F–SEMIGROUPS[1]

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Abstract. A semigroup S is called F- semigroup if there exists a group-congruence ρ on S such that every ρ-class contains a greatest element with respect to the natural partial order $\leq_S$ of S (see [8]). This generalizes the concept of F-inverse semigroups introduced by V. Wagner [12] and investigated in [7]. Five different characterizations of general F-semigroups S are given: by means of residuals, by special principal anticones, by properties of the set of idempotents, by the maximal elements in $(S, \leq_S)$ and finally, an axiomatic one using an additional unary operation. Also F-semigroups in special classes are considered; in particular inflations of semigroups and strong semilattices of monoids are studied.

1. Introduction and summary

The concept of F*–inverse* semigroup S was introduced by V. Wagner [12]: S is defined as an inverse semigroup such that for the least group–congruence σ on S every σ–class contains a greatest element with respect to the natural partial order of S. A theory of this class of inverse semigroups was developed in [7] (see also the corresponding chapters in [11] and [6]).

By [8] for every semigroup S the relation

$$a \leq_S b \text{ if and only if } a = xb = by, xa = a (= ay) \text{ for some } x, y \in S^1$$

is a partial order on S, the so called *natural partial order* . Note that for $e, f \in E_S$, $e \leq_S f$ iff $e = ef = fe$. If S is an inverse semigroup then $\leq_S$ coincides with the partial order mentioned above. Thus a generalization to non–inverse semigroups is made possible. First, in [3] *generalized* F–semigroups were defined as semigroups S, on which there exists a group–congruence ρ such that the identity ρ–class in the group S/ρ contains a greatest element with respect to the natural partial order of S, the pivot of S. Here the special case (corresponding to the F–inverse case) of F*–semigroups* is considered, which are defined as generalized F–semigroups S such that *every* ρ–class contains a greatest element with respect to $\leq_S$.

In the theory of partially ordered semigroups $(S, \cdot, \leq)$ F–semigroups appear under the name of *strong Dubreil–Jacotin* semigroups (see [1] and Theorem 3.5 below). Note that in this case the partial order $\leq$ given on S is supposed to be compatible with multiplication, that is, $a \leq b$ in S implies that $ac \leq bc$ and $ca \leq cb$ for every $c \in S$. In general, the natural partial order on a semigroup does not have this property (see [9]). Hence the results on strong Dubreil–Jacotin semigroups cannot be applied to our case and a theory for its own will be developed.

[1]Research supported by the Portuguese Foundation for Science and Technology through the research program POCTI.

In Section 2, examples of F–semigroups are given. The concept of residual – borrowed from the theory of partially ordered semigroups (see [1] or [2]) – is defined in Section 3. A characterization of F–semigroups by means of residuals of the greatest element of some ρ–class, in particular of the identity class, is provided. Also, an axiomatic characterization of F-semigroups due to M. Petrich is given. In Section 4, principal anticones (which characterize generalized F–semigroups, see [3]) are specialized thus obtaining another characterization of F–semigroups, which in turn gives rise to a characterization in terms of idempotents. The particular cases, where a greatest idempotent or an identity exists, are considered. Section 5 provides a characterization of F–semigroups S by means of the maximal elements in $(S, \leq_S)$ and the principal order ideals generated by them. Several special cases, as inverse, Clifford, monoid, centric, commutative, or eventually regular semigroups, are dealt with. The final section contains necessary and sufficient conditions for inflations of semigroups or strong semilatties of monoids, to be F–semigroups.

2. Definitions and examples

A semigroup $(S, \cdot)$ will be called *F–semigroup* if there exists a congruence ρ on S such that (1) $(S/\rho, \cdot)$ is a group, and (2) each ρ–class of S contains a greatest element with respect to the natural partial order $\leq_S$ of S. The maximum of the ρ–class, which constitutes the identity of S/ρ, is called the *pivot of S*.
In [3] a semigroup $(S, \cdot)$ was defined to be a *generalized F–semigroup* if there exists a group–congruence ρ on S such that the identity ρ–class of S admits a greatest element with respect to $\leq_S$. Hence every F–semigroup is a generalized F–semigroup. Therefore by [3], Proposition 3.7, Corollary 3.12, we have for every F–semigroup S:

1. E_S is a subsemigroup of S with greatest element;
2. S is E–inversive (i.e., for every $a \in S$ there exists $x \in S$ with $ax \in E_S$);
3. if S is a monoid then S is E–unitary (i.e., $e, ae \in E_S(e, ea \in E_S)$ implies that $a \in E_S$).

REMARK. The group–congruence ρ appearing in the definition of an F–semigroup S is uniquely determined: if τ is any group–congruence on S making S an F–semigroup then both ρ and τ make S a generalized F–semigroup, so that $\rho = \tau$ by [3], Theorem 3.3. In the following ρ will always denote this group–congruence and ϕ the natural homomorphism from S onto the group $G = S/\rho$.

EXAMPLES.
1. Every group G is an F–semigroup: the equality relation on G is a group-congruence, whose classes consist of a single element each.
2. Every semigroup S with greatest element is an F–semigroup: the universal relation on S is the desired group congruence. Semigroups with greatest element are characterized in [3], Theorem 3.5; in general, they are not regular.
3. A band B is an F–semigroup if and only if B has an identity: this holds by [3], Example 3 of Section 4. taking into account Example (2).
4. A trivially ordered semigroup S is an F–semigroup if and only if S is a group: this follows from [3], Example 5 of Section 4. and Example (1).

5. A semigroup S with zero is an F–semigroup if and only if S admits a greatest element: this holds by [3], Example 6, Section 4, and Example (2).
6. The direct product $S = T \times G$, where T is a semigroup with greatest element ω and G is a group, is an F–semigroup: the mapping $\phi : S \to G$, $(t,g)\phi = g$, is a surjective homomorphism, and for the corresponding group–congruence ρ on S an arbitrary ρ–class $(t,g)\rho \in S/\rho = G$ has $(\omega, g) \in S$ as greatest element. If T has no identity or is not regular then S has the same property.

Further examples are given in Section 6. below. The following is an example of a generalized F–semigroup which is *not* an F–semigroup.

7. Let $Y = \{\alpha, \beta, \gamma, \omega\}$ be the four–element Boolean lattice with $\alpha \| \beta, \gamma$ the least and ω the greatest element. Let $G_\alpha = \{1_\alpha, a\}, G_\beta = \{1_\beta, b\}, G_\gamma = \{1_\gamma, c\}$ be two–element groups and $G_\omega = \{1_\omega\}$ the trivial group. Let $1_\omega\phi_{\omega,\alpha} = 1_\alpha$, $1_\omega\phi_{\omega,\beta} = 1_\beta$, $1_\omega\phi_{\omega,\gamma} = 1_\gamma$; $1_\alpha\phi_{\alpha,\gamma} = 1_\gamma$, $a\phi_{\alpha,\gamma} = c$; $1_\beta\phi_{\beta,\gamma} = 1_\gamma$, $b\phi_{\beta,\gamma} = c$; then each of these maps is an injective homomorphism. Hence by [3], Corollary 4.7, the Clifford–semigroup $S = [Y, G_i, \phi_{i,j}]$ is a generalized F–semigroup. But S is not an F–semigroup. Indeed, for the (least) group–congruence σ on S (given by $x\sigma y$ if and only if $ex = ey$ for some $e \in E_S$) the σ–class $a\sigma \in S/\sigma$ has no greatest element: $a\sigma b$ since $1_\gamma a = 1_\gamma b = c$; if $m \in a\sigma$ is such that $a \leq_S m$, $b \leq_S m$, then by [5], Example V.10, $m \in G_\omega$ and $m = 1_\omega$ (since $\alpha \| \beta$); but $a \leq_S 1_\omega \in E_S$ implies by [9], Lemma 2.1, that $a \in E_S$, a contradiction. Note that σ makes S a generalized F–semigroup; hence by the Remark above, there is no other group–congruence on S which could make S an F–semigroup.

In a partially ordered set $(X, \leq)$, a subset having a greatest element is not necessarily a principal order ideal, that is, of the form $(a] = \{x \in X \mid x \leq a\}$. However, if S is an F–semigroup this holds for the ρ–classes with respect to $\leq_S$:

Lemma 2.1. *Let S be a semigroup such that there exist a group G and a surjective homomorphism $\phi : S \to G$. Then (i) $a \leq_S b$ implies $a\varphi = b\varphi$; (ii)for every $g \in G$, $g\phi^{-1}$ is a principal order ideal of $(S, \leq_S)$ if and only if $g\phi^{-1}$ admits a greatest element.*

PROOF. (i) holds by cancellation in G. (ii) This follows from the following fact: if $g \in G$ and $a \in g\phi^{-1}$, then for any $x \leq_S a$ $(x \in S)$ we have by (i), that $x\phi = a\phi = g$, whence $x \in g\phi^{-1}$.

3. Residuals

In the sequel we will use the following concepts, which are central in the theory of partially ordered semigroups (see [1],[2]). Let S be a semigroup and $s \in S$; for any $a \in S$ define $\langle s \,.\!\cdot\, a \rangle = \{x \in S \mid ax \leq_S s\}$. If $\langle s \,.\!\cdot\, a \rangle$ has a greatest element with respect to $\leq_S$ then this element is denoted by $s \,.\!\cdot\, a$ and is called the *right residual of s by a*. The left residual of $s \in S$ by $a \in S$ is defined dually and is denoted by $s \cdot\!. \, a$. If $s \,.\!\cdot\, a$ (resp. $s \cdot\!.\, a$) exists for every $a \in S$ then $s \in S$ is called *right* (resp. *left*) *residuated*; $s \in S$ is called *residuated* if it is both right– and left–residuated. Finally, $s \in S$ is called *equiresiduated* if it is residuated and $s \,.\!\cdot\, a = s \cdot\!. a$ for every $a \in S$; the latter element is denoted by $s : a$.

We give a first characterization of F–semigroups by means of residuals of the pivot, i.e., the greatest element of the identity ρ–class. Recall that ρ denotes the defining group–congruence and ϕ the corresponding natural homomorphism.

Proposition 3.1. *Let S be an F–semigroup and let $m \in S$ be the maximum element of its ρ–class. Then for any $a \in S$, $\langle m \;.\!\cdot\; a\rangle = \{x \in S \mid x\phi = (a\rho)^{-1}(m\rho)\}$, $\langle m \;\cdot\!.\; a\rangle = \{x \in S \mid x\phi = (m\rho)(a\rho)^{-1}\}$, both are principal order ideals of $(S, \leq_S)$, and $m \in S$ is residuated.*

PROOF. Let $a \in S$; then by Lemma 2.1:

$$x \in \langle m \;.\!\cdot\; a\rangle \Leftrightarrow ax \leq_S m \Leftrightarrow (ax)\phi = m\phi \Leftrightarrow x\phi = (a\rho)^{-1}(m\rho).$$

Thus for $g = (a\rho)^{-1}(m\rho) \in S/\rho$ we have that $\langle m \;.\!\cdot\; a\rangle = g\phi^{-1}$. Since $g\phi^{-1}$ has a greatest element so does $\langle m \;.\!\cdot\; a\rangle$, that is, $m \in S$ is right–residuated. Furthermore by Lemma 2.1, $\langle m \;.\!\cdot\; a\rangle$ is a principal order ideal of $(S, \leq_S)$; in fact $\langle m \;.\!\cdot\; a\rangle = (m \;.\!\cdot\; a]$. Similarly, $\langle m \;\cdot\!.\; a\rangle$ has a greatest element and $\langle m \;\cdot\!.\; a\rangle = (m \;\cdot\!.\; a]$. Hence m is residuated.

Corollary 3.2. *Let S be an F–semigroup with pivot ξ. Then $\langle \xi \;.\!\cdot\; a\rangle = [(a\rho)^{-1}]\phi^{-1} = \langle \xi \;\cdot\!.\; a\rangle = (\xi : a]$ for any $a \in S$, and ξ is equiresiduated.*

PROOF. Let $a \in S$. Since ξ is the greatest element of the identity ρ–class we have that $\xi\rho = 1_G$, the identity of $G = S/\rho$. It follows by Proposition 3.1, that ξ is residuated and that $\langle \xi \;.\!\cdot\; a\rangle = \{x \in S \mid x\phi = (a\rho)^{-1}\} = \langle \xi \;\cdot\!.\; a\rangle$ is a principal order ideal of $(S, \leq_S)$. Therefore $\xi \in S$ is equiresiduated and $\langle \xi \;.\!\cdot\; a\rangle = (\xi : a]$.

The following results are consequences of Corollary 3.2.

Corollary 3.3. *Let S be an F–semigroup with pivot ξ. Then for every $a \in S$,*

(1) the greatest element of $(a\rho)^{-1} \in S/\rho$ is $\xi : a$;

(2) the greatest element of $a\rho \in S/\rho$ is $\xi : (\xi : a)$.

PROOF. (1) Let $a \in S$; then by Corollary 3.2, $\xi : a$ exists and $[(a\rho)^{-1}]\phi^{-1} = (\xi : a]$. Hence $\xi : a$ is the greatest element of the ρ–class $(a\rho)^{-1} \in S/\rho$.
(2) Let $a \in S$; then by (1), $(a\rho)^{-1} = (\xi : a)\rho$. It follows that $a\rho = [(\xi : a)\rho]^{-1}$ has $\xi : (\xi : a)$ as greatest element.

Corollary 3.4. *Let S be an F–semigroup with pivot ξ. Then the corresponding group–congruence ρ on S is given by: $a \;\rho\; b$ if and only if $\xi : a = \xi : b$.*

PROOF. By Corollary 3.2, $\xi : a$ exists for every $a \in S$. It follows by Corollary 3.3(1), that

$$a\rho b \Leftrightarrow a\rho = b\rho \Leftrightarrow (a\rho)^{-1} = (b\rho)^{-1} \Leftrightarrow \xi : a = \xi : b.$$

Also using Corollary 3.2 we have a first characterization of F–semigroups in the class of all generalized F–semigroups (concerning the latter see [3]).

Theorem 3.5. *Let S be a semigroup. Then S is an F–semigroup if and only if S is a generalized F–semigroup whose pivot is right (left, equi) residuated.*

PROOF. *Necessity.* By Corollary 3.2, the pivot ξ of S is equiresiduated. *Sufficiency.* By hypothesis, there exists a group–congruence ρ on S and a surjective homomorphism $\phi : S \to S/\rho = G$ such that $1_G\phi^{-1} = (\xi]$, i.e., $\xi\rho = 1_G$. Let $g \in G$; we have to show that $g\phi^{-1} \subseteq S$ has a greatest element. Since ϕ is surjective, $g^{-1} = a\phi = a\rho$ for some $a \in S$. By the proof of Proposition 3.1 (taking $m = \xi$), $\langle \xi \mathrel{.\cdot} a\rangle = [(a\rho)^{-1}]\phi^{-1} = g\phi^{-1}$. Since by hypothesis, $\xi \mathrel{.\cdot} a$ exists it follows that $\xi \mathrel{.\cdot} a$ is the greatest element of $g\phi^{-1}$. Therefore by definition, S is an F–semigroup.

This characterization of an F–semigroup S uses residuation of the pivot of S, that is, of the greatest element of the identity ρ–class of S. There is an analoguous one in terms of residuation of the greatest element of *some* non–identity ρ–class of S (compare with [1], Theorem 25.8, on partially ordered semigroups). In order to prove this we first make the following observations concerning residuals.

If $(S, \leq, \cdot)$ is a *partially ordered* semigroup and if $a, m \in S$ are such that $m \mathrel{.\cdot} a$ exists, then $x \leq m \mathrel{.\cdot} a$ implies by multiplication on the left by $a \in S$ that $ax \leq a(m \mathrel{.\cdot} a) \leq m$ (see the definition of residuals in [2]). For a semigroup S and its natural partial order $\leq_S$ this implication does not hold, in general. In fact, consider the following

EXAMPLE. Let B be the three–element band given in [10], p.176, third multiplication table:

B	a	b	c
a	a	b	a
b	a	b	a
c	a	b	c

Then $S = B^1$ is an F–semigroup (see Example (3) of Section 2). For $a, c \in B$ we have $c \mathrel{.\cdot} a = 1$, since $a \cdot 1 = a \leq_S c$ and $1 \in S$ is the greatest element of $(S, \leq_S)$. For $x = b$ we have $x \leq_S 1 = c \mathrel{.\cdot} a$, but $ax = ab = b \not\leq_S c$ (since $bc = a \neq b$). Nevertheless, we have the following result:

Lemma 3.6. *Let S be a semigroup and let ρ be a group congruence on S. Assume that $a, m \in S$ are such that m is the greatest element of its ρ–class and that $m \mathrel{.\cdot} a$ (resp. $m \mathrel{\cdot.} a$) exists. Then $ax \leq_S m$ if and only if $x \leq_S m \mathrel{.\cdot} a$ (resp. $xa \leq_S m$ if and only if $x \leq_S m \mathrel{\cdot.} a$).*

PROOF. If $ax \leq_S m$ for some $x \in S$ then by definition, $x \leq_S m \mathrel{.\cdot} a$. Conversely, let $x \in S$ be such that $x \leq_S m \mathrel{.\cdot} a$, that is $x \in (m \mathrel{.\cdot} a]$. As in the proof of Proposition 3.1, $\langle m \mathrel{.\cdot} a\rangle = g\varphi^{-1}$ for some $g \in G = S/\rho$. Since $m \mathrel{.\cdot} a = \max \langle m \mathrel{.\cdot} a\rangle = \max\, (g\varphi^{-1})$ it follows by Lemma 2.1, that $g\varphi^{-1} = (m \mathrel{.\cdot} a]$. Therefore $x \in (m \mathrel{.\cdot} a] = \langle m \mathrel{.\cdot} a\rangle$ and $ax \leq_S m$.

This result allows to deduce a useful formula for residuals with respect to products (compare with [1], Theorem 25.5, on partially ordered semigroups).

Lemma 3.7. *Let S be a semigroup and let ρ be a group–congruence on S. If some ρ–class admits a greatest element m, which is also right (left) residuated, then for all $a, b \in S, m \mathrel{.\cdot} ab = (m \mathrel{.\cdot} a) \mathrel{.\cdot} b$ (resp. $m \mathrel{\cdot.} ab = (m \mathrel{\cdot.} b) \mathrel{\cdot.} a$).*

PROOF. By hypothesis, $m \cdot\!\cdot\ s$ exists for every $s \in S$. Let $a, b \in S$; then by definition, $ab(m \cdot\!\cdot\ ab) \leq_S m$. Hence $b(m \cdot\!\cdot\ ab) \leq_S m \cdot\!\cdot\ a$ and thus $m \cdot\!\cdot\ ab \in \langle (m \cdot\!\cdot\ a) \cdot\!\cdot\ b \rangle$. Let $x \in \langle (m \cdot\!\cdot\ a) \cdot\!\cdot\ b \rangle$; then $bx \leq_S m \cdot\!\cdot\ a$. Thus by Lemma 3.6, $a \cdot bx \leq_S m$, whence $x \leq_S m \cdot\!\cdot\ ab$. It follows that $m \cdot\!\cdot\ ab \in S$ is the greatest element of $\langle (m \cdot\!\cdot\ a) \cdot\!\cdot\ b \rangle$, which is denoted by $(m \cdot\!\cdot\ a) \cdot\!\cdot\ b$. Similarly for the left–residuals.

The announced characterization of (generalized) F–semigroups now follows.

Theorem 3.8. *Let S be a semigroup. Then*

(1) S is a generalized F–semigroup if and only if there exists a group–congruence ρ on S such that one of the ρ–classes of S contains a greatest element m and $(m \cdot\!\cdot\ m)$ (resp. $m \cdot\!\cdot\ m$) exists.

(2) S is an F–semigroup if and only if there exists a group–congruence ρ on S such that one of the ρ–classes of S contains a greatest element m and m is right (resp. left) residuated.

PROOF. (1) *Necessity.* The pivot ξ of S is the greatest element of the identity ρ–class of S. We show that $\xi \cdot\!\cdot\ \xi$ exists (similarly, $\xi \cdot\!\cdot\ \xi$ exists). First, $\xi \in \langle \xi \cdot\!\cdot\ \xi \rangle$ since by [3], Corollary 3.6, $\xi^2 \leq_S \xi$. Let $x \in \langle \xi \cdot\!\cdot\ \xi \rangle$; then $\xi x \leq_S \xi$ so that by Lemma 2.1, $(\xi x)\phi = \xi\phi$. Thus $(\xi\rho)(x\rho) = \xi\rho$ and $x\rho = 1_G$, the identity of $G = S/\rho$. Therefore $x \in 1_G\phi^{-1} = (\xi]$, whence $x \leq_S \xi$. It follows that $\xi = \xi \cdot\!\cdot\ \xi$.

Sufficiency. We show that $m \cdot\!\cdot\ m$ is the greatest element of the identity ρ–class in $G = S/\rho$. Let $x \in 1_G\phi^{-1}$; then $x\phi = 1_G$ and $(mx)\phi = (m\phi)(x\phi) = m\phi$. Thus $mx \in m\rho = (m]$ and $mx \leq_S m$, that is, $x \leq_S m \cdot\!\cdot\ m$. It follows by Lemma 2.1, that $1_G = x\phi = (m \cdot\!\cdot\ m)\phi$, whence $m \cdot\!\cdot\ m \in 1_G\phi^{-1}$. Therefore, S is a generalized F–semigroup with pivot $\xi = m \cdot\!\cdot\ m$.

(2)*Necessity.* holds by Proposition 3.1.
Sufficiency. By hypothesis, $m \cdot\!\cdot\ m$ exists (resp. $m \cdot\!\cdot\ m$ exists). Therefore by (1), S is a generalized F–semigroup with pivot $\xi = m \cdot\!\cdot\ m$ (see the proof of sufficiency in (1)). Let $a \in S$; then by Lemma 3.7, $m \cdot\!\cdot\ ma = (m \cdot\!\cdot\ m) \cdot\!\cdot\ a = \xi \cdot\!\cdot\ a$. Hence $\xi \cdot\!\cdot\ a$ exists, that is, ξ is right residuated. It follows by Theorem 3.5, that S is an F–semigroup.

We conclude this section with an axiomatic characterization of F-semigroups due to M. Petrich. It consists of four axioms concerning a unary operation on a semigroup, which reflect properties of the set of the greatest elements in the different ρ-class of an F-semigroup.

Theorem 3.9. (M. Petrich) *A semigroup S is an F-semigroup if and only if S has a unary operation $a \to a'$ satisfying*
(F1) $(ab)' = (a'b)' = (ab')'$,
(F2) $a \leq_S a'$,
(F3) if $a' = (a^2)'$ *and* $b' = (b^2)'$ *then* $a' = b'$,
(F4) for any $a \in S$ *there exists* $b \in S$ *such that* $a' = (aba)'$.

PROOF.*Necessity.* Let ρ be the defining group congruence on S and for any $a \in S$, a' the greatest element of $a\rho \in S/\rho$. Then for all $a, b \in S$, $a\rho a'$ and

$b\rho b'$ imply that $ab\ \rho\ a'b\ \rho\ ab'$, whence (F1) holds. (F2) follows from the definition of $a' \in S$ $(a \in S)$. If $a' = (a^2)'$ and $b' = (b^2)'$ then $a\rho a^2$ and $b\rho b^2$ so that by cancellation in the group S/ρ, $a\rho = 1_G = b\rho$; hence (F3) holds. Since S/ρ is a regular semigroup, (F4) is satisfied.

Sufficiency. Define a relation ρ on S by: $a\rho b \Leftrightarrow a' = b'$. Then by (F1), ρ is a congruence. By (F4), S/ρ is a regular semigroup. Let $a\rho, b\rho \in E_{S/\rho}$. Then $a\rho a^2$, $b\rho b^2$, whence $a' = (a^2)'$, $b' = (b^2)'$. Thus (F3) yields that $a' = b'$, and $a\rho = b\rho$. Therefore S/ρ is a group (see [11], Lemma II.2.10). Let $a\rho \in S/\rho$; we first show that $a' \in a\rho$. Since by (F2), $a \leq_S a'$ we have $a = xa' = xa$ for some $x \in S^1$. If $x = 1$ then $a' = a \in a\rho$. If $x \in S$ then $(x\rho)(a'\rho) = (x\rho)(a\rho)$ so that by cancellation, $a'\rho = a\rho$, and $a' \in a\rho$. Let $b \in a\rho$; then $b\rho a$ and $b' = a'$. Hence it follows by (F2), that $b \leq_S b' = a'$. Therefore, $a' \in S$ is the greatest element of the ρ-class $a\rho \in S/\rho$, and S is an F-semigroup.

4. Principal anticones

Generalized F–semigroups S were characterized in [3], Theorem 3.1, by the existence of a principal anticone $H = (\xi]$ in $(S, \cdot, \leq_S)$. Specializing H we obtain F–semigroups. Note that for the pivot ξ of S and any $a \in S$,

$$H : a = \{x \in S \mid ax \in H = (\xi]\} = \{x \in S \mid ax \leq_S \xi\} = \langle \xi \,.\!\cdot\, a\rangle.$$

Hence $\xi \,.\!\cdot\, a$ exists in S if and only if $H : a$ contains a greatest element. Taking into account Theorem 3.5, we thus have proved

Theorem 4.1. *Let S be a semigroup. Then S is an F–semigroup if and only if S has a principal anticone H such that for every $a \in S$, $H : a$ contains a greatest element with respect to $\leq_S$.*

As a consequence we obtain the following characterization.

Theorem 4.2. *Let S be a semigroup. Then S is an F–semigroup with pivot ξ if and only if*

(i) ξ is an upper bound of E_S and $\xi^2 \in E_S$,
(ii) $H = E_S \cup \{\xi\}$ is a unitary subset of S, i.e., $hx, h \in H$ or $xh, h \in H$ implies $x \in H$,
(iii) for every $a \in S$, $H : a$ has a greatest element.

PROOF. *Necessity.* Since S is also a generalized F–semigroup with pivot ξ, (i) and (ii) hold by [3], Theorem 3.8 and Corollary 3.6. Furthermore $(\xi] = E_S \cup \{\xi\} = H$ is a principal anticone of S. Thus by Theorem 4.1, also (iii) holds.

Sufficiency. First we show that S is E–inversive. Let $a \in S$; then by (iii), $H : a \neq \phi$. Hence there exists $x \in S$ such that $ax \in H = E_S \cup \{\xi\}$. If $ax \in E_S$ we are done; if $ax = \xi$ then $a \cdot x\xi = \xi^2 \in E_S$ by (i). Hence by the proof of Theorem 3.8 in [3], $H = E_S \cup \{\xi\}$ is a principal anticone of S. Together with (iii) it follows by Theorem 4.1, that S is an F–semigroup.

Two particular cases should be mentioned.

Theorem 4.3. *Let S be a semigroup containing a greatest idempotent e. Then S is an F–semigroup with pivot e if and only if (i) S is E–unitary, and (ii) for every $a \in S$ there exists a greatest $x \in S$ such that $ax \in E_S$.*

PROOF. *Necessity.* By Theorem 4.2, $H = E_S \cup \{e\} = E_S$ is a unitary subset of S, that is, S is E–unitary. Furthermore for every $a \in S$, $H : a = E_S : a = \{x \in S \mid ax \in E_S\}$ has a greatest element.
Sufficiency. By hypothesis, $e \in E_S$ is an upper bound of E_S with $e^2 = e \in E_S$. Also, $H = E_S \cup \{e\} = E_S$ is a unitary subset of S, by (i). Finally for every $a \in S$, $H : a = E_S : a$ has a greatest element, by (ii). Therefore by Theorem 4.2, S is an F–semigroup with pivot e.

If S is a *monoid* then the identity 1_S is the greatest idempotent of S. In this case the condition that S be E–unitary can be dropped. To see this we show

Lemma 4.4. *Let S be a monoid. Then S is E–unitary if and only if for every $e \in E_S$, $1_S \,.\!\cdot\, e$ (resp. $1_S \cdot\!.\, e$) exists in S.*

PROOF. *Necessity.* Let $e \in E_S$; then $e1_S \leq_S 1_S$ implies that $1_S \in \langle 1_S \,.\!\cdot\, e\rangle$. If $x \in \langle 1_S \,.\!\cdot\, e\rangle$ then $ex \leq_S 1_S$. Thus by [9], Lemma 2.1, $ex \in E_S$. Since S is E–unitary it follows that $x \in E_S$. Hence $x \leq_S 1_S$ and $1_S = 1_S \,.\!\cdot\, e$. Similarly $1_S \cdot\!.\, e = 1_S$.

Sufficiency. We show first that $1_S \,.\!\cdot\, e = 1_S$ for any $e \in E_S$. Indeed, $e1_S = e \leq_S 1_S$ implies that $1_S \leq_S 1_S \,.\!\cdot\, e$. But the identity of a semigroup is a maximal element with respect to $\leq_S$: $1_S \leq_S a \Rightarrow 1_S = xa = x1_S = x \Rightarrow 1_S = a$. Hence it follows that $1_S = 1_S \,.\!\cdot\, e$. Let $e, ex \in E_S$ for some $x \in S$. Then $ex \leq_S 1_S$ and so $x \leq_S 1_S \,.\!\cdot\, e = 1_S$. Thus by [9], Lemma 2.1, $x \in E_S$. Therefore S is E–unitary.

Using this result we obtain the second particular case.

Theorem 4.5. *Let S be a monoid. Then S is an F–semigroup if and only if for every $a \in S$ there exists a greatest $x \in S$ such that $ax \in E_S$.*

PROOF. *Necessity.* Since S is also a generalized F–semigroup, the pivot of S is $\xi = 1_S$ (1_S is maximal in $(S, \leq_S)$). Since 1_S is the greatest idempotent of S, the statement now follows by Theorem 4.3.

Sufficiency. First $E_S \neq \phi$ (see Section 2.(2)). Let $e \in E_S$; then by hypothesis, there exists a greatest $x \in S$ such that $ex \in E_S$, that is, $ex \leq_S 1_S$ ([9], Lemma 2.1). Therefore by definition, $1_S \,.\!\cdot\, e$ exists. It follows by Lemma 4.4, that S is E–unitary. Consequently by Theorem 4.3, S is an F–semigroup.

REMARK. By [3], Corollary 3.12, a monoid S is a *generalized* F–semigroup if and only if S is E–unitary and E–inversive. The latter means that for every $a \in S$ there exists *some* $x \in S$ such that $ax \in E_S$. The stronger version appearing in Theorem 4.5 – for every $a \in S$ there exists a *greatest* $x \in S$ such that $ax \in E_S$ – first implies that S is E–unitary and also that S is an F–semigroup.

5. Maximal elements

The aim of this section is a characterization of F–semigroups S by properties of the maximal elements m in $(S, \leq_S)$ and the principal order ideals

$(m] = \{x \in S \mid x \leq_S m\}$ generated by $m \in S$.

Lemma 5.1. *Let S be an F–semigroup. Then the maximal elements of $(S, \leq_S)$ are precisely the greatest elements in the different ρ–classes of S.*

PROOF. Let $a \in S$ be maximal in $(S, \leq_S)$ and let $m \in S$ be the greatest element of the ρ–class $a\rho \in S/\rho$. Then by Lemma 2.1, $a\rho = (m]$ and $a \leq_S m$. It follows that $a = m$. Conversely let $m \in S$ be the greatest element of its ρ–class and suppose that $m \leq_S x$ for some $x \in S$. Then $x \in x\rho = (n]$ for some $n \in S$. Hence $m \leq_S x \leq_S n$ and $m \in (n] = x\rho$. Thus $m \; \rho \; x \; \rho \; n$ and $n \in m\rho = (m]$. Therefore $n \leq_S m$ and $m = x$, that is, $m \in S$ is maximal in $(S, \leq_S)$.

Proposition 5.2. *Let S be an F–semigroup. Then S is the disjoint union of principal order ideals in $(S, \leq_S)$.*

PROOF. Let $a \in S$ and m be the greatest element of $a\rho \in S/\rho$. Then by Lemma 2.1, $a\rho = (m]$. Since ρ is an equivalence relation, it follows that the principal order ideals of $(S, \leq_S)$ corresponding to the different ρ–classes of S are disjoint (see Lemma 2.1).

REMARK. (1) A semigroup S is the disjoint union of principal order ideals of $(S, \leq_S)$ if and only if for every $a \in S$ there exists a unique maximal $m \in S$ with $a \leq_S m$.
(2) The converse of Proposition 5.2 does not hold. Consider for example, the two–element left–zero semigroup $S = \{a, b\}$. Then $\leq_S$ is the identity relation and so $S = \{a\} \cup \{b\} = (a] \cup (b]$. But S in not an F–semigroup since S is not a group (see Example (4) of Section 2).

The following additional conditions ensure that a semigroup is an F–semigroup, and conversely, they are also necessary.

Theorem 5.3. *A semigroup S is an F–semigroup if and only if*

(i) S is E–inversive,
(ii) for every $a \in S$ there exists a unique maximal $m \in S$ with $a \leq_S m$,
(iii) if $a, b \in S$ are included in the same maximal element, then so are ac, bc resp. ca, cb for any $c \in S$,
(iv) if $a \in S$, $e \in E_S$, then a, ae, ea are included in the same maximal element.

PROOF. *Necessity.* By Proposition 5.2, S is the disjoint union of principal order ideals $(m_i], i \in I$, in $(S, \leq_S)$, which are given by the ρ–classes of S and their maximum elements $m_i (i \in I)$. We have
(i) S is E–inversive by [3], Proposition 3.7.
(ii) Let $a \in S$; then $a \in a\rho = (m_i]$ for some $i \in I$. Thus $a \leq_S m_i$, where by Lemma 5.1, m_i is maximal in $(S, \leq_S)$. This $m_i \in S$ is unique since $a \in S$ belongs to a unique ρ–class.
(iii) Let $a, b, c \in S$ be such that $a, b \leq_S m_i$ for some $i \in I$. Then $a, b \in (m_i] = m_i\rho$ and $a\rho b$. Since ρ is a congruence, it follows that $ac \; \rho \; bc$ and $ac, bc \in m_j\rho = (m_j]$ for some $j \in I$. Thus $ac, bc \leq_S m_j$ where m_j is maximal in $(S \leq_S)$ by Lemma 5.1. Similarly for $ca, cb \in S$.

(iv) Let $a \in S, e \in E_S$. Then $e\rho$ is the identity 1_G of the group $G = S/\rho$ so that by (ii), for some $k \in I$: $(ae)\rho = (a\rho)(e\rho) = a\rho = (m_k] = (e\rho)(a\rho) = (ea)\rho$. Therefore $a, ae, ea \in (m_k]$ and $a, ae, ea \leq_S m_k$.

Sufficiency. Let $T = \{m_i \mid i \in I\}$ be the set of all maximal elements in $(S, \leq_S)$. Then by (ii), $T \neq \phi$ and S is the disjoint union of the principal order ideals $(m_i](i \in I)$ of $(S, \leq_S)$. We define:

$a \; \rho \; b$ if and only if $a, b \in (m_i]$ for some $i \in I$.

Using (ii) and (iii) it is easy to show that ρ is a congruence. The semigroup $(S/\rho, \cdot)$ has $e\rho$ $(e \in E_S)$ as identity: first $E_S \neq \phi$ by (i); let $a \in S$, then by (iv), $a, ae, ea \leq_S m_i$ for some $i \in I$; hence by definition of ρ, $a \; \rho \; ae \; \rho \; ea$; thus

$$(a\rho)(e\rho) = (ae)\rho = a\rho \text{ and } (e\rho)(a\rho) = (ea)\rho = a\rho.$$

Since the identity is unique, it follows that $e\rho = f\rho$ for all $e, f \in E_S$. Let $a\rho \in S/\rho$; then by (i), $ax = f \in E_S$ for some $x \in S$. Hence $(a\rho)(x\rho) = f\rho$, the identity of S/ρ. Thus $(S/\rho, \cdot)$ is a group. Furthermore by definition of ρ, every ρ–class of S contains a greatest element. It follows that S is an F–semigroup.

Corollary 5.4. *Let S be a semigroup with compatible natural partial order. Then S is an F–semigroup if and only if*

(i) S is E–inversive,
(ii) for every $a \in S$ there is a unique maximal $m \in S$ with $a \leq_S m$,
(iv) if $a \in S$, $e \in E_S$, then a, ae, ea are included in the same maximal element.

PROOF. Concerning sufficiency we show (iii) of Theorem 5.3. Let $a, b, c \in S$ be such that $a, b \leq_S m$ for some maximal $m \in S$. Then by hypothesis, $ac \leq_S mc$ and $bc \leq_S mc$. By (ii), $mc \leq_S p$ for some maximal $p \in S$. Hence $ac, bc \leq_S p$, and similarly, $ca, cb \leq_S q$ for a maximal $q \in S$.

The following special case provides a new characterization of F–*inverse* semigroups.

Corollary 5.5. *Let S be an inverse semigroup. Then S is an F–semigroup if and only if S has an identity and for every $a \in S$ there is a unique maximal $m \in S$ such that $a \leq_S m$.*

PROOF. Note that an inverse semigroup is E–inversive and has a compatible natural partial order. *Necessity.* By [3], Theorem 3.14, S is a monoid. The second property of S holds by Theorem 5.3. (ii)
Sufficiency. We show that (iv) of Corollary 5.4 holds: let $a \in S$, $e \in E_S$; then $e \leq_S 1_S$ implies that $ae \leq_S a1_S = a$ and $ea \leq_S 1_S a = a$; since by hypothesis, $a \leq_S m$ for some maximal $m \in S$ it follows that $a, ae, ea \leq_S m$.

Corollary 5.6. *Let S be a monoid with compatible natural partial order. Then S is an F–semigroup if and only if S is E–inversive and for every $a \in S$ there exists a unique maximal $m \in S$ such that $a \leq_S m$.*

Corollary 5.7. *Let S be a centric semigroup (i.e., $aS = Sa$ for every $a \in S$). Then S is an F–semigroup if and only if S is E–inversive and for every $a \in S$ there exists a unique maximal $m \in S$ such that $a \leq_S m$.*

PROOF. Notice that by [9], Corollary 3.7, $\leq_S$ is compatible. Concerning sufficiency we show (iv) of Corollary 5.4. Let $a \in S$, $e \in E_S$; then $ae = xa$ for some $x \in S$, whence $ae \leq_S a$. Similarly $ea \leq_S a$. Since $a \leq_S m$ for some maximal $m \in S$, we obtain that $ae, ea, a \leq_S m$.

As every commutative semigroup is centric we get

Corollary 5.8. *Let S be a commutative semigroup. Then S is an F–semigroup if and only if S is E–inversive and for every $a \in S$ there exists a unique maximal $m \in S$ with $a \leq_S m$.*

REMARK. Since every finite semigroup is E–inversive we obtain that a *finite commutative* semigroup S is an F–semigroup if and only if for every $a \in S$ there exists a *unique* maximal $m \in S$ such that $a \leq_S m$ (note that in a finite partially ordered set every element is contained in *some* maximal one).

Corollary 5.9. *Let S be a semigroup with compatible natural partial order and central idempotents (i.e., $ae = ea$ for every $a \in S, e \in E_S$). Then S is an F–semigroup if and only if S is E–inversive and for every $a \in S$ there exists a unique maximal $m \in S$ such that $a \leq_S m$.*

PROOF. In order to prove sufficiency we show that (iv) of Corollary 5.4 holds: let $a \in S$, $e \in E_S$; then $ae = ea \leq_S a$, whence $a, ae, ea \leq_S m$ for some maximal $m \in S$.

REMARK. Every *Clifford–semigroup* satisfies the general hypothesis of Corollary 5.9 and is E–inversive. Thus we obtain: a Clifford–semigroup S is an F–semigroup if and only if for every $a \in S$ there exists a unique maximal $m \in S$ such that $a \leq_S m$. Compare with Example (7) in Section 2: the Clifford–semigroup S given there is not an F–semigroup since for $c \in S$ we have $c <_S a$, $c <_S b$, where $a, b \in S$ are both maximal elements of $(S, \leq_S)$. Another characterization of Clifford–semigroups, which are F–semigroups, is given in Corollary 6.7 below, under the additional assumption that the underlying semilattice satisfies the ascending chain condition.

Corollary 5.10. *Let S be an eventually regular semigroup (i.e, for every $a \in S$ there exists $n > 0$ such that a^n is regular) with central idempotents. Then S is an F–semigroup if and only if for every $a \in S$ there exists a unique maximal $m \in S$ with $a \leq_S m$.*

PROOF. We show first that $a \leq_S b$ if and only if $a = eb$ for some $e \in E_{S^1}$. Following the proof of Corollary 1.4.6 in [4] we have

$$a <_S b \Rightarrow a = xb = by, xa = a \text{ for some } x, y \in S \Rightarrow a = x^n b = x^n a \text{ for every } n > 0.$$

By hypothesis, for $x \in S$ there exists $k > 0$ such that $x^k = x^k z x^k$ for some $z \in S$; thus $zx^k \in E_S$ and

$$a = x^k a = x^k z x^k \cdot a = zx^k \cdot x^k a = zx^k \cdot a = z \cdot x^k b = eb \text{ with } e = zx^k \in E_S.$$

Conversely, $a = eb$ with $e \in E_S$ implies by centrality of idempotents in S that $a = eb = be$, whence $a \leq_S b$. It follows easily that the natural partial order of S is compatible. Notice that an eventually regular semigroup is E–inversive. Consequently, the statement follows from Corollary 5.9.

REMARK. Every finite semigroup with central idempotents satisfies the general condition in Corollary 5.10.

6. F–semigroups in special classes

We give necessary and sufficient conditions for (1) Inflations of a semigroup, and (2) Strong semilattices of monoids, to be F–semigroups. Thus two methods for the construction of further examples of F–semigroups are provided.

(1) Inflations of semigroups.
Let $S = \bigcup_{\alpha \in T} T_\alpha$ be an inflation of a semigroup T such that for every $\alpha \in T$ there exist $\beta, \gamma \in T$ with $\alpha = \beta\alpha = \alpha\gamma$. Then

$$a \leq_S b \ (a \in T_\alpha, b \in T_\beta) \text{ if and only if } a = b \text{ or } a = \alpha \leq_T \beta.$$

In particular, if $a, b \in T_\alpha$ then $a \leq_S b$ if and only if $a = \alpha$ (see [3], Section 4).

Theorem 6.1. *Let $S = \bigcup_{\alpha \in T} T_\alpha$ be an inflation of the semigroup T in which for every $\alpha \in T$ there exist $\beta, \gamma \in T$ such that $\alpha = \beta\alpha = \alpha\gamma$. Then S is an F–semigroup if and only if*

(i) T is an F–semigroup,
(ii) $|T_\mu| \leq 2$ for every maximal $\mu \in T$,
(iii) $|T_\alpha| = 1$ for every non–maximal $\alpha \in T$.

PROOF. *Necessity.* (i) By definition, there exists a group G and a surjective homomorphism $\phi : S \to G$ such that for every $g \in G$, $g\phi^{-1}$ has a greatest element $u \in S$, say, whence by Lemma 2.1, $g\phi^{-1} = (u]$. Let $\psi = \phi|_T$; then $\psi : T \to G$ is a surjective homomorphism, too. We show that $g\psi^{-1} = (\omega]$ in $(T, \leq_T)$ if $u \in T_\omega$, say:

$$\begin{aligned} \alpha \in (\omega], \alpha \in T &\Rightarrow \alpha \leq_T \omega \leq_S u \Rightarrow \alpha \in (u] = g\phi^{-1} \text{ in } (S, \leq_S) \Rightarrow \\ &\Rightarrow \alpha\psi = \alpha\phi = g \Rightarrow \alpha \in g\psi^{-1}; \\ \alpha \in g\psi^{-1}, \alpha \in T &\Rightarrow \alpha\phi = \alpha\psi = g \Rightarrow \alpha \in g\phi^{-1} = (u] \text{ in } (S, \leq_S) \Rightarrow \\ &\Rightarrow \alpha \leq_S u \Rightarrow \alpha \leq_T \omega \Rightarrow \alpha \in (\omega]. \end{aligned}$$

It follows by definition, that T is an F–semigroup.
(ii) We show first that $|T_\beta| \leq 2$ for *every* $\beta \in T$. Assume that there exist $\beta \in T$ and $a, b \in T_\beta$ such that $a \neq b$, $a \neq \beta$, $b \neq \beta$. Then $\beta <_S a$ and $\beta <_S b$. But both a and b are maximal in $(S, \leq_S)$ since $a \leq_S x$ $(x \in S)$ implies that $a = \beta$. It follows by Theorem 5.3 that $a = b$, a contradiction. Thus we have shown (ii).
(iii) Assume that there exists $\alpha \in T$, not maximal with $|T_\alpha| \neq 1$. Then by the proof of (ii), $|T_\alpha| = 2$. Hence $T_\alpha = \{\alpha, a\}$ where $\alpha <_S a$. Again, $a \in S$ is maximal in $(S \leq_S)$. Since $\alpha \in T$ is not maximal in $(T, \leq_T)$, $\alpha <_T \beta$ for some $\beta \in T$. By Theorem 5.3 (ii), $\beta \leq_S m$ for some maximal $m \in S$; hence $\alpha <_T \beta \leq_S m$. It follows by Theorem 5.3 (ii), that $a = m$. Hence $\beta \leq_S m = a \in T_\alpha$ and so $\beta \leq_T \alpha$, a contradiction.

Sufficiency. By (i), there exists a group G and a surjective homomorphism $\psi : T \to G$ such that $g\psi^{-1}$ has a greatest element in $(T, \leq_T)$. Let $g \in G$ and $g\psi^{-1} = (\omega]$, where $\omega \in T$. By (i) and (ii), $|T_\omega| \leq 2$; thus $T_\omega = \{\omega, u\}$ where $\omega \leq_S u$. Let $\phi : S \to G$, $a\phi = \alpha\psi$ if $a \in T_\alpha$. Then ϕ is a surjective homomorphism. We show that $g\phi^{-1} = (u]$:

$a \in g\phi^{-1}, a \in T_\alpha$ say $\Rightarrow$ $\alpha\psi = a\phi = g \Rightarrow \alpha \in g\psi^{-1} = (\omega] \Rightarrow \alpha \leq_T \omega \leq_S u$;
if $\alpha = \omega$, $a \in T_\omega$ so that either $a = \omega$ or $a = u$, thus $a \leq_S u$ and $a \in (u]$;
if $\alpha <_T \omega |T_\alpha| = 1$ by (ii), and $a = \alpha$; thus $a = \alpha <_T \omega \leq_S u$ and $a \in (u]$;

$a \in (u], a \in T_\alpha$ say $\Rightarrow$ $a \leq_S u$; if $a = u$ then $a\phi = u\phi = \omega\psi = g$ and $a \in g\phi^{-1}$;
if $a <_S u$ then $a = \alpha <_T \omega$; by Lemma 2.1, $a\phi = \omega\phi = \omega\psi = g$, i.e.,$a \in g\phi^{-1}$.

It follows by definition, that S is an F–semigroup.

Corollary 6.2. *Let G be a group and let $S = \bigcup_{g\in G} T_g$ be an inflation of G. Then S is an F–semigroup if and only if $|T_g| \leq 2$ for every $g \in G$.*

PROOF. By Example (1) in Section 2, G is an F–semigroup. Since the natural partial order of any group is the identity relation, every element of G is maximal in $(G, \leq_G)$. Therefore, the statement follows from Theorem 6.1.

REMARK. (i) If T is a *finite* F-semigroup (more generally, if $(T, \leq_T)$ satisfies the ascending chain condition) then there are proper inflations of T, which are again F–semigroups.
(ii) The semigroups S given in Corollary 6.2 are non-regular F-semigroups without identity. Note that the pivot is $1_G \in S$ if $T_{1_G} = \{1_G\}$ and is $a \in S$ if $T_{1_G} = \{1_G, a\}$.

(2) Strong semilattices of monoids.
Let $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$ be a strong semilattice $(Y, \leq_Y)$ of monoids $S_\alpha (\alpha \in Y)$ with linking homomorphisms $\phi_{\alpha,\beta} : S_\alpha \to S_\beta (\alpha \geq_Y \beta)$. Then by the proof of Theorem 3.8 in [9]:

$$a \leq_S b \ (a \in S_\alpha, b \in S_\beta) \text{ if and only if } \alpha \leq_Y \beta \text{ and } a \leq_\alpha b\phi_{\beta,\alpha},$$

where $\leq_\alpha$ denotes the natural partial order of $S_\alpha (\alpha \in Y)$.

Theorem 6.3. *Let $S = [Y, S_\alpha, \phi_{\alpha,\beta}]$ be a strong semilattice of monoids. Then S is an F–semigroup if and only if*

(i) $(Y, \leq_Y)$ has a greatest element ω, and for every $\alpha \in Y$, $\phi_{\omega,\alpha}$ is a monoid homomorphism,
(ii) for every $\alpha \in Y$, S_α is E–unitary, and any $\phi_{\alpha,\beta}$ is idempotent pure,
(iii) for every $a \in S$, the set $T_a = \{x \in S | ax \in E_S\}$ has a greatest element.

PROOF. *Necessity.* (i) and (ii) hold by [3], Theorem 4.5. Furthermore by [3], Proposition 4.4, S is a monoid. Hence, (iii) holds by Theorem 4.5.
Sufficiency. First by (iii), for any $a \in S$, $a \in S_\alpha$ say, and $x \in T_a$, the element $y = x\varphi_{\beta,\alpha\beta} \in S_{\alpha\beta}$ satisfies $(a\varphi_{\alpha,\alpha\beta})y \in E(S_{\alpha\beta})$ with $\alpha\beta \leq_Y \alpha$. Therefore by [3],

Theorem 4.5, S is a generalized F-semigroup. Next by (i), $1_\omega = 1_S$ and the pivot of S is $\xi = 1_S$. Let $a \in S$; then by [9], Lemma 2.1,

$$\max T_a = \max\{x \in S | ax \in E_S\} = \max\{x \in S | ax \leq_S 1_S\} = \max\langle 1_S \mathbin{.\cdot} a\rangle,$$

which by (iii) shows that $1_S \mathbin{.\cdot} a$ exists in S. It follows by Theorem 3.5, that S is an F–semigroup.

Next we deduce a *necessary* condition for $S = [Y, S_\alpha, \phi_{\alpha,\beta}]$ to be an F–semigroup.

Lemma 6.4. *Let $S = [Y, S_\alpha, \phi_{\alpha,\beta}]$ be a strong semilattice of monoids. If S is an F–semigroup then the following holds: if $a, b \in S$ are two maximal elements then if $a \in S_\alpha, b \in S_\beta$ say, there is no $\gamma \leq_Y \alpha, \beta$ in Y such that $a\phi_{\alpha,\gamma}$, $b\phi_{\beta,\gamma} \in S_\gamma$ have a common lower bound in $(S_\gamma, \leq_\gamma)$.*

PROOF. Assume that there are maximal elements $a, b \in S, a \in S_\alpha, b \in S_\beta$ say, such that for some $\gamma \leq_Y \alpha, \beta$ in Y and some $c \in S_\gamma, c \leq_\gamma a\phi_{\alpha,\gamma}$ and $c \leq_\gamma b\phi_{\beta,\gamma}$. Since $a\phi_{\alpha,\gamma} \leq_S a$ and $b\phi_{\beta,\gamma} \leq_S b$ it follows that $c \leq_S a$ and $c \leq_S b$, contradicting Theorem 5.3 (ii).

REMARK. (1) The semigroup S in Example (7) of Section 2. does not have this property: $a \in G_\alpha, b \in G_\beta$ are maximal in $(S, \leq_S)$, but $a\phi_{\alpha,\gamma} = b\phi_{\beta,\gamma} = c$ and $c \in G_\gamma$ is a common lower bound of $a\phi_{\alpha,\gamma}, b\phi_{\beta,\gamma} \in G_\gamma$. Hence S is not an F–semigroup.

(2) For a strong semilattice of monoids, which is an F-semigroup, the linking homomorphisms are not injective, in general. Consider $S = [Y; S_\alpha, S_\omega, \varphi_{\omega,\alpha}]$ where S_α is a group, S_ω is a non trivial band with identity 1_ω, $Y : \alpha <_Y \omega$, and $e\varphi_{\omega,\alpha} = 1_\alpha$ for any $e \in S_\omega$. Then the relation ρ defined on S as the universal relation on E_S and as the identity relation on $S \backslash E_S$ is a group congruence whose classes are E_S and the sets $\{a\}$, $a \in S\backslash E_S$. Since $\max E_S = 1_\omega$ and $\max\{a\} = a$ for any $a \in S\backslash E_S$, S is an F-monoid with pivot $\xi = 1_\omega = 1_S$ (see also Theorem 6.3). But $\varphi_{\omega,\alpha} : S_\omega \to S_\alpha$ is not injective. Note that in case that each S_α in $S = [Y; S_\alpha; \varphi_{\alpha,\beta}]$ is a group and if S is an F-semigroup then by [3], Corollary 4.8, each $\varphi_{\alpha,\beta}$ is injective.

Some particular cases should be mentioned.

Theorem 6.5. *Let $S = [Y, S_\alpha, \phi_{\alpha,\beta}]$ be a strong semilattice of trivially ordered monoids. Then S is an F–semigroup if and only if (i) S is an E–inversive monoid, and (ii) for every $a \in S$ there exists a unique maximal $m \in S$ such that $a \leq_S m$.*

PROOF. First, by [9], Corollary 3.9, the natural partial order of S is compatible. If S is an F–semigroup then by [3], Proposition 4.4, S is an E-inverse monoid. The statement now follows by Corollary 5.6.

REMARK. Note that $a \in S, a \in S_\alpha$ say, is maximal in $S = [Y, S_\alpha; \varphi_{\alpha,\beta}]$ where each S_α is a trivially ordered monoid, if and only if $a \notin \operatorname{Im} \varphi_{\beta,\alpha}$ for any $\beta >_Y \alpha$ in Y. Also, by the Remark following Theorem 4.5 in [3], for such a semigroup S the components S_α $(\alpha \in Y)$ are not necessarily E-inversive, i.e., groups.

If the semilattice $(Y, \leq_Y)$ satisfies the ascending chain condition, conditions (i) and (ii) in Theorem 6.5 can be expressed in terms of properties of $(Y, \leq_Y)$ and the homomorphisms $\phi_{\alpha,\beta}$.

Corollary 6.6. *Let $S = [Y, S_\alpha, \phi_{\alpha,\beta}]$ be a strong semilattice of trivially ordered monoids such that $(Y, \leq_Y)$ satisfies the ascending chain condition (in particular, Y is finite). Then S is an F–semigroup if and only if*

(i) $(Y, \leq_Y)$ has a greatest element ω,

(ii) for every $a \in S, a \in S_\alpha$ say, there exists $\beta \leq_Y \alpha$ and $x \in S_\beta$ such that $(a\phi_{\alpha,\beta})x \in E(S_\beta)$,

(iii) if $a, b \in S$ are maximal in $(S, \leq_S)$, $a \in S_\alpha, b \in S_\beta$ say, then $a\phi_{\alpha,\gamma} \neq b\phi_{\beta,\gamma}$ for every $\gamma \leq_Y \alpha, \beta$.

PROOF. *Necessity.* (i) and (ii) hold by [3], Theorem 4.5.(iii) holds by Lemma 6.4.
Sufficiency. We show first that each $\phi_{\omega,\alpha}$ $(\alpha \in Y)$ is a monoid–homomorphism. Let $e \in E(S_\alpha)$; then $e \leq_\alpha 1_\alpha$ (the identity of S_α) so that by the trivial order of $S_\alpha, e = 1_\alpha$, that is, $E(S_\alpha) = \{1_\alpha\}$. Since $1_\omega\phi_{\omega,\alpha} \in E(S_\alpha)$ it follows that $1_\omega\phi_{\omega,\alpha} = 1_\alpha$. Therefore, $1_\omega \in S_\omega$ is the identity of all of S. Also, S is E-inversive by condition (ii), since $ax = (a\varphi_{\alpha,\alpha\beta})(x\varphi_{\beta,\alpha\beta}) = (a\varphi_{\alpha,\beta})x \in E_S$.
We proceed to show that for any $a \in S$ there exists a unique maximal $m \in S$ such that $a \leq_S m$. Let $a \in S, a \in S_\alpha$ say. If a is not maximal in $(S, \leq_S)$ then $a <_S b$ for some $b \in S$. Note that $b \notin S_\alpha$ since S_α is trivially ordered and $\leq_S$ restricted to S_α coincides with $\leq_\alpha$ (see the proof of Theorem 3.8 in [9]). Hence $b \in S_\beta$ for some $\beta >_Y \alpha$. If b is not maximal in $(S, \leq_S)$ then going on this way by the ascending chain condition on $(Y, \leq_Y)$, there exists a maximal $\xi \in Y$ such that $\alpha <_Y \beta <_Y \ldots <_Y \xi$, and $m \in S_\xi$ with $a <_S b <_S \ldots <_S m$. This $m \in S_\xi$ is maximal in $(S, \leq_S)$ since $m <_S x, x \in S_\gamma$ say, implies that $\xi <_Y \gamma$: contradiction. Furthermore, $m \in S_\xi$ is unique: let $a \leq_S n$ for some maximal $n \in S$, $n \in S_\eta$ say. Then again $\alpha \leq_Y \eta$ and $a \leq_\alpha n\phi_{\eta,\alpha}$. But $a <_S m$ implies that $\alpha <_Y \xi$ and $a \leq_\alpha m\phi_{\xi,\alpha}$. Since S_α is trivially ordered we obtain that $m\phi_{\xi,\alpha} = a = n\phi_{\eta,\alpha}$, contradicting (iii).
It follows by Theorem 6.5, that S is an F–semigroup.

Since every group is trivially ordered and E-inversive we obtain

Corollary 6.7. *Let $S = [Y, S_\alpha, \phi_{\alpha,\beta}]$ be a strong semilattice of groups such that $(Y, \leq_Y)$ satisfies the ascending chain condition (in particular, Y is finite). Then S is an F–semigroup if and only if*

(i) $(Y, \leq_Y)$ has a greatest element,

(ii) if $a, b \in S$ are maximal in $(S, \leq_S)$, $a \in S_\alpha, b \in S_\beta$ say, then $a\phi_{\alpha,\gamma} \neq b\phi_{\beta,\gamma}$ for every $\gamma \leq_Y \alpha, \beta$.

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