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Congruences on nilpotent-generated partial transformation semigroups

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Abstract

In 1987, Sullivan characterised the elements of the semigroup $NP(X)$ generated by the nilpotents in $P(X)$, the semigroup (under composition) consisting of all partial transformations of a set X; and, in 1999, Marques-Smith and Sullivan determined all the ideals of $NP(X)$ for arbitrary X. In this paper, we use that work to describe all the congruences on $NP(X)$.

1. Introduction

Throughout this paper, X is a non-empty set. In addition, $P(X)$ denotes the semigroup under composition of all *partial* transformations of X (that is, all transformations α whose *domain*, $\operatorname{dom}\alpha$, and *range*, $\operatorname{ran}\alpha$, are subsets of X). Note that $P(X)$ contains a zero (namely, the empty mapping $\emptyset$): we say $\alpha \in P(X)$ is *nilpotent* with *index* r if $\alpha^r = \emptyset$ and $\alpha^{r-1} \neq \emptyset$, and we let $NP(X)$ denote the semigroup generated by all nilpotents in $P(X)$. In like manner, if $I(X)$ denotes the symmetric inverse semigroup on X, we write $NI(X)$ for the semigroup generated by all nilpotents in $I(X)$.

In [4] the authors described the ideals of $NP(X)$ and $NI(X)$ as a prelude to determining all congruences on these semigroups. The congruences on $NI(X)$ were described in [5], and here we do the same for $NP(X)$. The case when X is finite is considered in section 3, and we cover the cases when X has infinite regular or singular cardinality in sections 4 and 5.

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2. Preliminary results

All notation and terminology will be from [1] and [4] unless specified otherwise. In particular, if $\alpha \in P(X)$, we let $r(\alpha)$ denote the *rank* of α (that is, $|X\alpha|$) and put

$$\begin{array}{ll} D(\alpha) = X\backslash X\alpha, & d(\alpha) = |D(\alpha)|, \\ G(\alpha) = X\backslash \text{dom } \alpha, & g(\alpha) = |G(\alpha)|. \end{array}$$

The cardinal numbers $d(\alpha)$ and $g(\alpha)$ are called the *defect* and the *gap* of α and were used by Sullivan in 1987 to characterise the elements of $NP(X)$ for arbitrary X.

To state his result for the infinite case, first we recall that a cardinal k is *regular* if $|\bigcup\{A_i : i \in I\}| = k$ implies either $|I| = k$ or some A_i has cardinal k; and k is *singular* if it is not regular. And, we say $\alpha \in P(X)$ is *spread over its rank* if for each cardinal $p < r(\alpha)$, there exists $y \in X$ with $|y\alpha^{-1}| > p$. The following two results summarise Corollary 3 and Theorem 4 in [6] and Lemmas 2.5 and 3.2 in [8].

Theorem 1. Let k be regular and $\alpha \in P(X)$. Then $\alpha \in NP(X)$ if and only if $g(\alpha) \neq 0$, $d(\alpha) = k$, and $g(\alpha) = k$ or $|y\alpha^{-1}| = k$ for some $y \in X$. Moreover, when this occurs, $NP(X)$ is a regular semigroup and each $\alpha \in NP(X)$ is a product of 3 or fewer nilpotents with index at most 3.

Theorem 2. Let k be singular and $\alpha \in P(X)$. Then $\alpha \in NP(X)$ if and only if $g(\alpha) \neq 0$, $d(\alpha) = k$, and either $g(\alpha) \geq r(\alpha)$ or α is spread over its rank. Moreover, when this occurs, $NP(X)$ is a regular semigroup and each $\alpha \in NP(X)$ is a product of 4 or fewer nilpotents with index at most 4.

For the finite case (see [6] Theorems 1 and 2), we need some additional notation.

If X is an arbitrary set with cardinal k and $1 \leq r \leq k$, we write

$$\begin{array}{rcl} P_r & = & \{\alpha \in P(X) : r(\alpha) < r\} \\ D_r & = & \{\alpha \in P(X) : r(\alpha) = r\} \end{array}$$

and recall that the P_r constitute all the proper ideals of $P(X)$ and that each D_r is a $\mathcal{D}$–class of $P(X)$. Moreover, if $k = n < \aleph_0$ then each $\alpha \in I(X) \cap D_{n-1}$ has a unique *completion* $\overline{\alpha} \in G(X)$, the symmetric group on X, defined by:

$$x\overline{\alpha} = \begin{cases} x\alpha, & \text{if } x \in \text{dom } \alpha, \\ b, & \text{if } x = a, \end{cases}$$

where $X \setminus \text{dom}\,\alpha = \{a\}$ and $X \setminus \text{ran}\,\alpha = \{b\}$ ([2] p 388). We write

$$E_{n-1} = \{\alpha \in I(X) \cap D_{n-1} : \overline{\alpha} \text{ is an even permutation}\}.$$

Theorem 3. Suppose $n \geq 3$ and $\alpha \in P(X)$.

(a) If n is even then $\alpha \in NP(X)$ if and only if $g(\alpha) \neq 0$.

(b) If n is odd then $\alpha \in NP(X)$ if and only if $g(\alpha) \neq 0$ and $\alpha \in P_{n-1} \cup E_{n-1}$.

In what follows, we extend the convention introduced in [1] vol 2, p 241: namely, if $\alpha \in P(X)$ is non-zero then we write

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, that the abbreviation $\{x_i\}$ denotes $\{x_i : i \in I\}$, and that $\operatorname{ran}\alpha = \{x_i\}$, $x_i\alpha^{-1} = A_i$ and $\operatorname{dom}\alpha = \bigcup\{A_i : i \in I\}$. In particular, if $\operatorname{dom}\alpha = A$ and $\operatorname{ran}\alpha = \{b\}$, we write α more simply as A_b, or a_b if $A = \{a\}$. Also, we let id_A denote the identity on A, and we write $Y = A \dot{\cup} B$ if $A \cap B = \emptyset$.

In passing, we note that, although $NI(X)$ and $NP(X)$ are nilpotent-generated, they are almost never isomorphic. This is because the first is an inverse semigroup, but the second is not. For example, if X is infinite, then $a_a \in NI(X)$ (since its gap and defect equal $|X|$), but a_a has more than one inverse in $NP(X)$: namely, if $a \in A \subseteq X$, then $a_a = a_a.A_a.a_a$ and $A_a = A_a.a_a.A_a$. Therefore, although the congruences on $NI(X)$ were determined in [5], to describe the congruences on $NP(X)$ is a related, but different, problem.

If $\alpha \in P(X)$, then $\alpha \circ \alpha^{-1}$ is an equivalence on $\operatorname{dom}\alpha$, hence it induces a partition $\{Y_i\}$ of $\operatorname{dom}\alpha$. We say A is a *cross-section* of $\alpha \circ \alpha^{-1}$ (or of the corresponding partition) if $A \subseteq \bigcup Y_i$ and $|A \cap Y_i| = 1$ for each i. If ρ is a congruence on a transformation semigroup, we often write $\alpha \sim \beta$ to mean $(\alpha, \beta) \in \rho$. Also, sometimes we write $x\alpha = \emptyset$ to mean $x \notin \operatorname{dom}\alpha$.

The following result is almost the same as [5] Lemma 1.

Lemma 1. Suppose $|X| \geq 3$ and let ρ be a non-identity congruence on $NP(X)$. Then $\emptyset\rho$, the ρ–class containing $\emptyset$, is an ideal of $NP(X)$ and it contains DP_1, the set of all constant maps in $NP(X)$.

Proof. Suppose $(\alpha, \beta) \in \rho$ where $\alpha \neq \beta$. Then $x\alpha \neq x\beta$ for some $x \in X$ and, without loss of generality, we can assume $x\alpha = y \neq \emptyset$. Let $a, b \in X$ and $\lambda = a_x, \mu = y_b$. Then λ and μ have non-zero gap (since $|X| \geq 3$) and it is easy to see that $\lambda, \mu \in NP(X)$. In fact, $\lambda\alpha\mu = a_b$ and $\lambda\beta\mu = \emptyset$ (even if $x \in \operatorname{dom}\beta$), hence $a_b \sim \emptyset$. If $Y \subsetneq X$ then $g(Y_a) \neq 0$ and, by one of the above Theorems, $Y_a \in NP(X)$. Now, $Y_b = Y_a.a_b$, so $Y_b \sim \emptyset$ and it follows that DP_1 is contained in $\emptyset\rho$, which is clearly an ideal of $NP(X)$. □

The proper ideals of $NP(X)$ were described in [4] Theorems 6 and 15 as follows. In [5] section 2, the authors remarked that, if X is infinite and $r \leq |X|$, then the proper ideals of $NI(X)$ are simply those of $I(X)$. However, this is not true for $NP(X)$, because each P_r contains *total* transformations (that is, $\alpha \in P(X)$ with $\operatorname{dom}\alpha = X$, so $g(\alpha) = 0$) and, by Theorems 1 and 2, these elements do not belong to $NP(X)$.

Theorem 4. For any set X with (finite or infinite) cardinal $k \geq 3$, the proper ideals of $NP(X)$ are precisely the sets

$$NP_r = \{\alpha \in NP(X) : r(\alpha) < r\}$$

where $1 \leq r \leq k$.

Thus, if ρ is a non-identity and non-universal congruence on $NP(X)$ then $\emptyset\rho = NP_r$ for some r such that $1 \leq r \leq |X|$: we call r the *primary rank* of ρ and denote it by $\eta(\rho)$. We also need the characterisation of Green's $\mathcal{D}$–relation on $NP(X)$ given in [4] Theorem 11 and p 312. We let DP_r denote the $\mathcal{D}$–class of $NP(X)$ which contains all elements with rank r.

Theorem 5. If X is any set (finite or infinite) and $\alpha, \beta \in NP(X)$ then $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in NP(X)$ if and only if $r(\beta) \leq r(\alpha)$. Hence, $\mathcal{D} = \mathcal{J}$ for $NP(X)$.

The proof of the next result closely follows the one for [5] Lemma 2, so we omit most of the details. Here, as in [1] vol 2, p 227, we let NP_r^* denote the Rees congruence on $NP(X)$ determined by the ideal NP_r.

Lemma 2. If ρ is a non-identity congruence on $NP(X)$ and $\eta = \eta(\rho)$ then

$$NP_\eta^* \subseteq \rho \subseteq NP_\eta^* \cup \mathcal{D}.$$

Proof. It is easy to see that $NP_\eta^* \subseteq \rho$, so we let $(\alpha, \beta) \in \rho$ and assume that $r(\beta) < r(\alpha) = r$.

(a) <u>r is infinite</u>. This means X is infinite and we note that the γ defined in case (a) for the proof of [5] Lemma 2 has gap and defect equal to $|X|$. Hence, by Theorems 1 and 2 above, this γ belongs to $NP(X)$ and, as before, we conclude that $r < \eta$.

(b) <u>r is finite</u>. In this case, X may be finite or infinite. However, for both possibilities, the γ and γ_i defined in case (b) for the proof of [5] Lemma 2 belong to $NP(X)$. Hence, that argument holds for this case, and we again conclude that $r < \eta$. □

The $\mathcal{L}$ and $\mathcal{R}$ relations on $P(X)$ are well-known: namely, $\alpha \; \mathcal{L} \; \beta$ if and only if $\operatorname{ran}\alpha = \operatorname{ran}\beta$; and $\alpha \; \mathcal{R} \; \beta$ if and only if $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$. And, if X is infinite, then $NP(X)$ is a regular subsemigroup of $P(X)$ by Theorems 1 and 2. Therefore, to prove a result which is analogous to [5] Lemma 3, we need to know that $NP(X)$ is regular when X is finite (see [6] p 341).

Lemma 3. If X is finite and $|X| = n \geq 3$ then $NP(X)$ is a regular semigroup.

Proof. Suppose $\alpha \in NP(X)$ and write $\operatorname{ran}\alpha = \{x_1, \ldots, x_r\}$. Let $A_i = x_i\alpha^{-1}$ and choose $a_i \in A_i$ for each $i = 1, \ldots, r$. If n is even then $g(\alpha) \neq 0$, so α is not surjective. Hence the map $\beta : x_i \mapsto a_i$ for $i = 1, \ldots, r$ belongs to $NP(X)$ and $\alpha = \alpha\beta\alpha$. The same argument can be applied when n is odd and $r(\alpha) \leq n - 2$. Also, if n is odd and $\alpha \in E_{n-1}$ and $g(\alpha) \neq 0$ then α is injective with rank $n - 1$: that is, $A_i = \{a_i\}$ and $\alpha : a_i \mapsto x_i$ for each i. Moreover, the completion of α is an even permutation. Clearly this implies $\alpha^{-1} \in E_{n-1}$ and so, in this case, α is also regular in $NP(X)$. □

Lemma 4. Let ρ be a congruence on $NP(X)$ and suppose $\eta(\rho)$ is finite. If $(\alpha, \beta) \in \rho$ and $\eta(\rho) \leq r(\alpha) < \aleph_0$ then $(\alpha, \beta) \in \mathcal{H}$.

Proof. The γ defined in the proof of [5] Lemma 3 belongs to $NP(X)$ (regardless of whether X is finite or infinite), hence we conclude, as before, that $\alpha \; \mathcal{L} \; \beta$.

To show $\alpha \mathrel{\mathcal{R}} \beta$, first we suppose $\operatorname{dom}\alpha \not\subseteq \operatorname{dom}\beta$. Choose $x \in \operatorname{dom}\alpha \setminus \operatorname{dom}\beta$, and let A be a cross-section of $\alpha \circ \alpha^{-1}$ which contains x. Then $\mathrm{id}_A \in NP(X)$ (since $|A| = r(\alpha)$, our justification for $\delta \in NI(X)$ in the proof of [5] Lemma 3 is also valid here). Moreover, $r(\mathrm{id}_A\,\alpha) = r$, but $r(\mathrm{id}_A\,\beta) \le r-1$ (since $x \notin \operatorname{dom}\beta$). Since $\mathrm{id}_A\,\alpha \sim \mathrm{id}_A\,\beta$, Lemma 2 implies $r < \eta(\rho)$, a contradiction. Therefore, $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$ and similarly $\operatorname{dom}\beta \subseteq \operatorname{dom}\alpha$, so $\operatorname{dom}\alpha = \operatorname{dom}\beta$.

Next we suppose $\alpha \circ \alpha^{-1} \not\subseteq \beta \circ \beta^{-1}$. Then there exists $(x, y) \in \alpha \circ \alpha^{-1} \setminus \beta \circ \beta^{-1}$ and we let B be a cross-section of $\beta \circ \beta^{-1}$ which contains x and y. Then $\mathrm{id}_B \in NP(X)$ (since $|B| = r(\beta) = r < \aleph_0$, so the same justification as before can be applied) and $r(\mathrm{id}_B\,\beta) = r$, but $r(\mathrm{id}_B\,\alpha) \le r - 1$ (since $x\alpha = y\alpha$). Like before, this is a contradiction since $\mathrm{id}_B\,\alpha \sim \mathrm{id}_B\,\beta$. Therefore, $\alpha \circ \alpha^{-1} \subseteq \beta \circ \beta^{-1}$, and similarly for the reverse inclusion, so we have shown $\alpha \mathrel{\mathcal{R}} \beta$. □

The next result is similar to [5] Lemma 4, but we include a proof for this new context.

Lemma 5. Let ρ be a non-identity congruence on $NP(X)$ and suppose $\eta(\rho)$ is finite. If $(\alpha, \beta) \in \rho$ where $\alpha \neq \beta$ and $\eta(\rho) \le r(\alpha) < \aleph_0$ then $r(\alpha) = \eta(\rho)$.

Proof. By Lemma 4, $(\alpha, \beta) \in \mathcal{H}$, so α and β have the same domain and range. Hence we can write

$$\alpha = \begin{pmatrix} A_1 & \dots & A_r \\ b_1 & \dots & b_r \end{pmatrix}, \quad \beta = \begin{pmatrix} A_1 & \dots & A_r \\ b_{1\pi} & \dots & b_{r\pi} \end{pmatrix}$$

for some permutation π of $\{1, \dots, r\}$. Let $\{a_i\}$ be a cross-section of $\{A_i\}$. Since $\alpha \neq \beta$, there exists i such that $i \neq i\pi$; and, since ρ is not the identity congruence, we know $\eta(\rho) \ge 2$ and thus $r \ge 2$. If γ is the identity on $\{a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_r\}$, then $\gamma \in NI(X)$ (via the usual justification when X is finite or infinite) and so $\gamma\alpha \sim \gamma\beta$. But, since $i\pi^{-1} \neq i$, $\operatorname{ran}(\gamma\beta)$ contains b_i, whereas $\operatorname{ran}(\gamma\alpha)$ does not. Therefore $(\gamma\alpha, \gamma\beta) \notin \mathcal{H}$ and so, by Lemma 4, $r(\gamma\alpha) = r - 1$ must be less than $\eta(\rho)$. Since $r(\alpha) = r \ge \eta(\rho)$ by supposition, it follows that $r = \eta(\rho)$. □

3. Finite primary rank

In [4] p 316, the authors observed that, if X is finite and $r < |X|$, then NI_{r+1}/NI_r is completely 0–simple. For what follows, we require a similar result for $NP(X)$ but one that is slightly more general: compare [5] Lemma 5. If r is any infinite cardinal then r' denotes the *successor* of r (that is, the least cardinal greater than r).

Lemma 6. If X is any set and $4 \le r < |X|$ then $NP_{r'}/NP_r$ is 0–bisimple, and it contains a primitive idempotent if and only if r is finite. Consequently, if r is finite then NP_{r+1}/NP_r is completely 0–simple.

Proof. Suppose $\alpha, \beta \in NP(X)$ and $r(\alpha) = r(\beta) = r$ (finite or infinite). Choose cross-sections $\{a_p\}$ and $\{b_p\}$ of $\alpha \circ \alpha^{-1}$ and $\beta \circ \beta^{-1}$, respectively, and write

$$\alpha = \begin{pmatrix} A_p \\ x_p \end{pmatrix}, \quad \beta = \begin{pmatrix} B_p \\ y_p \end{pmatrix}, \quad \gamma = \begin{pmatrix} B_p \\ x_p \end{pmatrix}, \quad \lambda = \begin{pmatrix} A_p \\ b_p \end{pmatrix}, \quad \lambda' = \begin{pmatrix} B_p \\ a_p \end{pmatrix}.$$

If $|X| = k \geq \aleph_0$, then $|P| = r < k$ implies $d(\gamma) = k$. Also, since $\gamma \circ \gamma^{-1} = \beta \circ \beta^{-1}$ and $r(\gamma) = r(\beta)$, the gap of γ satisfies the conditions of Theorem 1 or Theorem 2 (depending on the nature of k) and so $\gamma \in NP(X)$. Likewise, $\lambda, \lambda' \in NP(X)$. Also, $\alpha = \lambda\gamma$ and $\gamma = \lambda'\alpha$, thus $\alpha \ \mathcal{L} \ \gamma$ and similarly $\gamma \ \mathcal{R} \ \beta$. In other words, if X is infinite then all elements of $NP(X)$ with rank r are $\mathcal{D}$–related, and so $NP_{r'}/NP_r$ is 0–bisimple.

If $|X| = n < \aleph_0$, then $g(\gamma) = g(\beta) \neq 0$, so Theorem 3 implies that $\gamma \in NP(X)$ when n is even, and when n is odd and $r < n-1$. On the other hand, if n is odd and $r = n-1$, then α, β belong to E_{n-1} (since their gaps are non-zero). Moreover, in this case, $NP_n/NP_{n-1} = E_{n-1} \cup \{0\}$, and this is 0–bisimple by [5] Lemma 5.

Suppose r is finite and let $\alpha = \alpha\beta = \beta\alpha$ for non-zero idempotents $\alpha, \beta \in P(X)$, each with rank r. Then $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$, and both these sets contain r elements, so $\operatorname{ran}\alpha = \operatorname{ran}\beta$. Therefore, for each $x \in \operatorname{dom}\alpha$, $x\alpha = (x\beta)\alpha = x\beta$ (since $x\beta \in \operatorname{ran}\alpha$), hence $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$. Also, if $y \in \operatorname{dom}\beta$ then $y\beta = x\alpha$ for some $x \in \operatorname{dom}\alpha$, so $y\alpha\beta = y\beta\alpha = x\alpha^2 = x\alpha$ (since $x\alpha \in \operatorname{dom}\alpha$) and so $y \in \operatorname{dom}\alpha$. Thus, $\operatorname{dom}\alpha = \operatorname{dom}\beta$, and it follows that $\alpha = \beta$. In other words, every non-zero idempotent in NP_{r+1}/NP_r is primitive. Conversely, suppose β is a non-zero idempotent in $NP_{r'}/NP_r$ and assume $r \geq \aleph_0$. Then we can write

$$\beta = \begin{pmatrix} B_i \\ b_i \end{pmatrix}, \quad \alpha = \begin{pmatrix} B_j \\ b_j \end{pmatrix},$$

where $|I| = r$, $J = I \setminus \{0\}$ for some fixed $0 \in I$, and $b_i \in B_i$ for each i. Since $\beta \in NP(X)$, its gap satisfies the conditions of Theorems 1 or 2. Since $g(\alpha) \geq g(\beta)$ and $r(\alpha) = r(\beta)$, the same is true for α, and so $\alpha \in NP(X)$. In addition, $\alpha = \alpha\beta = \beta\alpha$. In other words, if $r \geq \aleph_0$ then no non-zero idempotent in $NP_{r'}/NP_r$ is primitive. □

Next we prove a result which is similar to [5] Lemma 6 and, in doing so, we do not assume any prior knowledge of the congruences on a completely 0–simple semigroup.

Lemma 7. Suppose X is any set and r is any positive integer with $r+1 \leq |X|$. If σ is a non-universal congruence on NP_{r+1}/NP_r, then the relation σ^+ defined on $NP(X)$ by

$$\sigma^+ = \mathrm{id}_{NP(X)} \cup [\sigma \cap (DP_r \times DP_r)] \cup (NP_r \times NP_r)$$

is a congruence on $NP(X)$.

Proof. Clearly σ^+ is an equivalence, so we aim to show it is left and right compatible with composition on $NP(X)$. To do this, we consider only the case when $(\alpha, \beta) \in \sigma$ and $r(\alpha) = r(\beta) = r$ (the other possibilities are easy to check). First suppose $|\operatorname{ran}\alpha \cap \operatorname{ran}\beta| = s < r$ and write $B = \operatorname{ran}\beta$. Then $\mathrm{id}_B \in DI_r$ (by the usual argument) and hence, in the semigroup NP_{r+1}/NP_r, $\alpha.\,\mathrm{id}_B = 0$ but $\beta.\,\mathrm{id}_B = \beta$. Since σ is a congruence on NP_{r+1}/NP_r, it follows that $(0, \beta) \in \sigma$ and hence σ is universal on NP_{r+1}/NP_r, a contradiction. Thus, $s = r$ and this implies $\operatorname{ran}\alpha = \operatorname{ran}\beta = Y$ say. Let $\mu \in NP(X)$, and note that the ranks of $\alpha\mu$ and $\beta\mu$ are equal and at most r. In fact, if $r(\alpha\mu) = r(\beta\mu) < r$, then $(\alpha\mu, \beta\mu) \in NP_r \times NP_r \subseteq \sigma^+$, as required. On the other hand, if $r(\alpha\mu) = r(\beta\mu) = r$, then $\operatorname{ran}\alpha$ is a cross-section of r (disjoint) sets in the partition of $\operatorname{dom}\mu$ determined by the equivalence $\mu \circ \mu^{-1}$ on $\operatorname{dom}\mu$. Hence, if $\mu' = \mu|Y$, then $g(\mu') \geq d(\alpha)$, and $\mu' = \mu$ if $|X| = n$ is finite and odd, and $r = n - 1$. That is, the usual argument shows that $\mu' \in DI_r$. Clearly, $\alpha\mu' = \alpha\mu$ and $\beta\mu' = \beta\mu$. Therefore, $(\alpha\mu, \beta\mu) \in \sigma \cap (DP_r \times DP_r) \subseteq \sigma^+$. Hence σ^+ is right compatible.

Now let $\lambda \in NP(X)$ and suppose $r(\lambda\alpha) = r(\lambda\beta) = r$ for the same α, β as at the start. Let $|\operatorname{dom}\alpha \cap \operatorname{dom}\beta| = t$ and $C = \operatorname{dom}\beta$. Then an argument similar to the one above leads us to conclude that $t = r$ and hence that $\operatorname{dom}\alpha = \operatorname{dom}\beta = Z$ say. Moreover, since $r(\lambda\alpha) = r = r(\alpha)$, there exists a subset A of $\operatorname{ran}\lambda$ which is a cross-section of $Z/(\alpha \circ \alpha^{-1})$. Let $\lambda_0 = \lambda|(A\lambda^{-1})$. Then

$$\{x\lambda_0^{-1} : x \in A\} \subseteq \{x\lambda^{-1} : x \in \operatorname{ran}\lambda\}$$

and $r(\lambda_0) = r(\lambda)$. Thus, when X is infinite, if $g(\lambda) \geq r(\lambda)$ or $|z\lambda^{-1}| \geq r(\lambda)$ for some $z \in A$, then λ_0 satisfies the same conditions and so $\lambda_0 \in DP_r$. Suppose λ is spread over its rank, but λ_0 is not: that is, there exists a cardinal $p < r(\lambda_0) \leq k$ such that $|x\lambda_0^{-1}| \leq p$ for all $x \in A$. This means $\operatorname{dom}\lambda_0 = \bigcup\{x\lambda_0^{-1} : x \in A\}$ has cardinal at most $p < k$, and hence $g(\lambda_0) = k$. Therefore, in this case, λ_0 also belongs to DP_r.

In fact, the same is true when $|X| = n < \aleph_0$, including when n is odd and $r = n-1$ (since then $\lambda \in NP(X)$, $g(\lambda) \neq 0$ and $r(\lambda) = n - 1$ together imply $\lambda \in E_{n-1}$, and hence $\lambda_0 = \lambda$). Since $\lambda_0\alpha = \lambda\alpha$ and $\lambda_0\beta = \lambda\beta$, we conclude that $(\lambda\alpha, \lambda\beta) \in \sigma^+$. □

Remark 1. Recall that every non-universal congruence ρ on a 0–simple semigroup is 0–*restricted*: that is, $0\rho = \{0\}$; and clearly, by Lemma 6, NP_{r+1}/NP_r is 0–simple for each (finite or infinite) $r \geq 4$. Consequently, in the above result, $\sigma_1^+ = \sigma_2^+$ implies $\sigma_1 = \sigma_2$. For, if $\sigma_1^+ = \sigma_2^+$ then, by their definition, $\sigma_1 \cap (DP_r \times DP_r) = \sigma_2 \cap (DP_r \times DP_r)$; and, since each σ_i is 0–restricted, this implies $\sigma_1 = \sigma_2$.

Using the results in section 2, we now determine all congruences ρ on $NP(X)$ for which $\eta(\rho)$ is finite. Again, our argument closely follows that for [5] Theorem 5, but we include all the details for this more general context.

Theorem 6. Let ρ be a non-identity and non-universal congruence on $NP(X)$ and suppose $r = \eta(\rho)$ is finite. Then $\rho = \sigma^+$ where σ is a non-universal congruence on NP_{r+1}/NP_r.

Proof. Suppose $(\alpha, \beta) \in \rho$. By the definition of $\eta(\rho)$, if one of α or β has rank less than r, then the other also has rank less than r, and thus $(\alpha, \beta) \in NP_r^*$. By Lemma 2, if the rank of α or β is at least r, then $r(\alpha) = r(\beta) = s$ say. We assert that if s is infinite then $\alpha = \beta$.

To see this, assume $s \geq \aleph_0$ and $x\alpha \neq x\beta$ for some $x \in \operatorname{dom}\alpha$ (without loss of generality). Write $x\alpha = a$ and choose a partial cross-section Y of $\alpha \circ \alpha^{-1}$ such that $x \in Y$, $|Y| = r$ and $a \notin Y\beta$ (this is possible since $s \geq \aleph_0$ and $r < \aleph_0$, and $x \notin a\beta^{-1}$). Let $Z = Y\alpha$ and observe that $\alpha' = \operatorname{id}_Y .\alpha. \operatorname{id}_Z$ has rank r, whereas $\beta' = \operatorname{id}_Y .\beta. \operatorname{id}_Z$ has rank at most $r - 1$ (since $a \in Z \setminus Y\beta$). Moreover, both id_Y and id_Z belong to $NI(X)$ since their ranks are finite. Therefore, $(\alpha', \beta') \in \rho$. Since this contradicts the choice of $r = \eta(\rho)$, the assertion follows.

Consequently, if $s \geq \aleph_0$ then $(\alpha, \beta) \in \operatorname{id}_{NP(X)}$. On the other hand, if $r \leq s < \aleph_0$ and $\alpha \neq \beta$, then Lemma 4 implies $r = s$. That is, $(\alpha, \beta) \in \rho \cap (DP_r \times DP_r)$. We assert that

$$\sigma = \rho \cap (DP_r \times DP_r) \cup \{(0,0)\}$$

is a congruence on NP_{r+1}/NP_r. For, clearly it is an equivalence on NP_{r+1}/NP_r. Also, if $(\alpha, \beta) \in \rho \cap (DP_r \times DP_r)$ and $\mu \in DP_r$ then $(\alpha\mu, \beta\mu) \in \rho$, where the ranks of $\alpha\mu$ and $\beta\mu$ are at most r. However, by the choice of $r = \eta(\rho)$, either $r(\alpha\mu) = r(\beta\mu) = r$ or both $r(\alpha\mu)$ and $r(\beta\mu)$ is less than r: in the former case, $(\alpha\mu, \beta\mu) \in \rho \cap (DP_r \times DP_r)$ and, in the latter case, $\alpha\mu = \beta\mu = 0$ in the Rees factor semigroup NP_{r+1}/NP_r. That is, σ is right compatible on NP_{r+1}/NP_r, and similarly it is left compatible. Thus, we have shown that $\rho \subseteq \sigma^+$ as defined in Lemma 7, and clearly $\sigma^+ \subseteq \rho$, so equality follows. Moreover, σ is non-universal on NP_{r+1}/NP_r: otherwise, $\rho \cap (DP_r \times DP_r) = DP_r \times DP_r$ and hence

$$\rho = \mathrm{id}_{NP(X)} \cup (DP_r \times DP_r) \cup (NP_r \times NP_r)$$

which is not a congruence on $NP(X)$ (for example, if $|A| = |B| = r < \aleph_0$ and $A \cap B = \emptyset$ then $(\mathrm{id}_A, \mathrm{id}_B) \in \rho$, but $(\mathrm{id}_A \,.\, \mathrm{id}_A, \mathrm{id}_A \,.\, \mathrm{id}_B) \notin \rho$ by the definition of $\eta(\rho)$). $\square$

Given the above result, we need more information about the congruences on NP_{r+1}/NP_r. In fact, by Lemma 6, NP_{r+1}/NP_r is a completely 0–simple semigroup for finite $r \geq 4$, and thus all of its congruences can be described (see [1] section 10.7). To avoid the complication which that entails, we prove the following result.

Lemma 8. Suppose X is any set and $4 \leq r < |X|$, and let σ be a non-universal congruence on NP_{r+1}/NP_r. Then, for each $Y \subseteq X$ with cardinal r, there exists $N \lhd G(Y)$ such that

$$\sigma = \{(\lambda.\,\mathrm{id}_Y\,.\mu, \lambda.\gamma.\mu) : \lambda, \mu \in DP_r \text{ and } \gamma \in N\} \cup \{(0,0)\}.$$

Proof. Clearly, NI_{r+1}/NI_r is a subsemigroup of NP_{r+1}/NP_r. Hence, the restriction $\overline{\sigma}$ of σ to NI_{r+1}/NI_r is a congruence on NI_{r+1}/NI_r. Moreover, $\overline{\sigma}$ is non-universal: otherwise, $(\alpha, 0) \in \overline{\sigma} \subseteq \sigma$ for some $\alpha \in DI_r$ and then, by Lemma 6, each $\beta \in DP_r$ equals $\lambda\alpha\mu$ for some $\lambda, \mu \in DP_r$, which implies $(\beta, 0) \in \sigma$, and thus σ is universal, a contradiction. Therefore, by [5] Lemma 7, for each $Y \subseteq X$ with cardinal r, there exists $N \lhd G(Y)$ such that

$$\overline{\sigma} = \{(\lambda'.\,\mathrm{id}_Y\,.\mu', \lambda'.\gamma.\mu') : \lambda', \mu' \in DI_r \text{ and } \gamma \in N\} \cup \{(0,0)\}.$$

We assert that, for this $N \lhd G(Y)$, σ equals the relation:

$$\tau = \{(\lambda.\,\mathrm{id}_Y\,.\mu, \lambda.\gamma.\mu) : \lambda, \mu \in DP_r \text{ and } \gamma \in N\} \cup \{(0,0)\}.$$

To see this, note that $\overline{\sigma} \subseteq \sigma$ and, in particular, $(\mathrm{id}_Y, \gamma) \in \sigma$ for all $\gamma \in N$. Hence, $\tau \subseteq \sigma$. Conversely, suppose $(\alpha, \beta) \in \sigma$. In the proof of Lemma 7, we showed that $\operatorname{ran}\alpha = \operatorname{ran}\beta$, and that similarly $\operatorname{dom}\alpha = \operatorname{dom}\beta$. In fact, since r is finite, we can adapt the argument in the last paragraph of the proof of Lemma 4 to show that $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$. Thus we can write

$$\alpha = \begin{pmatrix} A_1 & \dots & A_r \\ x_1 & \dots & x_r \end{pmatrix} \quad \sim_\sigma \quad \beta = \begin{pmatrix} A_1 & \dots & A_r \\ x_{1\pi} & \dots & x_{r\pi} \end{pmatrix},$$

where π is a permutation of $\{1,\ldots,r\}$. Clearly, if $Y=\{y_1,\ldots,y_r\}$, then

$$\alpha=\begin{pmatrix}A_i\\ y_i\end{pmatrix}\circ \mathrm{id}_Y\circ\begin{pmatrix}y_i\\ x_i\end{pmatrix},\quad \beta=\begin{pmatrix}A_i\\ y_i\end{pmatrix}\circ\begin{pmatrix}y_i\\ y_{i\pi}\end{pmatrix}\circ\begin{pmatrix}y_i\\ x_i\end{pmatrix},$$

where the first and last mappings in these expressions for α and β are elements of DP_r, by a now-standard argument (as usual, the exceptional case occurs when $|X|=n$ is odd and $r=n-1$, but then $NP_{r+1}/NP_r=E_{n-1}\cup\{0\}$ and this was discussed fully in the proof of [5] Lemma 7). Moreover, if $a_i\in A_i$ for each $i=1,\ldots,r$, then

$$\mathrm{id}_Y=\begin{pmatrix}y_i\\ a_i\end{pmatrix}\circ\alpha\circ\begin{pmatrix}x_i\\ y_i\end{pmatrix}\quad\sim_\sigma\quad\begin{pmatrix}y_i\\ a_i\end{pmatrix}\circ\beta\circ\begin{pmatrix}x_i\\ y_i\end{pmatrix}=\begin{pmatrix}y_i\\ y_{i\pi}\end{pmatrix}=\gamma\ (\text{say}).$$

Since this pair belongs to $\overline{\sigma}$, it follows that $\gamma\in N$ and thus $(\alpha,\beta)\in\tau$. □

The next result extends [5] Corollary 1 to arbitrary sets.

Corollary 1. For any set X, the set of all congruences on $NP(X)$ with finite primary rank forms a chain with respect to $\subseteq$.

Proof. Let ρ_1 and ρ_2 be distinct congruences on $NP(X)$, neither of which equals the identity or the universal congruence on $NI(X)$, and write $\eta(\rho_i)=r_i$, where r_i are positive integers for $i=1,2$. Then $\rho_i=\sigma_i^+$ for some (unique) congruence σ_i on NP_{r_i+1}/NP_{r_i}. If $r_1<r_2$ then $NP_{r_1}\subsetneq NP_{r_2}$ and

$$\sigma_1\cap(DP_{r_1}\times DP_{r_1})\subsetneq NP_{r_2}\times NP_{r_2},$$

from which we deduce that $\rho_1\subseteq\rho_2$. Suppose $r_1=r_2=r$, say. By Lemma 8, σ_1 is determined by some $N_1\triangleleft G(Y)$ and σ_2 by some $N_2\triangleleft G(Y)$ where $|Y|=r$ (note: the same Y can be used). Since the normal subgroups of $G(Y)$ form a chain, it follows from Lemma 8 that $\sigma_1\subseteq\sigma_2$ or $\sigma_2\subseteq\sigma_1$, and hence that $\rho_1\subseteq\rho_2$ or $\rho_2\subseteq\rho_1$. □

4. Infinite primary rank for $NP(X)$ when $|X|$ is regular

Henceforth, X is an infinite set with cardinal k.

Suppose ρ is a congruence on $NP(X)$ and let

$$\overline{\rho}=\rho\cap[NI(X)\times NI(X)].$$

Clearly, $\overline{\rho}$ is a congruence on $NI(X)$; and, if $\eta(\rho)$ is infinite, then $\eta(\overline{\rho})$ is also (for example, if $\eta(\rho)\geq\aleph_0$ then $NP_{\aleph_0}\times NP_{\aleph_0}\subseteq\rho$ and thus $NI_{\aleph_0}\times NI_{\aleph_0}\subseteq\overline{\rho}$, so $\eta(\overline{\rho})\geq\aleph_0$). In this event, [5] Theorem 8 enables us to describe $\overline{\rho}$ in terms of a finite number of Rees congruences and Malcev congruences, as follows.

Theorem 7. Suppose $|X|=k\geq\aleph_0$. If $\overline{\rho}$ is a non-universal congruence on $NI(X)$ for which $\eta(\overline{\rho})\geq\aleph_0$ then

$$\overline{\rho}=I^*_{\eta_1}\cup[\Delta_{\xi_1}\cap I^*_{\eta_2}]\cup\cdots\cup[\Delta_{\xi_{r-1}}\cap I^*_{\eta_r}]\cup[\Delta_n\cap(DI_k\times DI_k)] \tag{1}$$

where $\eta_1=\eta(\overline{\rho})$ and the cardinals ξ_i,η_i form a sequence:

$$n\leq\xi_{r-1}<\cdots<\xi_1\leq\eta_1<\cdots<\eta_r\leq k,$$

in which ξ_{r-1} is infinite, either $n=1$ or n is infinite, and if $n\geq\aleph_0$ then $\eta_r=k$.

Conversely, if $\overline{\rho}$ is a relation on $NI(X)$ defined as in (1) for a sequence of cardinals with the above properties, then $\overline{\rho}$ is a non-universal congruence on $NI(X)$.

In the above, for each proper ideal $I_r = I(X) \cap P_r = NI_r$ of $NI(X)$, I_r^* denotes the corresponding *Rees congruence* on $NI(X)$: compare [1] vol 1, p 17 and vol 2, p 227. Also, as in [5], DI_r denotes the $\mathcal{D}$–class of $NI(X)$ which contains all elements with rank r. In addition, for each $\alpha, \beta \in P(X)$ and $n \geq \aleph_0$, we let

$$D(\alpha,\beta) = \{x \in X : x\alpha \neq x\beta\}, \quad \mathrm{dr}\,(\alpha,\beta) = \max\,(|D(\alpha,\beta)\alpha|, |D(\alpha,\beta)\beta|)$$

$$\Delta_n = \{(\alpha,\beta) \in P(X) \times P(X) : \mathrm{dr}\,(\alpha,\beta) < n\}.$$

Then, by [7] Theorem 3.1, each Δ_n is a so-called *Malcev congruence* on $P(X)$. Note that for the definition of $D(\alpha,\beta)$, we use the convention: $x\alpha = \emptyset$ if and only if $x \notin \mathrm{dom}\,\alpha$.

Since $NI(X) \subseteq NP(X)$ and $\overline{\rho} \subseteq \rho$, we know each term in (1) is contained in ρ. We assert that, if $|X| = k$ is regular, then

$$\rho = NP_{\eta_1}^* \cup [\Delta_{\xi_1} \cap NP_{\eta_2}^*] \cup \cdots \cup [\Delta_{\xi_{r-1}} \cap NP_{\eta_r}^*] \cup [\Delta_n \cap (DP_k \times DP_k)] \qquad (2)$$

where the cardinals ξ_i, η_i are the same as those corresponding to $\overline{\rho}$ in (1).

In fact, since $\overline{\rho} \subseteq \rho$, we know $\eta(\overline{\rho}) \leq \eta(\rho)$. For the reverse inequality, suppose $(\alpha,\emptyset) \in \rho$ for some $\alpha \in NP(X)$ and let A be a cross-section of $\alpha \circ \alpha^{-1}$. Since k is regular and $\alpha \in NP(X)$, Theorem 1 implies that $g(\alpha) \neq 0$, and either $g(\alpha) = k$ or $|z\alpha^{-1}| = k$ for some $z \in X$. Clearly, in each case, $\mathrm{id}_A \in NI(X)$ and so $(\mathrm{id}_A.\alpha, \emptyset) \in \rho$, where $\mathrm{id}_A.\alpha$ belongs to $NI(X)$ and has the same rank as α. This implies $\eta(\rho) \leq \eta(\overline{\rho})$ and equality follows. In addition, since $I_{\eta_1}^* \subseteq \rho$, we know $(\mathrm{id}_A, \emptyset) \in \rho$ for each $A \subseteq X$ with cardinal less than η_1. Consequently, if $\alpha \in NP(X)$ has range A, then $\alpha = \alpha.\,\mathrm{id}_A$ and so $(\alpha,\emptyset) \in \rho$. In other words, $NP_{\eta_1}^* \subseteq \rho$.

To consider the other terms in (1), we will need the following result: see [7] Lemma 3.4.

Lemma 9. If $\alpha, \beta \in P(X)$ and $\mathrm{dr}\,(\alpha,\beta) = \zeta \geq \aleph_0$ then there exists $Y \subseteq D(\alpha,\beta)$ such that $Y\alpha \cap Y\beta = \emptyset$ and $\max\,(|Y\alpha|, |Y\beta|) = \zeta$.

The next result will simplify some of our argument regarding (2): we omit a proof since it is exactly the same as that for [5] Lemma 11.

Lemma 10. If the ranks of $\alpha, \beta \in NP(X)$ are not equal, and at least one of them is infinite, then $\mathrm{dr}\,(\alpha,\beta) = \max\,(r(\alpha), r(\beta))$.

Remark 2. This result implies that, if $\aleph_0 \leq \xi \leq \eta$ and $(\alpha,\beta) \in \Delta_\xi \cap NP_\eta^*$, where $r(\alpha) > r(\beta)$ and $r(\alpha) \geq \aleph_0$, then $r(\alpha) = \mathrm{dr}\,(\alpha,\beta) < \xi$, and so $(\alpha,\beta) \in NP_\xi^*$. Moreover, since $\xi \geq \aleph_0$, the same conclusion holds if $r(\alpha)$ and $r(\beta)$ are both finite (since, for example, $D(\alpha,\beta)\alpha \subseteq \mathrm{ran}\,\alpha$). In other words, suppose $(\alpha,\beta) \in \Delta_\xi \cap NP_\eta^*$, where $r(\alpha) \geq r(\beta)$ and $r(\alpha) \geq \aleph_0$. If we can show that there exists $\lambda \in NI_\eta$ for which $\lambda\alpha, \lambda\beta \in NI(X)$ and $r(\lambda\alpha) = r(\alpha)$, then either $(\lambda\alpha, \lambda\beta) \in NI_\xi^*$ if $r(\lambda\alpha) > r(\lambda\beta)$, or $(\lambda\alpha,\lambda\beta) \in \Delta_\xi \cap NI_\eta^*$ if $r(\lambda\alpha) = r(\lambda\beta)$.

We now return to the argument regarding (2). If $(\alpha, \beta) \in \rho$ and $\operatorname{dr}(\alpha, \beta) = d \geq \aleph_0$ then, without loss of generality, there exists $Y = \{y_i\} \subseteq D(\alpha, \beta)$ such that $Y\alpha \cap Y\beta = \emptyset$ and $|Y\alpha| = d$. Clearly, although α may not be injective, we can assume Y is a partial cross-section of $\alpha \circ \alpha^{-1}$, and then $|I| = d$. Let $D = D(\alpha, \beta)$ and $C = D\alpha \cup D\beta$. Then $\operatorname{ran}\alpha \setminus C = \operatorname{ran}\beta \setminus C = \{e_j\}$ say, and, for each j, there exists $r_j \in \operatorname{dom}\alpha \cap \operatorname{dom}\beta$ such that $r_j\alpha = e_j = r_j\beta$ (this is true by our convention: $x\alpha = \emptyset$ if and only if $x \notin \operatorname{dom}\alpha$, mentioned above).

Let λ be the identity on $Y \cup \{r_j\}$. Again, since k is regular and $\alpha \in NP(X)$, Theorem 1 implies that $g(\alpha) \neq 0$, and either $g(\alpha) = k$ or $|z\alpha^{-1}| = k$ for some $z \in X$. In the first case, $g(\lambda) \geq g(\alpha)$ implies $\lambda \in NI(X)$; and, in the second case, if z equals $y_i\alpha$ or $r_j\alpha$ for some i or j, then $z\alpha^{-1} \cap \operatorname{dom}\lambda$ equals y_i or r_j, hence $g(\lambda) \geq |z\alpha^{-1}|$ and so $\lambda \in NI(X)$ (clearly, if $z \notin Y\alpha \cup \{r_j\alpha\}$, then the same conclusion holds). It follows that $\operatorname{dr}(\lambda\alpha, \lambda\beta) = d$ and

$$\lambda\alpha = \begin{pmatrix} y_i & r_j \\ a_i & e_j \end{pmatrix} \quad \sim_\rho \quad \lambda\beta = \begin{pmatrix} y_i & r_j \\ b_i & e_j \end{pmatrix}, \tag{3}$$

where b_i may not exist for some i (that is, when $y_i \notin \operatorname{dom}\beta$) and the b_i may not be distinct (for example, if β is not injective on Y). If $|\{b_i\}| = d$, write $\{b_i\} = \{b_\ell\}$ where the b_ℓ are distinct and fix $y_\ell \in Y$ such that $y_\ell\beta = b_\ell$. If λ' is the identity on $\{y_\ell\} \cup \{r_j\}$ then, as before, $\lambda' \in NI(X)$ and we obtain

$$\lambda'\alpha = \begin{pmatrix} y_\ell & r_j \\ a_\ell & e_j \end{pmatrix} \quad \sim_\rho \quad \lambda'\beta = \begin{pmatrix} y_\ell & r_j \\ b_\ell & e_j \end{pmatrix}, \tag{4}$$

and these are elements of $NI(X)$ whose difference rank equals $|L| = d$. On the other hand, if $|\{b_i\}| < d$ then $\{a_i\} \setminus \{b_i\} = \{a_\ell\}$, say, has cardinal d. In this event, if μ is the identity on $\{a_\ell\} \cup \{e_j\}$ then $\mu \in NI(X)$ (since $d(\mu) \geq d(\alpha) = k$) and from (3) we obtain:

$$\lambda\alpha\mu = \begin{pmatrix} y_\ell & r_j \\ a_\ell & e_j \end{pmatrix} \quad \sim_\rho \quad \lambda\beta\mu = \begin{pmatrix} r_j \\ e_j \end{pmatrix}. \tag{5}$$

Hence, again we find a pair in $\overline{\rho}$ whose difference rank equals $|L| = d$. In other words, if ρ contains a pair of elements which differ at $d \geq \aleph_0$ places, then $\overline{\rho}$ does also.

Note that, with the above notation, $r(\beta) \leq r(\alpha) = r$, say, and

$$Y\alpha \subseteq D\alpha = D\alpha \cap \operatorname{ran}\alpha \quad \text{and} \quad D\beta \cap \operatorname{ran}\alpha \subseteq D\beta.$$

Hence, $|C \cap \operatorname{ran}\alpha| = |(D\alpha \cap \operatorname{ran}\alpha) \cup (D\beta \cap \operatorname{ran}\alpha)| = d$, and

$$r(\alpha) = |C \cap \operatorname{ran}\alpha| + |\operatorname{ran}\alpha \setminus C| = |I| + |J| = r(\lambda\alpha) \geq r(\lambda\beta).$$

Clearly, we will reach the same conclusion if λ' or μ are used in the above argument.

Therefore, by Remark 2, if $\aleph_0 \leq \xi \leq \eta$ and $(\alpha, \beta) \in \Delta_\xi \cap NP^*_\eta$, then $(\lambda\alpha, \lambda\beta) \in \Delta_\xi \cap I^*_\eta$ for some $\lambda \in NI(X)$. In other words, we have shown that: if there exists $(\alpha, \beta) \in \rho$ for which $r(\beta) \leq r(\alpha) = r$ and $\operatorname{dr}(\alpha, \beta) = d \leq r$, then there exists $(\overline{\alpha}, \overline{\beta}) \in \overline{\rho}$ for which $r(\overline{\beta}) \leq r(\overline{\alpha}) = r$ and $\operatorname{dr}(\overline{\alpha}, \overline{\beta}) = d$. Clearly, the converse also holds since $\overline{\rho} \subseteq \rho$, and $I_\eta = NI_\eta \subseteq NP_\eta$ implies that $\Delta_\xi \cap I^*_\eta \subseteq \Delta_\xi \cap NP^*_\eta$.

In addition, since $\Delta_{\xi_{i-1}} \cap I^*_{\eta_i} \subseteq \rho$ for each $i = 1, \ldots, r$, we know $(\mathrm{id}_{A\cup B}, \mathrm{id}_A) \in \rho$, where $X = A \dot\cup B \dot\cup Z$, $|A| < \eta_i$, $|B| < \xi_{i-1} < \eta_i$ and $|Z| = k$. Consequently, if $\alpha \in NP(X)$ has range $A \cup B$, then $\alpha . \mathrm{id}_{A\cup B} = \alpha$ and $(\alpha, \beta) \in \rho$, where $\beta = \alpha . \mathrm{id}_A$, $r(\alpha) = r(\beta) = |A|$ and $\mathrm{dr}\,(\alpha, \beta) = |B|$. From this, it follows that $\Delta_{\xi_{i-1}} \cap NP^*_{\eta_i} \subseteq \rho$ for each $i = 1, \ldots, r$.

It remains to consider the last term in (1) and the corresponding one in (2).

If $n = 1$ in (1), then no pair of distinct elements of $NI(X)$ with rank k are $\overline{\rho}$–equivalent. Suppose there exists $(\alpha, \beta) \in \rho \cap (DP_k \times DP_k)$ where $\alpha \neq \beta$. Without loss of generality, we assume that $a\alpha \neq a\beta$ for some $a \in \mathrm{dom}\,\alpha$, and let $A = \{a_i\}$ be a cross-section of $\alpha \circ \alpha^{-1}$ which contains $a = a_0$, say. Then, as before, $\mathrm{id}_A \in NI(X)$ and we have:

$$\mathrm{id}_A . \alpha = \begin{pmatrix} a_i \\ a_i\alpha \end{pmatrix} \quad \sim_\rho \quad \mathrm{id}_A . \beta = \begin{pmatrix} a_i \\ a_i\beta \end{pmatrix}, \tag{6}$$

where the $a_i\beta$ are not necessarily distinct. If $|\{a_i\beta\}| = k$, write $\{a_i\beta\} = \{a_j\beta\}$ where the $a_j\beta$ are distinct (if non-empty), $0 \in J$ and $|J| = k$. Let $B = \{a_j\}$. Then $\mathrm{id}_B \in NI(X)$ and

$$\mathrm{id}_B . \alpha = \begin{pmatrix} a_j \\ a_j\alpha \end{pmatrix} \quad \sim_\rho \quad \mathrm{id}_B . \beta = \begin{pmatrix} a_j \\ a_j\beta \end{pmatrix}.$$

Since $a_0 \in B$, $\mathrm{id}_B . \alpha \neq \mathrm{id}_B . \beta$ and these are $\overline{\rho}$–equivalent elements of $NI(X)$ with rank k, contradicting our initial assumption that $n = 1$.

Hence, if $n = 1$ then $|\{a_i\beta\}| < k$ and so $\{a_i\alpha\} \setminus \{a_i\beta\} = \{a_j\alpha\} = Z$, say, has cardinal k. Then $\mathrm{id}_Z \in NP(X)$ (since $|X \setminus Z| \geq d(\alpha) = k$) and from (6) we obtain:

$$\mathrm{id}_A . \alpha . \mathrm{id}_Z = \begin{pmatrix} a_j \\ a_j\alpha \end{pmatrix} \quad \sim_\rho \quad \mathrm{id}_A . \beta . \mathrm{id}_Z = \emptyset.$$

It follows that $\eta(\rho) = k'$ and ρ is universal, contradicting our basic supposition.

Suppose instead that $n \geq \aleph_0$ in (1), and hence that $\eta_r = k$ (by the condition on the cardinals). This means that, if $X = A \dot\cup B \dot\cup Z$, $|A| = |Z| = k$ and $|B| < n$, then

$$(\mathrm{id}_{A\cup B}, \mathrm{id}_A) \in \Delta_n \cap (DI_k \times DI_k) \subseteq \overline{\rho}.$$

From this, like before, it follows that $\Delta_n \cap (DP_k \times DP_k) \subseteq \rho$.

Consequently, we have proved half of the following result. For its converse, we note that, just as in [5], Lemma 10 can be used to show that ρ is a congruence on $NP(X)$, provided the cardinals have the properties stated: the difference between the last paragraph in the proof of [5] Theorem 8 and the current one is simply a matter of notation (that is, 'I' and 'NI' become 'NP').

Theorem 8. Suppose $|X| = k \geq \aleph_0$ and k is regular. If ρ is a non-universal congruence on $NP(X)$ for which $\eta(\rho) \geq \aleph_0$ then

$$\rho = NP^*_{\eta_1} \cup [\Delta_{\xi_1} \cap NP^*_{\eta_2}] \cup \cdots \cup [\Delta_{\xi_{r-1}} \cap NP^*_{\eta_r}] \cup [\Delta_n \cap (DP_k \times DP_k)] \tag{7}$$

where $\eta_1 = \eta(\rho)$ and the cardinals ξ_i, η_i form a sequence:

$$n \leq \xi_{r-1} < \cdots < \xi_1 \leq \eta_1 < \cdots < \eta_r \leq k,$$

in which n is infinite and $\eta_r = k$.

Conversely, if ρ is a relation on $NP(X)$ defined as in (8) for a sequence of cardinals with the above properties, then ρ is a non-universal congruence on $NP(X)$.

5. Infinite primary rank for $NP(X)$ when $|X|$ is singular

In this section, X is an infinite set whose cardinal k is singular: that is, according to [3] Lemma 10.2.2, $k = \sum k_m$ for some distinct infinite cardinals k_m, where $|M| < k$ and $k_m < k$ for each $m \in M$. To describe all the congruences on $NP(X)$ for such X, we closely follow the argument in section 4. In fact, here the only differences will occur when we need to ensure that a specific transformation belongs to $NP(X)$: that is, it satisfies the conditions of Theorem 2.

Like before, given a congruence ρ on $NP(X)$, we let $\overline{\rho}$ denote the restriction of ρ to $NI(X)$ and observe that if $\eta(\rho)$ is infinite, then $\eta(\overline{\rho})$ is also. In fact, since $\overline{\rho} \subseteq \rho$, we know $\eta(\overline{\rho}) \leq \eta(\rho)$. For the reverse inequality, suppose $(\alpha, \emptyset) \in \rho$ for some $\alpha \in NP(X)$ and let A be a cross-section of $\alpha \circ \alpha^{-1}$. Since k is singular and $\alpha \in NP(X)$, Theorem 2 implies that $g(\alpha) \neq 0$, and either $g(\alpha) \geq r(\alpha)$ or α is spread over its rank. If $r(\alpha) < k$ then $|A| < k$, so $|X \setminus A| = k$ and hence $\mathrm{id}_A \in NI(X)$. Suppose $r(\alpha) = k$. If $g(\alpha) \geq r(\alpha)$ then $|X \setminus A| \geq g(\alpha) = k$; and, if α is spread over its rank then, for each $m \in M$ (see the start of this section), there exists $y_m \in X$ such that $|y_m\alpha^{-1}| > k_m$. Since A contains exactly one element from each $y_m\alpha^{-1}$, we see that, for each m, $|y_m\alpha^{-1} \setminus A| > k_m$. Hence, $k = \sum k_m \leq \sum |y_m\alpha^{-1} \setminus A|$, and it follows that $|X \setminus A| = k$. Thus, $\mathrm{id}_A \in NI(X)$ in all cases and, as in section 4, we deduce that $\eta(\rho) \leq \eta(\overline{\rho})$ and equality follows. Moreover, since $\eta(\rho) = \eta_1 < k$, we know $|X \setminus A| = k$ for each $A \subseteq X$ with cardinal less than η_1, hence $\mathrm{id}_A \in NI_{\eta_1}$ and so, like before, we conclude that $NP^*_{\eta_1} \subseteq \rho$.

Next, both Lemma 9 and Lemma 10 hold for any set X, so they can be applied in the present situation. In particular, Remark 2 remains valid.

Now, using the same notation as before, we let λ be the identity on $B = Y \cup \{r_j\}$. Since k is singular and $\alpha \in NP(X)$, Theorem 2 implies that $g(\alpha) \neq 0$, and either $g(\alpha) \geq r(\alpha)$ or α is spread over its rank. If $r(\alpha) < k$ then $|I| + |J| < k$, hence $g(\lambda) = k$ and $\lambda \in NI(X)$. Suppose instead that $r(\alpha) = k$. Then, the above argument for the set A applies equally here for the set B, and we deduce that $\lambda \in NI(X)$ in all cases. As at (3), this implies that $(\lambda\alpha, \lambda\beta) \in \rho$, where $\mathrm{dr}\,(\lambda\alpha, \lambda\beta) = d$ and, as before, the same proviso holds. Then the same λ' belongs to $NI(X)$ (since $\{y_\ell\} \cup \{r_j\} \subseteq Y \cup \{r_j\} = B$) and we again obtain (4). On the other hand, if μ is the identity on the set $\{a_\ell\} \cup \{e_j\}$ specified before, then $\mu \in NI(X)$ (since, by Theorem 2, $d(\mu) \geq d(\alpha) = k$) and thus we again obtain (5).

Consequently, when k is singular, we have shown that: there exists $(\alpha, \beta) \in \rho$ for which $r(\beta) \leq r(\alpha) = r$ and $\mathrm{dr}\,(\alpha, \beta) = d \leq r$ if and only if there exists $(\overline{\alpha}, \overline{\beta}) \in \overline{\rho}$ for which $r(\overline{\beta}) \leq r(\overline{\alpha}) = r$ and $\mathrm{dr}\,(\overline{\alpha}, \overline{\beta}) = d$. And, like in section 4, it follows that $\Delta_{\xi_{i-1}} \cap NP^*_{\eta_i} \subseteq \rho$ for each $i = 1, \ldots, r$.

Finally, we compare the last term in (1) with the corresponding one in (2). We have already seen that, if k is singular and A is a cross-section of $\alpha \circ \alpha^{-1}$, then $\mathrm{id}_A \in NI(X)$ and thus we obtain (6). By continuing to follow the argument in section 4, we see that $B = \{a_j\} \subseteq A$, hence $|X \setminus B| = k$ and so $\mathrm{id}_B \in NI(X)$. This gives a contradiction like before. Since the rest of the previous argument holds verbatim, we conclude that $n \geq \aleph_0$ in (1) and hence that $\eta_r = k$. Like before, it then easily follows that $\Delta_n \cap (DP_k \times DP_k) \subseteq \rho$.

Thus, we have proved a result which is exactly the same as Theorem 8, except that $|X| = k$ is a singular cardinal.

We now deduce a result similar to [5] Corollary 2. Our proof follows the one for $NI(X)$ but, since it depends on Theorem 8 (and the corresponding result for singular cardinals), we include all the details.

Corollary 2. Suppose $|X| = k \geq \aleph_0$ and write $\Delta_k^+ = \Delta_k \cap [NP(X) \times NP(X)]$. Then Δ_k^+ is the only maximal congruence on $NP(X)$, and hence $NP(X)/\Delta_k^+$ is a congruence-free nilpotent-generated regular semigroup.

Proof. First we note that Δ_k^+ is a non-universal congruence on $NP(X)$: for example, if $X = A \dot\cup B$ where $|A| = |B| = k$, then $\mathrm{id}_A \in NP(X)$ and $\mathrm{dr}\,(\mathrm{id}_A, \emptyset) = k$, so $(\mathrm{id}_A, \emptyset) \notin \Delta_k^+$.

Since $NP(X)$ is nilpotent-generated and regular (by Theorems 1 and 2), and Δ_k^+ is a congruence on $NP(X)$, it follows that $NP(X)/\Delta_k^+$ is also nilpotent-generated and regular.

Suppose $\Delta_k^+ \subseteq \rho$ for some non-universal congruence on $NP(X)$. Now, $\eta(\rho)$ equals the least cardinal greater than $r(\alpha)$ for each $\alpha \in NP(X)$ such that $(\alpha, \emptyset) \in \rho$. But, if $A \subseteq X$ has cardinal less than k, then $d(\mathrm{id}_A) = k$ and $g(\mathrm{id}_A) = k > |A| = r(\mathrm{id}_A)$, so $(\mathrm{id}_A, \emptyset) \in \Delta_k^+ \subseteq \rho$. In particular, since $\aleph_0 \leq |A| < k$ can occur, we deduce that $\eta(\rho) \geq \aleph_0$. Therefore, ρ has the form displayed in (7), regardless of whether k is regular or singular. Clearly, $(\alpha, \emptyset) \in \Delta_k^+ \subseteq \rho$ for each $\alpha \in NP_k$, so $\eta_1 = k$. Moreover, if $X = A \dot\cup B \dot\cup C$ where $|A| = |C| = k$ and $|B| < k$, then both $\mathrm{id}_{A \cup B}$ and id_A have gap and defect equal to k, so they belong to DP_k and hence

$$(\mathrm{id}_{A \cup B}, \mathrm{id}_A) \in \Delta_k \cap [DP_k \times DP_k].$$

It follows that $n \geq k$. Since $NP_k^* \subseteq \Delta_k^+$, this implies that each term in (7) is contained in Δ_k^+, hence $\rho \subseteq \Delta_k^+$ and equality follows.

Finally, suppose ρ is a maximal congruence on $NP(X)$ for which there exists $(\alpha, \beta) \in \rho$ with $\mathrm{dr}\,(\alpha, \beta) = k$. Then $r(\alpha) = r(\beta) = k$ (by the definition of 'difference rank'). Since such pairs (α, β) do not belong to the congruences described in Theorem 6, we deduce that $\eta(\rho) \geq \aleph_0$. However, then (7) implies that $n = k'$, and so we have a contradiction:

$$k' \leq \xi_{r-1} < \cdots < \xi_1 \leq \eta_1 < \cdots < \eta_r \leq k.$$

Thus, $\mathrm{dr}\,(\alpha, \beta) < k$ for all $(\alpha, \beta) \in \rho$, hence $\rho \subseteq \Delta_k^+$, and equality follows by the maximality of ρ and the fact that Δ_k^+ is non-universal. □

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