\}$ and $\{b_i\}$ are bases for $\operatorname{ran}\alpha$ and $\operatorname{ran}\beta$, respectively. Now expand $\{b_i\}$ to a basis for V, say $\{b_i\} \dot\cup \{b_\ell\}$, and define $\lambda \in T(V)$ by

$$\lambda = \begin{pmatrix} b_\ell & b_i \\ 0 & a_i \end{pmatrix}.$$

Since $d(\lambda) = d(\alpha)$, it follows that $\lambda \in OE(p,q)$. Also, $\alpha = \beta\lambda$. Similarly, expand $\{a_i\}$ to a basis $\{a_i\} \dot\cup \{a_k\}$ for V and define $\mu \in T(V)$ by

$$\mu = \begin{pmatrix} a_k & a_i \\ 0 & b_i \end{pmatrix}.$$

Clearly, $d(\mu) = d(\beta)$ and so $\mu \in OE(p,q)$. Also, $\beta = \alpha\mu$. Hence, α, β are $\mathcal{R}$–related in $OE(p,q)$. □

Next, we characterise the $\mathcal{R}$–relation on $OM(p,q)$. To do this, we need [7] Lemma 4 which we quote below for convenience.

Lemma 4. If $\alpha, \beta, \lambda \in T(V)$ satisfy $\alpha = \beta\lambda$ then

$$d(\beta) \leq n(\lambda) + \dim(\operatorname{ran}\lambda/\operatorname{ran}\alpha).$$

In fact, if we also have $\ker\alpha = \ker\beta$, then $d(\beta) = n(\lambda) + \dim(\operatorname{ran}\lambda/\operatorname{ran}\alpha)$.

Lemma 5. Let $\alpha \in OM(p,q)$ and denote the $\mathcal{R}$–class of $OM(p,q)$ containing α by R_α. Then,

(a) $\alpha \notin OE(p,q)$ implies $R_\alpha = \{\alpha\}$;

(b) $\alpha \in OE(p,q)$ implies $R_\alpha = \{\beta \in OM(p,q) : \beta \in OE(p,q) \text{ and } \ker\beta = \ker\alpha\}$.

Proof. Suppose $\alpha \notin OE(p,q)$ and $\alpha \mathcal{R} \beta$ in $OM(p,q)$ for some $\beta \neq \alpha$. Then, $\ker\alpha = \ker\beta$ and $\beta = \alpha\mu$ for some $\mu \in OM(p,q)$. Thus, we have $d(\alpha) < q$ and $n(\mu) \geq q$, and by Lemma 4 we have a contradiction: namely, $d(\alpha) = n(\mu) + \dim(\operatorname{ran}\mu/\operatorname{ran}\beta) \geq q$.

To see that (b) holds, suppose $\alpha \in OM(p,q) \cap OE(p,q)$ and $\alpha \mathcal{R} \beta$ in $OM(p,q)$. Then, as usual, this implies $\ker\alpha = \ker\beta$. Moreover, if $\beta \notin OE(p,q)$ then (a) implies $\alpha \in R_\alpha = R_\beta = \{\beta\}$, hence $\alpha = \beta$, contradicting our supposition.

Conversely, suppose $\beta \in OM(p,q) \cap OE(p,q)$ and $\ker\beta = \ker\alpha$. Since $OM(p,q) \cap OE(p,q)$ is a regular subsemigroup of $OE(p,q)$ (by Theorem 2), Hall's Theorem ([2], Proposition II.4.5) implies that the $\mathcal{R}$–relation on $OM(p,q) \cap OE(p,q)$ is the restriction of the $\mathcal{R}$–relation on $OE(p,q)$ to $OM(p,q) \cap OE(p,q)$. In other words, since $\alpha, \beta \in OM(p,q) \cap OE(p,q)$ and $\ker\alpha = \ker\beta$, we deduce from Lemma 3 that $\alpha \mathcal{R} \beta$ in $OM(p,q) \cap OE(p,q)$ and hence $\alpha \mathcal{R} \beta$ in $OM(p,q)$. That is, $\beta \in R_\alpha$ as required. □

As observed above, the $\mathcal{L}$–relations on $OM(p,q)$ and on $AM(p,q)$ have identical characterisations (compare Lemma 2 and [7] Lemma 2), but the same does not happen for the $\mathcal{R}$–relations on these two semigroups (compare the previous Lemma and [7] Lemma 6).

Analogously, it is easy to see the similarity between the characterisations of the $\mathcal{R}$–relations on $OE(p,q)$ and on $AE(p,q)$ (compare Lemma 3 and [7] Lemma 3), but there is a substantial difference between the characterisations of the $\mathcal{L}$–relations on these semigroups. As before, in order to describe the $\mathcal{L}$–relation on $OE(p,q)$ we need [7] Lemma 7 (quoted below) as a preliminary Lemma.

Lemma 6. If $\alpha, \beta, \lambda \in T(V)$ satisfy $\alpha = \lambda\beta$, then

$$n(\beta) \leq d(\lambda) + \dim(\ker\alpha/\ker\lambda).$$

In fact, if $\operatorname{ran}\alpha = \operatorname{ran}\beta$ then $n(\beta) = d(\lambda) + \dim(\ker\alpha/\ker\lambda)$.

Lemma 7. Let $\alpha \in OE(p,q)$ and denote the $\mathcal{L}$-class of $OE(p,q)$ containing α by L_α. Then,

(a) $\alpha \notin OM(p,q)$ implies $L_\alpha = \{\alpha\}$;

(b) $\alpha \in OM(p,q)$ implies $L_\alpha = \{\beta \in OE(p,q) : \beta \in OM(p,q) \text{ and } \operatorname{ran}\beta = \operatorname{ran}\alpha\}$.

Proof. First suppose $\alpha \notin OM(p,q)$. If $\beta \in OE(p,q)$ is such that $\alpha \mathcal{L} \beta$ and $\beta \neq \alpha$, then there exist $\lambda, \mu \in OE(p,q)$ such that $\alpha = \lambda\beta$ and $\beta = \mu\alpha$, and so $\operatorname{ran}\alpha = \operatorname{ran}\beta$. By Lemma 6, we have $q > n(\alpha) = d(\mu) + \dim(\ker\beta/\ker\mu) \geq q$, a contradiction. Thus, (a) holds.

To see that (b) holds, suppose $\alpha \in OM(p,q) \cap OE(p,q)$ and $\alpha \mathcal{L} \beta$ in $OE(p,q)$. Then, as usual, this implies $\operatorname{ran}\alpha = \operatorname{ran}\beta$. Moreover, if $\beta \notin OM(p,q)$ then (a) implies $\alpha \in L_\alpha = L_\beta = \{\beta\}$, hence $\alpha = \beta$, contradicting our supposition.

Now suppose $\beta \in OM(p,q) \cap OE(p,q)$ and $\operatorname{ran}\beta = \operatorname{ran}\alpha$. Since $OM(p,q) \cap OE(p,q)$ is a regular subsemigroup of $OM(p,q)$, Hall's Theorem implies that the $\mathcal{L}$–relation on $OM(p,q) \cap OE(p,q)$ is the restriction of the $\mathcal{L}$–relation on $OM(p,q)$ to $OM(p,q) \cap OE(p,q)$. In other words, since $\alpha, \beta \in OM(p,q) \cap OE(p,q)$ and $\operatorname{ran}\alpha = \operatorname{ran}\beta$, we deduce from Lemma 2 that $\alpha \mathcal{L} \beta$ in $OM(p,q) \cap OE(p,q)$ and hence $\alpha \mathcal{L} \beta$ in $OE(p,q)$. That is, $\beta \in L_\alpha$ as required. □

We proceed to describe the $\mathcal{D}$ and $\mathcal{J}$ relations on $OM(p,q)$, and the characterisation of its ideals follows from this.

Theorem 3. If $\alpha, \beta \in OM(p,q)$ then $\alpha \ \mathcal{D} \ \beta$ in $OM(p,q)$ if and only if one of the following occurs.

(a) $\alpha, \beta \in OE(p,q)$ and $r(\alpha) = r(\beta)$,

(b) $\alpha, \beta \notin OE(p,q)$ and $\operatorname{ran}\alpha = \operatorname{ran}\beta$.

Proof. Suppose $\alpha \ \mathcal{L} \ \gamma \ \mathcal{R} \ \beta$ in $OM(p,q)$. By Lemma 5(b), if $\beta \in OE(p,q)$ then $\gamma \in OE(p,q)$ and $\ker\beta = \ker\gamma$. Suppose $\{e_j\}$ is a basis for $\ker\beta = \ker\gamma$ and expand it to a basis $\{e_j\} \dot\cup \{e_i\}$ for V. Then $\{e_i\beta\}$ and $\{e_i\gamma\}$ are bases for $\operatorname{ran}\beta$ and $\operatorname{ran}\gamma$, respectively, and hence $r(\beta) = r(\gamma)$. By Lemma 2, $\operatorname{ran}\alpha = \operatorname{ran}\gamma$, so $r(\alpha) = r(\beta)$; also $d(\alpha) = d(\gamma) \geq q$, so $\alpha \in OE(p,q)$. Conversely, suppose $\alpha, \beta \in OM(p,q) \cap OE(p,q)$ and $r(\alpha) = r(\beta)$. Let $\{e_k\}$ and $\{f_j\}$ be bases for $\ker\alpha$ and $\ker\beta$, respectively, with $|K| = n(\alpha) \geq q$ and $|J| = n(\beta) \geq q$. Expand these sets to two bases for V, say $\{e_k\} \dot\cup \{e_i\}$ and $\{f_j\} \dot\cup \{f_\ell\}$, respectively. Then, $\{e_i\alpha\}$ is a basis for $\operatorname{ran}\alpha$ and $\{f_\ell\beta\}$ is a basis for $\operatorname{ran}\beta$. Since $r(\alpha) = r(\beta)$, we have $|I| = |L|$, so we can write $\{f_i\}$ and $\{f_i\beta\}$ instead of $\{f_\ell\}$ and $\{f_\ell\beta\}$, respectively. Now define $\lambda \in T(V)$ by

$$\lambda = \begin{pmatrix} f_j & f_i \\ 0 & e_i\alpha \end{pmatrix}.$$

Since $n(\lambda) = n(\beta) \geq q$ and $d(\lambda) = d(\alpha) \geq q$, we have $\lambda \in OM(p,q) \cap OE(p,q)$. In fact, $\operatorname{ran}\alpha = \operatorname{ran}\lambda$ and $\ker\lambda = \ker\beta$, hence $\alpha \ \mathcal{L} \ \lambda \ \mathcal{R} \ \beta$ by Lemmas 2 and 5(b). In other words, we have shown that $\alpha \ \mathcal{D} \ \beta$ in $OM(p,q)$.

Now suppose $\alpha \ \mathcal{L} \ \gamma \ \mathcal{R} \ \beta$ in $OM(p,q)$ and $\beta \notin OE(p,q)$. Then, $\beta = \gamma$ by Lemma 5(a), and so $\alpha \ \mathcal{L} \ \beta$. Hence, $\operatorname{ran}\alpha = \operatorname{ran}\beta$ and $\alpha \notin OE(p,q)$. Conversely, if $\alpha, \beta \notin OE(p,q)$ and $\operatorname{ran}\alpha = \operatorname{ran}\beta$ then $\alpha \ \mathcal{L} \ \beta$ (by Lemma 2), and the result follows. □

Theorem 4. If $\alpha, \beta \in OM(p,q)$ then $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in OM(p,q)^1$ if and only if $r(\alpha) \leq r(\beta)$. Consequently, $\alpha \ \mathcal{J} \ \beta$ in $OM(p,q)$ if and only if $r(\alpha) = r(\beta)$.

Proof. Let $\alpha, \beta \in OM(p,q)$ and suppose $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in OM(p,q)^1$. Then, $r(\alpha) \leq r(\beta\mu) \leq r(\beta)$. Conversely, suppose $r(\alpha) \leq r(\beta)$ and let $\{e_j\}$ and $\{f_k\}$ be bases for $\ker\alpha$ and $\ker\beta$, respectively, with $|J| = n(\alpha) \geq q$ and $|K| = n(\beta) \geq q$. Expand $\{e_j\}$ to a basis $\{e_j\} \dot\cup \{e_i\}$ for V and write $e_i\alpha = a_i$ for each i. Clearly, $\{a_i\}$ is a basis for $\operatorname{ran}\alpha$. Similarly, expand $\{f_k\}$ to a basis for V, say $\{f_k\} \dot\cup \{f_\ell\}$, and write $\{f_\ell\} = \{g_i\} \dot\cup \{g_m\}$ (note that $\{f_\ell\beta\}$ is a basis for $\operatorname{ran}\beta$ and $r(\beta) \geq r(\alpha) = |I|$). Now write $g_i\beta = b_i$ and $g_m\beta = b_m$ for each i and each m. Since $\{b_i\} \dot\cup \{b_m\}$ is a basis for $\operatorname{ran}\beta$, it can be extended to a basis $\{b_i\} \dot\cup \{b_m\} \dot\cup \{b_r\}$ for V, where $|R| = d(\beta)$.

If $|M| \geq q$ or $|R| \geq q$, define $\lambda, \mu \in T(V)$ by

$$\lambda = \begin{pmatrix} e_j & e_i \\ 0 & g_i \end{pmatrix}, \quad \mu = \begin{pmatrix} b_i & b_m & b_r \\ a_i & 0 & 0 \end{pmatrix}.$$

Then $n(\lambda) = n(\alpha) \geq q$ and $n(\mu) = |M| + |R| \geq q$, so $\lambda, \mu \in OM(p,q)$. Also, $\alpha = \lambda\beta\mu$. On the other hand, suppose that $|M| < q$ and $|R| < q$. Then, $r(\beta) + d(\beta) = p$ implies $|I| = p$. Since p is infinite, we can write $\{g_i\} = \{u_i\} \dot\cup \{v_i\}$. For every i, write $u_i\beta = c_i$ and $v_i\beta = d_i$, and define λ, μ in $T(V)$ by

$$\lambda = \begin{pmatrix} e_j & e_i \\ 0 & u_i \end{pmatrix}, \quad \mu = \begin{pmatrix} c_i & d_i & b_m & b_r \\ a_i & 0 & 0 & 0 \end{pmatrix}.$$

Clearly, $n(\lambda) = n(\alpha) \geq q$ and $n(\mu) = \dim\langle d_i, b_m, b_r\rangle = p \geq q$. Hence, $\lambda, \mu \in OM(p,q)$. Also, $\alpha = \lambda\beta\mu$. □

It is well known that the ideals of $T(V)$ are precisely the sets

$$I_\xi = \{\alpha \in T(V) : r(\alpha) < \xi\}$$

where $1 \leq \xi \leq p'$ and p' denotes the successor of p (compare [3] Vol. 2, section IX.9). As remarked in section 2, each $OM(p,\xi)$, with $\aleph_0 \leq \xi \leq p$, is a right ideal of $T(V)$ and each $OE(p,\zeta)$, with $\aleph_0 \leq \zeta \leq p$, is a left ideal of $T(V)$. Hence, $OE(p,\zeta).OM(p,\xi)$ is an ideal of $T(V)$ for all cardinals ζ and ξ such that $\aleph_0 \leq \zeta, \xi \leq p$. Next we show that, in fact, $OE(p,\zeta).OM(p,\xi) = T(V)$. To do this, let $\alpha \in T(V)$ and write, in the usual way,

$$\alpha = \begin{pmatrix} e_j & e_i \\ 0 & a_i \end{pmatrix}.$$

If $|J| \geq \zeta$ and $|J| \geq \xi$ then define $\beta \in T(V)$ by

$$\beta = \begin{pmatrix} e_j & e_i \\ 0 & e_i \end{pmatrix}.$$

Clearly, $\beta \in OE(p,\zeta)$ and $\alpha = \beta\alpha$. Since $\alpha \in OM(p,\xi)$ in this case, it follows that $\alpha \in OE(p,\zeta).OM(p,\xi)$. Now if $|J| < \zeta \leq p$ or $|J| < \xi \leq p$, then $|I| = p$, hence we can write $\{e_i\} = \{f_i\} \,\dot\cup\, \{g_i\}$. Define $\delta, \gamma \in T(V)$ by

$$\delta = \begin{pmatrix} e_j & e_i \\ 0 & g_i \end{pmatrix}, \quad \gamma = \begin{pmatrix} e_j & f_i & g_i \\ 0 & 0 & a_i \end{pmatrix}.$$

Then, $d(\delta) = \dim\langle e_j, f_i\rangle = p \geq \zeta$ and $n(\gamma) = \dim\langle e_j, f_i\rangle = p \geq \xi$. Since $\alpha = \delta\gamma$, we have $\alpha \in OE(p,\zeta).OM(p,\xi)$. Therefore, $OE(p,\zeta).OM(p,\xi) = T(V)$.

If $\xi \leq p$ then $I_\xi \subseteq OM(p,q)$ (since $r(\alpha) < p$ and $p = n(\alpha) + r(\alpha)$ imply $n(\alpha) = p \geq q$). Hence, each I_ξ, with $1 \leq \xi \leq p$, is an ideal of $OM(p,q)$, and clearly it is a proper subset of $OM(p,q)$. The next result shows that these are exactly the proper ideals of $OM(p,q)$.

Theorem 5. The proper ideals of $OM(p,q)$ are precisely the sets I_ξ where $1 \leq \xi \leq p$. Moreover, the set I_ξ is a principal ideal of $OM(p,q)$ if and only if ξ is a successor cardinal.

Proof. By the remark above, each I_ξ, with $1 \leq \xi \leq p$, is a proper ideal of $OM(p,q)$. Conversely, let I be any proper ideal of $OM(p,q)$ and let ξ be the least cardinal greater than $r(\beta)$ for every $\beta \in I$ (possible since the cardinals are well-ordered). Then, $1 \leq \xi \leq p'$ and $I \subseteq OM(p,q) \cap I_\xi$. Given $\alpha \in OM(p,q) \cap I_\xi$, we know $n(\alpha) \geq q$ and $r(\alpha) < \xi$. Thus, there exists $\beta \in I$ such that $r(\alpha) \leq r(\beta)$: otherwise, $r(\beta) < r(\alpha) < \xi$ for every $\beta \in I$, and this contradicts our choice of ξ. Therefore, by Theorem 4, $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in OM(p,q)^1$ and so $\alpha \in I$. Hence $I = OM(p,q) \cap I_\xi$, and this equals I_ξ precisely when $\xi \neq p'$ (since $I_\xi = T(V)$ when $\xi = p'$).

Next we determine all principal ideals of $OM(p,q)$. To do this, let ξ be a successor cardinal, say $\xi = \eta'$, and choose $\alpha \in I_\xi$ with $r(\alpha) = \eta$. Then $r(\beta) \leq \eta$ for every $\beta \in I_\xi$

(otherwise, $r(\beta) > \eta$ implies $r(\beta) \geq \eta' = \xi$, a contradiction). Therefore, by Theorem 4, $\beta \in J(\alpha)$, the principal ideal of $OM(p,q)$ generated by α. Hence, $I_\xi \subseteq J(\alpha)$ and clearly the reverse inclusion also holds. Thus, I_ξ is principal. Conversely, suppose $I_\xi = J(\alpha)$ for some $\alpha \in OM(p,q)$. Let $r(\alpha) = \eta$ and suppose $\eta < \chi < \xi \leq p$ for some cardinal χ. Clearly, $\chi = r(\beta)$ for some $\beta \in OM(p,q)$ and, by Theorem 4, $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in OM(p,q)^1$. Therefore, $J(\alpha) \subsetneq J(\beta) \subseteq I_\xi$, contradicting our supposition. In other words, ξ is the least cardinal greater than η, and so $\xi = \eta'$. □

From the Theorem above, it follows that the semigroup $OM(p,q)$ is neither left 0-simple nor right 0-simple (recall a remark before Theorem 1). In passing, we note that this is true even if q is finite. For, in this case, the sets I_ξ with $1 < \xi \leq p$ still are non-zero proper ideals of $OM(p,q)$.

Similarly, we can determine the ideals of $OE(p,q)$. To do so, we first describe the $\mathcal{D}$ and $\mathcal{J}$ relations on this semigroup.

Theorem 6. If $\alpha, \beta \in OE(p,q)$ then $\alpha\ \mathcal{D}\ \beta$ in $OE(p,q)$ if and only if one of the following occurs.

(a) $\alpha, \beta \in OM(p,q)$ and $r(\alpha) = r(\beta)$,

(b) $\alpha, \beta \notin OM(p,q)$ and $\ker\alpha = \ker\beta$.

Proof. Suppose $\alpha\ \mathcal{L}\ \gamma\ \mathcal{R}\ \beta$ in $OE(p,q)$. By Lemma 7(b), if $\alpha \in OM(p,q)$ then $\gamma \in OM(p,q)$ and $\operatorname{ran}\alpha = \operatorname{ran}\gamma$. By Lemma 3, $\ker\beta = \ker\gamma$, so $r(\beta) = r(\gamma)$ and $n(\beta) = n(\gamma) \geq q$. Therefore, $\beta \in OM(p,q)$ and $r(\alpha) = r(\beta)$. Conversely, if $\alpha, \beta \in OM(p,q) \cap OE(p,q)$ and $r(\alpha) = r(\beta)$, then the same argument as that used in the proof of Theorem 3(a) shows that $\alpha\ \mathcal{D}\ \beta$ in $OE(p,q)$.

Now suppose $\alpha\ \mathcal{L}\ \gamma\ \mathcal{R}\ \beta$ in $OE(p,q)$ and $\alpha \notin OM(p,q)$. Then, $\alpha = \gamma$ by Lemma 7(a), and so $\alpha\ \mathcal{R}\ \beta$. By Lemma 3, $\ker\alpha = \ker\beta$, hence $\beta \notin OM(p,q)$. Conversely, if $\alpha, \beta \notin OM(p,q)$ and $\ker\alpha = \ker\beta$ then $\alpha\ \mathcal{R}\ \beta$ (by Lemma 3), and the result follows.□

Theorem 7. If $\alpha, \beta \in OE(p,q)$ then $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in OE(p,q)^1$ if and only if $r(\alpha) \leq r(\beta)$. Consequently, $\alpha\ \mathcal{J}\ \beta$ in $OE(p,q)$ if and only if $r(\alpha) = r(\beta)$.

Proof. Suppose $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in OE(p,q)^1$. Then, as before, $r(\alpha) \leq r(\beta)$. Conversely, assume $r(\alpha) \leq r(\beta)$ and write, in the usual way,

$$\alpha = \begin{pmatrix} e_j & e_i \\ 0 & a_i \end{pmatrix}, \quad \beta = \begin{pmatrix} f_k & g_i & g_m \\ 0 & b_i & b_m \end{pmatrix}$$

(note that this is possible since $r(\beta) \geq r(\alpha) = |I|$). Clearly, $\{a_i\}$ and $\{b_i\} \dot\cup \{b_m\}$ are bases for $\operatorname{ran}\alpha$ and $\operatorname{ran}\beta$, respectively. Hence, they can be expanded to bases for V, say $\{a_i\} \dot\cup \{a_s\}$ and $\{b_i\} \dot\cup \{b_m\} \dot\cup \{b_r\}$, respectively, where $|S| = d(\alpha) \geq q$ and $|R| = d(\beta) \geq q$. If $|K| \geq q$ or $|M| \geq q$, then define $\lambda, \mu \in T(V)$ by

$$\lambda = \begin{pmatrix} e_j & e_i \\ 0 & g_i \end{pmatrix}, \quad \mu = \begin{pmatrix} b_i & b_m & b_r \\ a_i & 0 & 0 \end{pmatrix}.$$

Since $d(\lambda) = \dim\langle f_k, g_m \rangle \geq q$ and $d(\mu) = d(\alpha) \geq q$, it follows that $\lambda, \mu \in OE(p,q)$. Also, $\alpha = \lambda\beta\mu$. Now suppose $|K| < q$ and $|M| < q$. Then, $n(\beta) + r(\beta) = p$ implies

$|I| = p$. Therefore, we can write $\{g_i\} = \{u_i\} \dot\cup \{v_i\}$ (because p is infinite). Let $u_i\beta = c_i$ and $v_i\beta = d_i$ for each i and define $\lambda, \mu \in T(V)$ by

$$\lambda = \begin{pmatrix} e_j & e_i \\ 0 & u_i \end{pmatrix}, \quad \mu = \begin{pmatrix} c_i & d_i & b_m & b_r \\ a_i & 0 & 0 & 0 \end{pmatrix}.$$

Clearly, $d(\lambda) = \dim\langle f_k, v_i, g_m\rangle = p \geq q$ and $d(\mu) = d(\alpha) \geq q$. Hence, $\lambda, \mu \in OE(p, q)$. Also, $\alpha = \lambda\beta\mu$. $\square$

The following result determines the proper ideals of $OE(p, q)$: they are exactly the proper ideals of $T(V)$.

Theorem 8. The proper ideals of $OE(p, q)$ are precisely the sets I_ξ where $1 \leq \xi \leq p$. Moreover, the set I_ξ is a principal ideal of $OE(p, q)$ if and only if ξ is a successor cardinal.

Proof. If $1 \leq \xi \leq p$ then $I_\xi \subseteq OE(p, q)$ (since $r(\alpha) < \xi \leq p$ and $p = r(\alpha) + d(\alpha)$ imply $d(\alpha) = p \geq q$). Since each I_ξ, with $1 \leq \xi \leq p$, is an ideal of $T(V)$ (see a remark before Theorem 5) it is an ideal of $OE(p, q)$. For the converse, let I be an ideal of $OE(p, q)$ and let ξ be the least cardinal greater than $r(\beta)$ for every $\beta \in I$ (this is possible since the cardinals are well-ordered). Then, $1 \leq \xi \leq p'$ and $I \subseteq OE(p, q) \cap I_\xi$. Let $\alpha \in OE(p, q) \cap I_\xi$. Then, $d(\alpha) \geq q$ and $r(\alpha) < \xi$, hence (as before) there exists $\beta \in I$ such that $r(\alpha) \leq r(\beta)$. Therefore, by Theorem 7, $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in OE(p, q)$[1] and so $\alpha \in I$. Hence, $I = OE(p, q) \cap I_\xi$ and this equals I_ξ precisely when $\xi \neq p'$ (since $I_\xi = T(V)$ when $\xi = p'$). Using an argument similar to that in the proof of Theorem 5, we conclude that I_ξ is principal if and only if ξ is a successor cardinal; and in this case, $I_\xi = J(\alpha)$ for some α such that $r(\alpha)' = \xi$. $\square$

It is now easy to see that the semigroup $OE(p, q)$ is neither left 0-simple nor right 0-simple (recall a remark before Theorem 1), and this is true even if q is finite (since, in this case, the sets I_ξ with $1 < \xi \leq p$ still are non-zero proper ideals of $OE(p, q)$).

Given the results above on the ideals of the semigroups $OM(p, q)$ and $OE(p, q)$ and the results obtained in [7] section 3 on the ideals of $AM(p, q)$ and $AE(p, q)$, we end this section by illustrating the ideal structure of these four semigroups: it is now easy to see that their ideal structures are extremely connected. Clearly, the three columns below the first row in the following diagram are mutually disjoint.

$$\begin{array}{ccccccc}
AM(p,q) & & OE(p,q) & & OM(p,q) & & AE(p,q) \\
& \searrow \swarrow & & \searrow \swarrow & & \searrow \swarrow & \\
& AM(p,q) \cap OE(p,q) & & OE(p,q) \cap OM(p,q) & & OM(p,q) \cap AE(p,q) & \\
& \vdots & & \downarrow & & \vdots & \\
& AM(p,q) \cap OE(p,\xi) & & I_p & & OM(p,\xi) \cap AE(p,q) & \\
& \vdots & & \vdots & & \vdots & \\
& AM(p,q) \cap OE(p,p) & & I_1 & & OM(p,p) \cap AE(p,q) &
\end{array}$$

4. Maximal right cancellative subsemigroups

In [6], the author studied basic properties of the semigroup $KN(p, q)$ consisting of all injective linear transformations $\alpha \in T(V)$ for which $d(\alpha) \geq q$. She showed that $KN(p, q)$ is a right cancellative semigroup without idempotents; and if $p > q$ then its right ideals form a chain and it has no maximal principal left ideals. Also, by [6] Theorem 6, any semigroup with these properties can be embedded in some $KN(p, q)$, where $p = |S|$. Clearly $KN(p, q) \subseteq OE(p, q)$. In fact, it is a maximal right cancellative subsemigroup of $OE(p, q)$, as we proceed to show.

Lemma 8. Let S be a subsemigroup of $T(V)$ containing $KN(p, q)$ and at least one non-injective element of $T(V)$. Then, there exists some $\alpha \in S$ such that $n(\alpha) \geq 2$.

Proof. Suppose $n(\alpha) < 2$ for every $\alpha \in S$. Since S is not contained in the semigroup of all injective linear transformations of V, there exists $\gamma \in S$ such that $n(\gamma) = 1$. Let $a \in \ker \gamma$ be non-zero and suppose $\{a\} \dot\cup \{a_i\}$ is a basis for V, with $|I| = p$. Write $\{a_i\} = \{b\} \dot\cup \{c_i\}$ (note that this is possible since p is infinite). Clearly, $\{b\gamma\} \dot\cup \{c_i\gamma\}$ is a basis for $\operatorname{ran}\gamma$, hence it can be extended to a basis for V, say $\{b\gamma\} \dot\cup \{c_i\gamma\} \dot\cup \{d_\ell\}$, where $|L| = d(\gamma) \leq p$. Now write $\{c_i\} = \{e_i\} \dot\cup \{f_i\} \dot\cup \{g_\ell\}$ and define $\lambda, \mu \in T(V)$ by

$$\lambda = \begin{pmatrix} a & b & c_i \\ b & a & e_i \end{pmatrix}, \quad \mu = \begin{pmatrix} b\gamma & c_i\gamma & d_\ell \\ a & e_i & g_\ell \end{pmatrix}.$$

Then $n(\lambda) = 0 = n(\mu)$ and $d(\lambda) = p = d(\mu)$, so $\lambda, \mu \in KN(p, q) \subseteq S$. Therefore, $\lambda\gamma\mu\gamma \in S$ and we have $a\lambda\gamma\mu\gamma = 0 = b\lambda\gamma\mu\gamma$. Since $\{a, b\}$ is linearly independent, it follows that $n(\lambda\gamma\mu\gamma) \geq 2$, a contradiction. Hence, there exists some $\alpha \in S$ such that $n(\alpha) \geq 2$. □

Theorem 9. The semigroup $KN(p, q)$ is a maximal right cancellative subsemigroup of $OE(p, q)$.

Proof. Suppose $KN(p, q) \subseteq M \subseteq OE(p, q)$, where M is a right cancellative subsemigroup of $OE(p, q)$. If M contains some non-injective element, then there exists $\beta \in M$ such that $n(\beta) \geq 2$ (by Lemma 8). Suppose $\{e_j\}$ is a basis for $\ker\beta$ and let $a, b \in \{e_j\}$, $a \neq b$. Now expand $\{a, b\}$ to a basis $\{a, b\} \dot\cup \{e_i\}$ for V, with $|I| = p$, and write $\{e_i\} = \{f_i\} \dot\cup \{g_i\}$ (possible since p is infinite). Define $\lambda, \mu \in T(V)$ by

$$\lambda = \begin{pmatrix} a & b & e_i \\ a & b & f_i \end{pmatrix}, \quad \mu = \begin{pmatrix} a & b & e_i \\ b & a & f_i \end{pmatrix}.$$

Then $n(\lambda) = 0 = n(\mu)$ and $d(\lambda) = p = d(\mu)$, hence $\lambda, \mu \in KN(p, q) \subseteq M$. Clearly $\lambda\beta = \mu\beta$ and, since M is right cancellative, it follows that $\lambda = \mu$, a contradiction. Therefore, all elements of M are one-to-one and, since $M \subseteq OE(p, q)$, it follows that $M \subseteq KN(p, q)$. Hence, $M = KN(p, q)$ and we have the required result. □

The following example illustrates the fact that $OE(p, q)$ contains maximal right cancellative subsemigroups which do not equal $KN(p, q)$.

Example 1. Let $V = U \oplus \langle x_i \rangle$ where $\dim U = q$ and $|I| = p$, and let H denote the set of all $\alpha \in T(V)$ with the form:

$$\alpha = \begin{pmatrix} U & x_i \\ 0 & x_{i\pi} \end{pmatrix}$$

for some $\pi \in G(I)$, the symmetric group on the set I. It is easy to see that H is a subgroup of $OE(p,q)$, and clearly H contains no injective elements. Moreover, if $\mathcal{F}$ denotes the family of all right cancellative subsemigroups of $OE(p,q)$ that contain H, then $\mathcal{F}$ is non-empty. Thus, we can use Zorn's Lemma to show that $\mathcal{F}$ contains a maximal element, H' say. Then H' is a maximal right cancellative subsemigroup of $OE(p,q)$ which does not equal $KN(p,q)$.

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