Lecture notes on
Ordinary Differential Equations

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October 13, 2011

Abstract

This is not a book! These are personal notes written while preparing lectures on several courses at the UM. They are rather informal and may even contain mistakes. I tried to be as synthetic as I could, without missing the observations that I consider important.

I probably will not lecture all I wrote, and did not write all I plan to lecture. So, I included empty or sketched paragraphs, about material that I think should/could be lectured within the same course. The last section, "Bestiario", is a list of famous examples proposed to students for their projects.

References contain some introductory manuals, some classics, and other books where I have learnt things in the past century. Besides, good material and further references can easily be found on the web, for example in Wikipedia.

Pictures were made with "Grapher" on my MacBook.

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1 Introduction

1.1 What is it about

This is a first introduction to solving differential equations. "Solving differential equations is useful" is V.I. Arnold's translation [Ar85] of Isaac Newton's anagram

"6accdae13eff7i3l9n4o4qrr4s8t12vx"

(*Data aequatione quotcunque fluentes quantitates involvente fluxiones invenire et vice versa*) contained in a letter to Gottfried Leibniz in 1677.

1.2 Models and laws of physics

Laws of nature, or empirical models of physical phenomena, are relations between observables.

Some, as Kepler's third law $T^2/a^3 \simeq 3 \times 10^{-19}\ \mathrm{s}^2\mathrm{m}^{-3}$ (relating the period T of revolution of a planet to the semi-major axis a of its orbit) or the perfect gas equation $PV = nRT$ (saying that the product of the pressure P by the volume V of an ideal gas is proportional to the temperature T), simply say that the actual value of a certain observable is equal to some function of the actual values of other observables.

Many of them are equations that contain derivatives of some observable w.r.t. others observables, and as such are called *differential equations*. The typical situation is that of "dynamics", some observable changing in time according to a law that prescribes the behavior of some of its time derivatives.

The archetypical example is Newton law "force = mass × acceleration"

$$m\,\ddot{q}(t) = F(t,q)\,.$$

It says that the trajectories $t \mapsto q(t)$ of a moving particle in an inertial frame are not arbitrary curves, but curves that have second derivative $\ddot{q}(t)$ proportional to a given function $F(t,q)$ called force, the proportionality factor being the "inertial mass" of the particle. Given the force, and given the initial position $q(0)$ and velocity $\dot{q}(0)$ of the particle, Newton equation prescribes the future and the past history of the particle. Hence, being able to solve the Newton equation amounts to being able to make predictions.

1.3 First classification and examples

Differential equations are classified according to their "form" and to "methods" at our disposal to solve them.

A first dichotomy is *ordinary* versus *partial* differential equations.

Ordinary differential equations. An *Ordinary Differential Equation* (later on referred to as *ODE*) is a differential equation where the unknown function only depends on one real variable. The *order* of a differential equation is the biggest order of the derivatives entering in the equation. Examples are

- Newton equation
$$m\,\ddot{q} = F(q,t)$$
satisfied by the trajectory $r(t)$ of a moving particle of mass m subject to a force F),
- the (consequence of the) Kirchoff's law
$$L\dot{I} + RI = V(t)$$
(satisfied by the current $I(t)$ in a LR circuit driven by a tension $V(t)$),
- or the Lotka-Volterra system
$$\begin{cases} \dot{x} = ax - bxy \\ \dot{y} = -cy + dxy \end{cases}$$
(modeling the competition between x preys and y predators in the same territory).

Partial differential equations. A *Partial Differential Equation* (later on referred to as *PDE*) is a differential equation where the unknown function depends on two or more real variables (hence the derivatives have to be partial derivatives). Examples are

- the wave equation
$$\frac{\partial^2 u}{\partial t^2} - c\frac{\partial^2 u}{\partial x^2} = 0$$
which describes, for example, small oscillations of the displacement $u(x,t)$ of a string from its equilibrium position,

- the Poisson and Laplace's equations
$$\Delta V = 4\pi\rho \quad \text{and} \quad \Delta V = 0$$
satisfied by the electric potential V in a region with or without charges, respectively, where ρ is the charge density and the "Laplacian" Δ in the 3-dimensional Euclidean space is the differential operator $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$),

- the heat equation
$$\frac{\partial \rho}{\partial t} - \beta\Delta\rho = 0$$
which describes propagation of heat in a homogeneous medium,

- or the Schrödinger equation
$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi + V\psi$$
satisfied by the complex valued wave function ψ of an electron subject to a potential field V.

The two classes, ODEs and PDEs, require conceptually distinct techniques to be dealt with.

Homework. Look for differential equation in your field.

1.4 Notations and facts from analysis

Números. $\mathbb{N} := \{0,1,2,3,\dots\}$, $\mathbb{Z} := \{0,\pm1,\pm2,\pm3,\dots\}$, $\mathbb{Q} := \{p/q \text{ com } p,q,\in\mathbb{Z}\,,\ q\neq 0\}$. $\mathbb{R}$ e $\mathbb{C}$ são os corpos dos númeors reais e complexos, respectivamente.

Espaço Euclidiano. $\mathbb{R}^n$ denota o espaço Euclidiano de dimensão n. Fixada a base canónica $e_1,\dots,e_n$, os pontos de $\mathbb{R}^n$ são os vectores $x = (x_1,x_2,\dots,x_n) := x_1e_1+x_2e_2+\cdots+x_ne_n$ de coordenadas $x_i \in \mathbb{R}$, com $i = 1,2,\dots,n$. As coordenadas no plano Euclidiano ou no espaço 3-dimensional são também denotadas, conforme a tradição, por $(x,y)\in\mathbb{R}^2$ ou $(x,y,z)\in\mathbb{R}^3$.

O *produto interno Euclidiano* em $\mathbb{R}^n$, denotado por $\langle x,y\rangle$ ou $x\cdot y$, é
$$\langle x,y\rangle := x_1y_1 + x_2y_2 + \cdots + x_ny_n\,,$$
e a norma Euclidiana é $\|x\| := \sqrt{\langle x,x\rangle}$. A distância Euclidiana entre os pontos $x,y\in\mathbb{R}^n$ é definida pelo teorema de Pitágoras
$$\mathrm{d}(x,y) := \|x-y\| = \sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}\,.$$
A bola aberta de centro $a\in\mathbb{R}^n$ e raio $r>0$ é o conjunto $B_r(a) := \{x\in\mathbb{R}^n \text{ s.t. } \|x-a\| < r\}$. Um subconjunto $A\subset\mathbb{R}^n$ é aberto em $\mathbb{R}^n$ se cada seu ponto $a\in A$ é o centro de uma bola $B_\varepsilon(a)\subset A$, com $\varepsilon>0$ suficientemente pequeno.

Caminhos. Se $t\mapsto x(t) = (x_1(t),x_2(t),\dots,x_n(t))\in\mathbb{R}^n$ é uma função diferenciável do "tempo" $t\in I\subset\mathbb{R}$, ou seja, um caminho diferenciável definido num intervalo $I\subset\mathbb{R}$ com valores no espaço Euclidiano $\mathbb{R}^n$, então as suas derivadas são denotadas por
$$\dot{x} := \frac{dx}{dt}\,, \qquad \ddot{x} := \frac{d^2x}{dt^2}\,, \qquad \dddot{x} := \frac{d^3x}{dt^3}\,, \qquad \dots$$
Em particular, $v(t) := \dot{x}(t)$ é dita "velocidade" (da trajectória x no instante t), a sua norma $\|v(t)\|$ "velocidade escalar" , e $a(t) := \ddot{x}$ "aceleração".

Campos. Um *campo escalar* é uma função real $u : X \subset \mathbb{R}^n \to \mathbb{R}$ definida num domínio $X \subset \mathbb{R}^n$. Um *campo vectorial* é uma função $f : X \subset \mathbb{R}^n \to \mathbb{R}^k$, cujas coordenadas $f_1(x), f_2(x), \dots, f_k(x)$ são k campos escalares.

A derivada do campo diferenciável $f : X \subset \mathbb{R}^n \to \mathbb{R}^k$ no ponto $x \in X$ é a aplicação linear $df(x) : \mathbb{R}^n \to \mathbb{R}^k$ tal que

$$f(x+v) = f(x) + df(x) \cdot v + o(\|v\|)$$

para todos os vectores $v \in \mathbb{R}^n$ de norma $\|v\|$ suficientemente pequena, definida em coordenadas pela matriz Jacobiana $(\partial f_i / \partial x_j) \in \mathrm{Mat}_{k \times n}(\mathbb{R})$. Em particular, o *diferencial* do campo escalar $u : X \subset \mathbb{R}^n \to \mathbb{R}$ no ponto $x \in X$ é a forma linear $du(x) : \mathbb{R}^n \to \mathbb{R}$,

$$du(x) := \frac{\partial u}{\partial x_1}(x)\, dx_1 + \frac{\partial u}{\partial x_2}(x)\, dx_2 + \cdots + \frac{\partial u}{\partial x_n}(x)\, dx_n$$

(onde dx_k, o diferencial da função coordenada $x \mapsto x_k$, é a forma linear que envia o vector $v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ em $dx_k \cdot v := v_k$). A derivada do campo escalar diferenciável $u : X \subset \mathbb{R}^n \to \mathbb{R}$ na direção do vector $v \in \mathbb{R}^n$ (aplicado) no ponto $x \in X \subset \mathbb{R}^n$, é igual, pela regra da cadeia, a

$$(\pounds_v u)(x) := \left. \frac{d}{dt} u(x+vt) \right|_{t=0} = du(x) \cdot v \, .$$

O *gradiente* do campo escalar diferenciável $u : X \subset \mathbb{R}^n \to$ é o campo vectorial $\nabla u : X \subset \mathbb{R}^n \to \mathbb{R}^n$ tal que $du(x) \cdot v = \langle \nabla u(x), v \rangle$ para todo os vectores (tangentes) $v \in \mathbb{R}^n$ (aplicados no ponto $x \in X$).

2 Ordinary differential equations

2.1 Ordinary differential equations as problems

Differential equations are actually problems that we are asked to solve, to make predictions and to take decisions. Let us illustrate this with the following examples.

e.g. **Free particle.** The trajectory $t \mapsto q(t) \in \mathbb{R}^3$ of a free particle of mass m in an inertial frame is modeled by the Newton equation

$$\frac{d}{dt}(mv) = 0\,, \qquad \text{i.e., if } m \text{ is constant,} \qquad ma = 0\,,$$

where $v(t) := \dot{q}(t)$ denotes the *velocity* and $a(t) := \ddot{q}(t)$ denotes the *acceleration* of the particle. In particular, the *linear momentum* $p := mv$ is a constant of the motion (i.e. $\frac{d}{dt}p = 0$), in accordance with Galileo's principle of inertia or Newton's first law[1].

The solutions of Newton equation are the affine lines

$$q(t) = s + vt\,,$$

where $s, v \in \mathbb{R}^3$ are arbitrary vectors, the initial position and the initial velocity.

Thus, for example, the trajectory of a free particle starting at $q(0) = (3, 2, 1)$ with velocity $\dot{q}(0) = (1, 2, 3)$ is $q(t) = (3, 2, 1) + (1, 2, 3)t$.

e.g. **Solving the Newton equation of free fall near the Earth surface.** The Newton equation

$$m\ddot{x} \simeq -G\frac{mM_\oplus}{R_\oplus^2}$$

models the free fall of a particle of mass m near the Earth surface. Here $x(t)$ is the height of the particle at time t (measured from some reference height, e.g. the sea level), $G \simeq 6.67\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2}$ is the gravitational constant, $M_\oplus$ and $R_\oplus$ are the mass and radius of the Earth, respectively. We are assuming that $x \ll R_\oplus$. Since inertial and gravitational masses are (experimentally) equal, the mass m cancels out and we get the equation

$$\ddot{x} = g\,,$$

where $g := GM_\oplus/R_\oplus^2 \simeq 9.8\ \mathrm{m\,s}^{-2}$ is the the gravitational acceleration near the Earth surface, independent on the falling object!

A function with constant second derivative equal to $-g$ is $-gt^2/2$. But it is not the unique solution. We may add to it any function with zero second derivative, that is any constant s and any linear function vt. This means that also any

$$x(t) = s + vt - \frac{1}{2}gt^2$$

is a solution of our Newton equation, for any s and any v. The first arbitrary constant s is the initial position $x(0)$ (and this physically corresponds to the homogeneity of space: Newtonian physics is independent on the place where the laboratory is placed). The second arbitrary constant v is the initial velocity $\dot{x}(0)$ (and this physically corresponds to Galilean invariance: we cannot distinguish between two inertial laboratories moving at constant speed one from each other).

The moral is that the Newton equation alone does not have a "unique" solution. It has a whole "family of solutions", depending on two parameters s and v. On the other side, once we fix the initial position $x(0)$ and the initial velocity $\dot{x}(0)$, the solution turns out to be unique (we'll prove it soon! meanwhile, you may try to prove that the difference of any two solutions with the same initial conditions is constant and equal to zero). In other words, once known the initial "state" of the particle, i.e. its position and its velocity, the Newton equation uniquely determines the "future" and "past" history of the particle.

[1] "Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare" [Isaac Newton, *Philosophiae Naturalis Principia Mathematica*, 1687.]

ex: **Free fall near the Earth surface.** Consider the above Newton equation $\ddot{x} = -g$ as a model for the free fall, and solve the following problems.

- A stone is left falling from the top of the Pisa tower, about 56 meters high, with zero initial speed. Compute the height of the stone after 1 second, and determine the time needed for the stone to hit the ground.
- With which initial upward velocity a stone should be thrown in order to reach an height of 20 meters?
- With which initial upward velocity a stone should be thrown in order to fall back after 10 seconds?

The above situation is quite typical. Here is another example, actually a very important one!, where we can easily prove the uniqueness of the solution given an initial data.

e.g. **A differential equation for the exponential function.** Consider the first order ODE

$$\dot{x} = x$$

where $\dot{x}$ denotes the derivative of $x(t)$ w.r.t. the real variable t.

An obvious solution is $x(t) = 0$. Besides, computation shows that the exponential function e^t satisfies the equation, since you may have learnt in "Calculus" that the exponential is equal to its own derivative. Indeed, the (natural) exponential is defined by the power series

$$\exp(t) := \sum_{n \geqslant 0} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \dots$$

(remember that $0! = 1$), which converges uniformly in any bounded interval. You may check, deriving the power series term by term, that $\exp' = \exp$.

But we can multiply it by any constant b and still get a solution, hence any function $x(t) = be^t$ satisfies the above identity. If we set $t = 0$, we notice that b is the value of $x(0)$.

We claim that $x(t) = x_0 e^t$ is the "unique" solution of the differential equation $\dot{x} = x$ with initial data $x(0) = x_0$. Indeed, let $y(t)$ be any other solution. Since the exponential is never zero, we can divide by e^t and define the function $h(t) = y(t)e^{-t}$. Deriving we get

$$\dot{h} = (\dot{y} - y)\, e^{-t} \, .$$

But y solves the equation, hence the first derivative of h is everywhere zero. By the mean value theorem h is a constant function, and, since $y(0) = x_0$ too, its value at the origin is $h(0) = y(0)e^{-0} = x_0$. This implies that $y(t)$ is indeed equal to $x(t)$.

The problems posed by a ODE. We can formulate as follows the basic problems posed by a generic ODE of order k which can be solved for the biggest order derivative. Consider the ODE

$$x^{(k)} = F\left(t, x, \dot{x}, \ddot{x}, ..., x^{(k-1)}\right) ,$$

where F is some real valued function of $k+1$ real variables, $\dot{x} = dx/dt$, $\ddot{x} = d^2x/dt^2$, ... and $x^{(k)} = d^kx/dt^k$.

A *solution*, or integral curve, of the equation is a function $t \mapsto \varphi(t)$, defined in some interval I of the real line, which once inserted inside F gives the above identity

$$\varphi^{(k)}(t) = F\left(t, \varphi(t), \varphi'(t), \varphi''(t), ..., \varphi^{(k-1)}(t)\right)$$

for any t in the chosen interval I. Of course, we must ask that φ has so many derivatives as needed, hence that it is at least a k-times differentiable function.

As we have seen, a differential equation usually admits more than one solution (a one-dimensional family for the exponential growth, or a two-dimensional family for the Newton equation). Finding the *general solution* means writing formulas for the whole family, depending on a certain number of parameters. But this is in general a helpless task.

It turns out that in the good cases the number of free parameter is equal to the order of the ODE. Once we have fixed them, the solution is unique. Since most ODEs of physics are dynamical equations describing the time evolution of some observable/s x, it is natural to relate such free parameters to "initial conditions", and to pose the problem whether fixing initial conditions we are able to predict the future and the past of the system. This is called *initial value problem*, or also *Cauchy problem*.

For example, the Cauchy problem for the above generic k-th order ODE is: find a solution $t \mapsto \varphi(t)$ such that

$$\varphi(t_0) = x_0 \quad \varphi'(t_0) = x_1 \quad \varphi''(t_0) = x_2 \quad \dots \quad \varphi^{(k-1)}(t_0) = x_{k-1}$$

Above, t_0 is some point in the interval where the equation is defined, which you may think as an "initial time", and x_0, x_1, x_2, ..., x_{k-1} are the "initial values" of x and its first $k-1$ derivatives at time t_0.

Depending on the context, namely on the physical question you want to answer, other problems may arise: the free parameters may be related to different kinds of "boundary conditions". Here are some examples.

e.g. **Equilibrium profile of a star.** The gravitational equilibrium profile of a star is described, as a first approximation, by the *Lane-Emden equation*

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^p \,.$$

Here ξ is a reduced radius, $\theta(\xi)$ is proportional to the density at radius ξ, and p is a parameter which depends on the polytropic equation of state $P = K\rho^{1+1/p}$ of the "gas" forming the star (cold star, white dwarf, neutron star, ...). The physically relevant problem is to find the solution with initial conditions $\theta(0) = 1$ and $d\theta/d\xi(0) = 0$. The point where the first zero of the solution is attained is then interpreted to be the radius of the star.

e.g. **Sending a rocket to the Moon.** If you want to send a rocket of initial mass $m(0)$ to the Moon in time T, you must solve the suitable Newton equation

$$\frac{d}{dt}\left(m\frac{dr}{dt}\right) = -G\frac{mM_\oplus}{|r-R_\oplus|^3}(r-R_\oplus) - G\frac{mM_☾}{|r-R_☾|^3}(r-R_☾) + \dots \text{ friction and other perturbations}$$

with boundary conditions $r(0) =$"Cape Canaveral" and $r(T) =$"Moon".

ex: **Radioactive decay.** The rate of radioactive decay is observed to be proportional to the amount of radioactive substance present. This means that the amount $N(t)$ of radioactive substance present at time t satisfies the autonomous first order ODE

$$\dot{N} = -\beta N$$

for some positive "decay constant" β (its inverse, $1/\beta$, is the mean-life of each nucleus, the decay being modeled with an exponential random variable X for the life-time, with law $\mathbf{P}(X \leqslant t) = 1 - e^{-\beta t}$ for $t \geqslant 0$.), where $\dot{N}$ denotes first derivative of N w.r.t. time t.

- Find the general solution (keeping in mind that physically meaningful solutions must be positive).
- Find the formula for the solution of the Cauchy problem with initial data $N(0) = N_0$.
- The "half-time" of a radioactive substance is defined as the time needed for the amount of substance to become half of the initial amount, i.e. it is that time T such that $N(T)/N(0) = 1/2$. Find the relation between the half-time T and the decay constant β, and show that the half-time is well defined, i.e. it does not depend on the initial data $N(0)$.
- Radiocarbon ^{14}C (which decays as ${}^{14}_{6}C \to {}^{14}_{7}N + e^- + \overline{\nu}_e$) has a mean-life $1/\beta \simeq 8033$ years. Show how to date fossils assuming that the ratio of radiocarbon in a living being is fixed and known[2].
- Assume that Solar radiation produces ^{14}C in the atmosphere at a given fixed rate α (which is not the case, due to Solar variations). Then the amount of radiocarbon in our atmosphere follows the law
$$\dot{N} = -\beta N + \alpha$$
Show that $\overline{N} = \alpha/\beta$ is a equilibrium solution. Set $x(t) = N(t) - \overline{N}$, solve the differential equation for x, and show that $N(t) \to \overline{N}$ as $t \to \infty$, independently of the initial condition $N(0)$.

[2] J.R. Arnold and W.F. Libby, Age determinations by Radiocarbon Content: Checks with Samples of Known Ages, *Sciences* **110** (1949), 1127-1151.

ex: **Exponential growth.** The growth of a population in an unlimited environment is modeled with the first order ODE

$$\dot{N} = \lambda N$$

where $N(t)$ is the amount of specimen present at time t, λ is a positive "fertility constant", and N' denotes the derivative of N w.r.t. time t.

- Find the general solution as a function of the (positive) initial data $N(0) = N_0$.
- If a population of bacteria double in one hour, how much does it grow in two hours?
- If a predator kills the specimen at a fixed rate α, then the population grows as

  $$\dot{N} = \lambda N - \alpha$$

  Show that $\overline{N} = \alpha/\lambda$ is an equilibrium solution. Say what happens to the other solutions for large times.

ex: **Chandrasekhar's solutions of the Lane-Emden equation.** Show that[3]

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2, \quad \theta(\xi) = \frac{\sin \xi}{\xi} \quad \text{and} \quad \theta(\xi) = \frac{1}{\sqrt{1 + \frac{1}{3}\xi^2}}$$

are solutions of the Lane-Emden equation $\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^p$, for $p = 0$, 1 and 5, respectively.

2.2 Almost all ODEs have order one!

Here we claim that an ODE of arbitrary order $n > 1$ which can be solved for the n-th derivative is equivalent to a first order ODE, provided we allow the new unknown function to be vector valued. This means that, at least in principle, the study of a large class of ODEs can be reduced to the study of first order ODEs.

Reduction to order one. Indeed, consider the ODE

$$y^{(n)} = F\left(t, y, \dot{y}, \ddot{y}, ..., y^{(n-1)}\right), \tag{2.1}$$

of order $n > 1$, where, for example, $y(t)$ is a real valued function. Define a new variable $x = (x_1, x_2, ..., x_{n-1})$, taking values in some domain $X \subset \mathbb{R}^n$, as

$$x_1 := y, \qquad x_2 := \dot{y}, \qquad x_3 := \ddot{y}, \qquad ... \qquad x_n := y^{(n-1)}.$$

Then the above ODE (2.1) is equivalent to the "system" of n one-dimensional ODEs

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= x_3 \\ &\vdots \\ \dot{x_{n-1}} &= x_n \\ \dot{x_n} &= F\left(t, x_1, x_2, ..., x_{n-1}, x_n\right) \end{aligned} .$$

The space X where x takes its values is called *phase space* of the system. It is convenient to write the system in a compact form, namely as a first order ODE

$$\dot{x} = v\left(t, x\right),$$

for the unknown vector valued function x, where

$$v(t, x) = \left(x_1, x_2, ..., x_{k-1}, F\left(t, x_1, x_2, ..., x_n\right)\right)$$

is now a vector valued function of t and x, called *velocity field*. The initial value problem for the system is simply: find a solution with initial data $x(t_0) = x_0$, for some $x_0 \in X \subset \mathbb{R}^n$.

[3] Subrahmanyan Chandrasekhar, *Introduction to the theory of stellar structure*, Dover, New York 1942.

2.3 Conservative systems: from Newton to Lagrange and Hamilton

Newtonian mechanics. According to greeks, the "velocity" $\dot{q} = \frac{dq}{dt}$ of a planet, where $q = (q_1, q_2, q_3) \in \mathbb{R}^3$ is its position in the Euclidean space we think we live in and t is time, was determined by gods or whatever forced planets to move around their orbits. Then came Galileo, and showed that gods could at most determine the "acceleration" $\ddot{q} = \frac{d^2q}{dt^2}$, since the laws of physics should be written in the same way by observers in any reference system at uniform rectilinear motion with respect to the fixed stars. Finally came Newton, who decided that what gods determined was to be called "force", and discovered that the trajectories of planets, fulfilling Kepler's empirical three laws, were solutions of his famous second order differential equation

$$m\ddot{q} = F \,, \tag{2.2}$$

where m is the mass of the planet and the attractive force F between the planet and the Sun is proportional to the product of their masses and inverse proportional to the square of their distance.

Later, somebody noticed that many observed forces were "conservative", could be written as $F = -\nabla U$, for some real valued function $U(q)$ called *potential energy.* There follows that Newton equations for a particle of mass m in a conservative force field can be written as

$$m\ddot{q} = -\nabla U \,. \tag{2.3}$$

The quantity

$$T(q, \dot{q}) := \frac{1}{2} m \left|\dot{q}\right|^2 = \frac{1}{2} m \sum_{i=1}^{3} \dot{q}_i^2$$

is called *kinetic energy* of the particle, and the sum

$$E(q, \dot{q}) := \frac{1}{2} m \left|\dot{q}\right|^2 + U(q)$$

is called *(total) energy* of the system. The total energy is a conserved quantity, namely is constant along solutions of the Newton equation, since

$$\begin{aligned} \frac{d}{dt} E(q, \dot{q}) &= \langle \dot{q}, m\ddot{q} \rangle + \langle \dot{q}, \nabla U \rangle \\ &= \langle \dot{q}, m\ddot{q} + \nabla U \rangle = 0 \qquad \text{(by Newton equation (2.3))} \,. \end{aligned}$$

Variational principle, Euler-Lagrange equations. An alternative, and indeed useful, formulation of Newtonian mechanics is the one named after Lagrange. The *Lagrangian* of the (conservative) system is

$$L(q, \dot{q}) := T - U = \frac{1}{2} m \left|\dot{q}\right|^2 - U(q) \,, \tag{2.4}$$

A *trajectory* γ is a C^1 curve $q : [t_1, t_2] \to \mathbb{R}^3$, given explicitly by a continuously differentiable map $t \mapsto q(t)$, defined for $t \in [t_1, t_2]$. The *action* of the trajectory γ is

$$S(\gamma) := \int_{t_1}^{t_2} L(q(t), \dot{q}(t)) \, dt$$

So, the action is a function, or "functional" (to remember that its argument if a space of functions!), on the space of possible trajectories. It turns out that Nature choose those trajectories which minimize, at least locally, the action. More precisely, we first observe that

Proposition 2.1. *The critical points of the action are the solutions of the Euler-Lagrange equation*

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i} \qquad \textit{for } i = 1, 2, 3.$$

Proof. For simplicity we consider the one-dimensional case. Let $q(t) + \delta(t)$ be a small variation of the path $q(t)$, with the constraints $\delta(t_1) = \delta(t_2) = 0$. The first variation of the action is

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta(t) + \frac{\partial L}{\partial \dot{q}} \dot{\delta}(t) \right) dt \\ &= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right) \delta(t) \, dt \qquad \text{(integrating by parts)} \end{aligned}$$

and the last integral vanishes for all variations $\delta(t)$ iff the expression inside parenthesis vanishes for all times t inside the given interval. $\square$

Now, you may check that the Euler-Lagrange equations for the Lagrangian (2.4) are Newton equations (2.3).

e.g. **Free motion and straight lines.** Free motion in the Euclidean space $\mathbb{R}^3$ is a critical point of the action $S(\gamma) = \int T\, dt$, obtained integrating the kinetic energy $T = \frac{1}{2}|\dot{q}|^2$. Solutions of the Euler-Lagrange equation $\frac{d}{dt}\dot{q} = 0$ are, as expected, straight lines $q(t) = s + vt$. They are as well minimizers of the length $\ell(\gamma) := \int \sqrt{2T}\, dt = \int_{t_1}^{t_2} |\dot{q}|\, dt$ of the trajectory, that is *geodesics* of the Euclidean space.

One could change the metric, and consider $T = \frac{1}{2}\sum_{i,j} g_{ij}(x)\dot{x}_i\dot{x}_j$.

Hamiltonian mechanics. The vector $p = m\dot{q}$, with coordinates $p_i = \partial L/\partial \dot{q}_i$, is called "(linear) momentum". If there are no forces, the linear momentum is conserved, since Newton equations reduce to $\frac{d}{dt}p = 0$. The space $\mathbb{R}^3 \times \mathbb{R}^3$ with coordinates (q, p) is called "phase space" of the mechanical system. As a function of p, the kinetic energy is $K(p) = |p|^2/2m$ and its gradient is p/m, so that Hamilton could write Newton's second order differential equations as the system of first order differential equations

$$\dot{q} = \nabla K \quad \dot{p} = -\nabla U\,.$$

If we define the "Hamiltonian" of the system as

$$H(q, p) = K(p) + U(q)\ ,$$

which is nothing but the total energy written as a function of the phase space variables q and p, the above system takes the elegant form

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \text{for } i = 1, 2, 3$$

called "Hamilton's equations" of motion.

In the very same way one can describe the Newtonian motion of N point-like particles under conservative interactions, and the result are Hamilton's equations in a $6N$-dimensional phase space.

ex: **Hooke's law.** Write the Hamilton's equations corresponding to *Hooke's law*

$$m\ddot{q} = -kq$$

(call $\omega = \sqrt{k/m}$) and find the energy as a function of (q, p).

ex: **Mathematical pendulum.** Do the same for the *mathematical pendulum*

$$\ddot{\theta} = -\omega^2 \sin(\theta)$$

where $\omega = \sqrt{g/\ell}$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is an angle.

2.4 Vector fields and ODEs

As we have seen, a rather general class of ODEs (those that can be solved for the higher order derivative) is given by

$$\dot{x} = v(t, x)\,,$$

where the unknown function is $x(t)$, $\dot{x}$ denotes dx/dt, and v is some function of the two variables.

You can imagine that x is the "position" of a moving particle and t is "time". The problem posed is then that of determine the *trajectory* (or "time law") $t \mapsto x(t)$ of the moving particle once known its velocity $v(x, t)$ at every time and every position. With this interpretation in mind, we'll refer to v as a *velocity field*.

Phase space and extended phase space. The space X where x takes its values is called *phase space* of the system. For some time we'll only consider systems with one-dimensional phase space, hence in the next sections X will be the real line $\mathbb{R}$, or some interval of the real line (e.g. the allowed range of a temperature is a half-line $[0, \infty)$, the allowed range of a velocity of a massive particle in special relativity is a finite interval $(-c, c)$). You must keep in mind that interesting physics deals with more than one real valued observables at a time (as a set of positions and linear momenta each one in Euclidean 3-dimensional space), hence with phase spaces which are subsets of some Euclidean space $\mathbb{R}^n$.

The cartesian product of "time" (which we model as the real line) by the phase space, namely $\mathbb{R} \times X$ with coordinates (t, x), is called *extended phase space*. It is the space where graphs of solutions, also called *integral curves* of the equation, live.

Images of solutions in the phase space are called *phase curves*, or *orbits* of the system.

Directions field. Now, look at the equation. It says that the time derivative of $x(t)$ at time t is equal to $v(t, x)$. This means that the graph of a solution $t \mapsto \varphi(t)$, when seen in the extended phase space, must be a curve having slope $v(t, \varphi(t))$ in correspondence with the point $(t, \varphi(t))$.

You may think that, attached to any point (t, x) in the extended phase space, the equation prescribes a line $\ell_{(t,x)}$ with slope $v(t, x)$ with must be the tangent line to solutions passing through that point. This correspondence $(t, x) \mapsto \ell_{(t,x)}$ is called *lines* (or *directions*) *field* of the equation. Drawing the lines field may help in guessing how solutions behave.

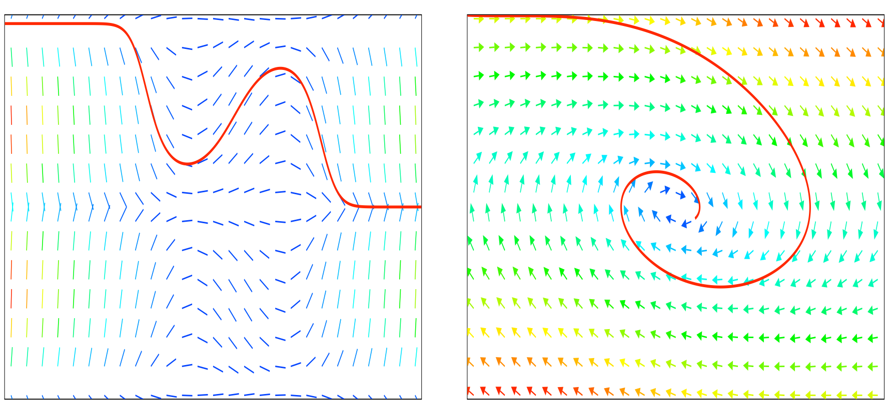
Initial value problem. Solving the initial value problem (or Cauchy problem) for $\dot{x} = v(t, x)$ with initial condition $x(t_0) = x_0$ means finding the/those trajectory/ies $t \mapsto \varphi(t)$ such that $\varphi(t_0) = x_0$. Their graphs in the extended phase space are curves that pass through the point (t_0, x_0).

Solutions need not be defined all over the time line $\mathbb{R}$, in general. We may content with *local solutions*, defined in some interval I containing the initial time t_0. If everything goes right, namely if we are able to prove an "existence and uniqueness" theorem, through every initial condition $(t_0, x_0) \in \mathbb{R} \times X$ will pass one and only one such curve. In this case, solutions may be defined over a maximal interval of times.

Vector fields. An important class of ODEs may be written as

$$\dot{x} = v(x)$$

for some velocity field $v(x)$ which does not depend on time. Theu are called *autonomous* since they correspond to physical (Newtonian) systems which are isolated, no external forces. Here x takes values in some domain $X \subset \mathbb{R}^n$, or in some manifold, Then v defines a *vector field* on X, a vector $v(x)$ attached to each point $x \in X$, which prescribe the velocity of the solution passing through the given point. Indeed, solutions are curves $t \mapsto x(t)$ such that $\frac{d}{dt}x(t) = v(x(t))$.



The lines field of $\dot{x} = \sin(x)(1 - t^2)$, and the vector field of the damped pendulum, $\dot{q} = p$, $\dot{p} = -\sin(q) - p/2$, together with one solution of each.

ex: Draw directions fields, trajectories and phase curves of the differential equations considered in the previous examples and exercises.

2.5 Simulations

Numerical integration The first observation is that a function $x(t)$ is a solution of the Cauchy problem for $\dot{x} = v(t, x)$ with initial condition $x(t_0) = x_0$ iff

$$x(t) = x_0 + \int_{t_0}^{t} v\,(s, x(s))\, ds\,.$$

Euler lines and method What the differential equation $\dot{x} = v(t, x)$, written in Leibnitz' notation

$$\frac{dx}{dt} = v(t, x)\,,$$

wants to say is that to a "small displacement" dt of time there corresponds a "small displacement" dx of the value of x which is proportional to dt by the factor $v(t, x)$. Namely, the very definition of derivative as a limit is suggesting that if δt is sufficiently small,

$$x(t + \delta t) = x(t) + v(t, x(t))\delta t + \text{something small}$$

where the "something small" is much less than δt. If you don't mind to disregard the "something small" above, you'll get a recursive procedure to find approximate solutions with a given initial data $x(t_0) = x_0$. Indeed, a good approximation of $x(t_0 + \delta t)$ is

$$x(t_0) + v(t, x(t_0))\delta t\,.$$

But then you can bet that a good approximation of $x(t_0 + \delta t + \delta t)$ is

$$x(t_0 + \delta t) + v(t_0 + \delta t, x(t_0 + \delta t))\delta t\,,$$

on so on. After n iterations you'll get a guess for the "true" value of the solution $x(t_0 + n\delta t)$.

This recipe is called *Euler method* (or *tangent line method*) to approximate/simulate solutions of first order ODEs. The polygonal lines it produces are called *Euler lines.*

What in Euler's times needed weeks of laborious handmade computations can nowadays be made in a few nano-seconds of CPU time with your personal computer. Fix an initial condition $t_0 =$ `time` and $x(t_0) =$ `x`, fix a small (depending on yours' machine possibilities) integration step `dt`, and define the velocity field `v(time, x)` you want to integrate. Then a `c++` cycle like

```
while (time < t)
  {
    x += v(time, x) * dt ;
    time += dt ;
  }
```

will "return" a value `x` which is an approximation of $x(t)$. What is maybe surprising is that the method actually converges (in some sense which we'll not discuss here, see Peano's existence theorem) to a true solution as the integration step `dt` goes to zero, provided some smoothness conditions on the velocity field v.

e.g. **The discrete exponential.** Use the Euler method to solve $\dot{x} = x$ with initial condition $x(0) = 1$. If ε denotes the step, you'll get the estimates

$$x(\varepsilon) \simeq 1 + \varepsilon\,, \quad x(2\varepsilon) \simeq (1 + \varepsilon) + (1 + \varepsilon)\varepsilon = (1 + \varepsilon)^2\,, \quad \dots, \quad x(n\varepsilon) \simeq (1 + \varepsilon)^n$$

so that

$$x(t) \simeq \left(1 + \frac{t}{n}\right)^n$$

where $n \simeq t/\varepsilon$ is the number of steps necessary to go from 0 to t. The limit for $\varepsilon \to 0$ is

$$\lim_{n\to\infty} \left(1 + \frac{t}{n}\right)^n\,,$$

the well known formula for $\exp(t)$.

Estimating the error. The error we commit in each step is of order $(\delta t)^2$. Since we need $n = (t-t_0)/\delta t$ steps to simulate the value of the solution after time $t-t_0$, we espect an overall error of order $\sim (t-t_0)\cdot \delta t$.

Homework. Write a code, in your favorite language, to integrate first order ODEs using Euler method. Better if you get a graphic answer in the extended phase space. Then bring it into the classroom and compare with exact solutions you'll learn to find.

Runge-Kutta 4 . Call $x_n = x(t_0 + (n+1)\delta t)$, $t_n = t_0 + n\varepsilon$, where $\varepsilon = \delta t$ the step. Then the method is

$$x_{n+1} = x_n + \frac{\varepsilon}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

where

$$k_1 = v(t_n, x_n) \quad k_2 = v\left(t_n + \varepsilon/2, x_n + k_1\varepsilon/2\right) \quad k_3 = v\left(t_n + \varepsilon/2, x_n + k_2\varepsilon/2\right) \quad k_4 = v(t_n + \varepsilon, x_n + \varepsilon k_3)$$

ex: Simulações com software proprietário. Existem software proprietários que permitem resolver analiticamente, quando possível, ou fazer simulações numéricas de equações diferenciais ordinarias e parciais. Por exemplo, a função `ode45` do MATLAB®, ou a função `NDSolve` do Mathematica®, calculam soluções aproximadas de EDOs $\dot{x} = v(t,x)$ utilizando variações do método de Runge-Kutta.

- Verifique se os PC do seu Departamento/da sua Universidade têm accesso a um dos software proprietários MATLAB® ou Mathematica®.
- Em caso afirmativo, aprenda a usar as funções `ode45` ou `NDSolve`.

3 First order ODEs on the line

Here we study first order ODEs with one-dimensional phase space of the form

$$\dot{x} = v(t, x)\,,$$

where x is a real valued function of time t, and $\dot{x}$ denotes dx/dt.

At the end of the story we'll see that there are two classes of equations that we can solve, at least provided that we are able to compute integrals: linear equations and separable equations.

3.1 Integrating simple ODEs

Simple ODEs. The simplest case occurs when the velocity field v does not depend on the phase space variable x, so the equation is

$$\dot{x} = v(t)\,,$$

where $v(t)$ is some given function of time. This just says that x is a primitive of v, and the fundamental theorem of calculus (i.e. Leibniz and/or Newton's discovery) tells us how to compute such a primitive: just integrate the function v from some initial time t_0 up to a final time t. Indeed, provided v is a continuous function, the derivative of $\int_{t_0}^{t} v(s)ds$ at the point t is $v(t)$. This explains the current use of the expression "integrate" a differential equation instead of "solving" a differential equation, as well as the meaning of Newton's quoted anagram.

Primitives are not unique, but are defined modulo an additive constant. This arbitrary constant can be matched with the initial condition, so that the solution of $\dot{x} = v(t)$ with initial condition $x(t_0) = x_0$ is

$$x(t) = x_0 + \int_{t_0}^{t} v(s)ds\,.$$

Here you may observe that this class of ODEs have "symmetries". The line field does not depend on x, hence slopes of solutions are the same along horizontal lines ($t =$ constant) in the extended phase space. There follows that any translate $\varphi(t) + c$ of a solution $\varphi(t)$ is still a solution. This is but a geometrical interpretation of the arbitrary constant in the primitive of v.

ex: **Training.** Integrate the following equations:

$$\dot{x} = 2\sin(t) \qquad \dot{x} = e^{-t} \qquad \dot{x} = t^2 - t\,.$$

ex: **Newtonian motion in a time dependent force field.** The one-dimensional motion of a particle of mass m subject to a force $F(t)$ is modeled by the Newton equation

$$m\ddot{x} = F(t)\,.$$

- Call $v = \dot{x}$ the velocity of the particle, and derive the first order ODE satisfied by the velocity v.
- Solve the equation for the velocity, given a force $F(t) = F_0 \sin(\gamma t)$ and an initial condition $v(0) = v_0$.
- Use the above solution $v(t)$ to find the trajectory $x(t)$ of the particle, given an initial position $x(0) = x_0$.

ex: **Rockets.** Se um foguetão (no espaço vazio, sem forças gravitacionais!) expulsa combustível a uma velocidade relativa constante $-V$ e a uma taxa constante $\dot{m} = -\alpha$, então a sua trajectória num referencial inercial (uni-dimensional) é modelada pela equação de Newton

$$\frac{d}{dt}(mv) = \alpha(V - v)\,, \qquad \text{ou seja}\,, \qquad \dot{m}v + m\dot{v} = \alpha(V - v)\,.$$

- Resolva a EDO $\dot{m} = -\alpha$ para a massa do foguetão, com massa inicial $m(0) = m_0$, e substitua o resultado na equação de Newton, obtendo

$$\dot{v} = \frac{\alpha V}{m_0 - \alpha t}$$

  (valida se $0 \leqslant t < m_0/\alpha$).
- Calcule a trajectória do foguetão com velocidade inicial $v(0) = v_0$ e posição inicial $q(0) = 0$, válida para tempos t inferiores ao tempo necessário para acabar o combustível.

3.2 Autonomous first order ODEs and flows

Autonomous ODEs. A first order ODE of the form

$$\dot{x} = v(x)\,,$$

where the velocity field v does not depend on time, is called *autonomous*. We already encountered examples in the models of radioactive decay and population growth. Most fundamental equations of physics (those describing closed systems, without external forces) can be written as autonomous first order ODEs, and this corresponds to time-invariance of physical laws.

Here you may notice symmetries again. The line field v of an autonomous equation is constant along vertical lines (x = constant) of the extended phase space. Hence any translate $\varphi(t+s)$ of a solution $\varphi(t)$ is still a solution. This is the manifestation of time-invariance of a law codified by an autonomous ODE. This also implies that there is no loss of generality in restricting to initial value problems with initial time $t_0 = 0$.

Equilibrium solutions. First, we observe that an autonomous equation may admit constant solutions. Indeed, if x_0 is a *singular point* of the vector field v, i.e. a point where $v(x_0) = 0$, then the constant function

$$x(t) = x_0$$

obviously solves the equation. Such solutions, which do not change with time, are called *equilibrium*, or *stationary*, solutions.

Solutions near non-singular points. Let x_0 be a *non-singular point* of the velocity field $v(x)$, i.e. a point x_0 where $v(x_0) \neq 0$. We want to solve $\dot{x} = v(x)$ with initial condition $x(t_0) = x_0$. First, rewrite the equation $dx/dt = v(x)$ formally as "$dx/v(x) = dt$" (multiply by dt and divide by $v(x)$, so that all x's are on the left and all t's are on the right). Instead of trying to make sense to this last expression (which is possible, of course, and here you can appreciate the beauty of Leibniz' notation dx/dt for derivatives!), observe that it is suggesting that $\int dx/v(x) = \int dt$. Now assume that the velocity field v is continuous and let $J = (x_-, x_+)$ be the maximal interval containing x_0 where v is different from zero. Integrating, from x_0 to $x \in J$ on the left and from t_0 to t on the right, we obtain a differentiable function $x \mapsto t(x)$ defined as

$$t(x) - t_0 = \int_{x_0}^{x} \frac{dy}{v(y)}$$

for any $x \in J$. Now, observe that the derivative dt/dx is equal to $1/v$. Since, by continuity, $1/v$ does not change its sign in J, our $t(x)$ is a strictly monotone continuously differentiable function. We can invoke the inverse function theorem and conclude that the function $t(x)$ is invertible. This prove that the above relation defines actually a continuously differentiable function $t \mapsto x(t)$ in some interval $I = t(J)$ of times around t_0. Finally, you may want to check that the function $t \mapsto x(t)$ solves the Cauchy problem: just compute the derivative (using the inverse function theorem),

$$\begin{aligned} \dot{x}(t) &= 1/\left(\frac{dt}{dx}(x(t))\right) \\ &= v(x)\,, \end{aligned}$$

and check the initial condition. Observe that the function $t(x) - t_0$ has then the interpretation of the "time needed to go from x_0 to x".

At the end of the story, if you are lucky enough and know how to invert the function $t(x)$, you'll get an explicit solution as

$$x(t) = F^{-1}\left(t - t_0 + F(x_0)\right)\,,$$

where F is any primitive of $1/v$. Close inspection of the above reasoning shows that the local solution you've found is indeed the unique one. Namely, we have the following

Proposition 3.1. (Existence and uniqueness theorem for autonomous ODEs near a non-singular point) *Let $v(x)$ be a continuous velocity field and let x_0 be a non-singular point of v. Then there exist one and only one solution of $\dot{x} = v(x)$ with initial condition $x(t_0) = x_0$ in some sufficiently small interval I around t_0. Moreover, the solution $x(t)$ is the inverse function of*

$$t(x) = t_0 + \int_{x_0}^{x} \frac{dy}{v(y)}\,,$$

defined in some small interval J around x_0.

Proof. Here we give the pedantic proof. Let J be as above. Define a function $H : \mathbb{R} \times J \to \mathbb{R}$ as

$$H(t,x) = t - t_0 - \int_{x_0}^{x} \frac{dy}{v(y)} \,.$$

If $t \mapsto \varphi(t)$ is a solution of the Cauchy problem, then computation shows that $\frac{d}{dt} H\left(t, \varphi(t)\right) = 0$ for any time t. There follows that H is constant along the solutions of the Cauchy problem. Since $H(t_0, x_0) = 0$, we conclude that the graph of any solution belongs to the level set $\Sigma = \{(t,x) \in \mathbb{R} \times J \text{ s.t. } H(t,x) = 0\}$. Now observe that H is continuously differentiable and that its differential $dH = dt + dx/v(x)$ is never zero. Actually, both partial derivatives $\partial H/\partial t$ and $\partial H/\partial x$ are always different from zero. Hence we can apply the implicit function theorem and conclude that the level set Σ is, in some neighborhood $I \times J$ of (t_0, x_0), the graph of a unique differentiable function $x \mapsto t(x)$, as well as the graph of a unique differentiable function $t \mapsto x(t)$, the inverse of t, which as we have already seen solves the Cauchy problem. □

On the failure of uniqueness near singular points. The interval $I = t(J)$ where the solution is defined need not be the entire real line: solutions may reach the boundary of J, i.e. one of the singular points $x_\pm$ of the velocity field, in finite time. Since singular points are themselves equilibrium solutions, this imply that solutions of the Cauchy problem at singular points may not be unique, under such mild conditions (continuity) for the velocity field. Later we'll see Picard's theorem, which prescribes stronger regularity conditions on v under which the Cauchy problem admits unique solutions for any initial condition in the extended phase space.

e.g. **Two solutions with the same initial condition!** Both the curves $x(t) = 0$ and $x(t) = t^3$ solve the equation

$$\dot{x} = 3x^{2/3}$$

with initial condition $x(0) = 0$. The problem here is that the velocity field $v(x) = 3x^{2/3}$, although continuous, is not differentiable and not even Lipschitz at the origin. You may notice that the solution starting, for example, at $x_0 = 1$ reaches (or better comes from) the singular point $x_- = 0$ in finite time, since

$$\begin{aligned} t(x_-) - t(x_0) &= \int_1^0 \frac{1}{3} y^{-2/3} dy \\ &= -1 \,. \end{aligned}$$

e.g. **Leibniz's tractrix.** Leibniz's *tractrix* is the solution of the differential equation

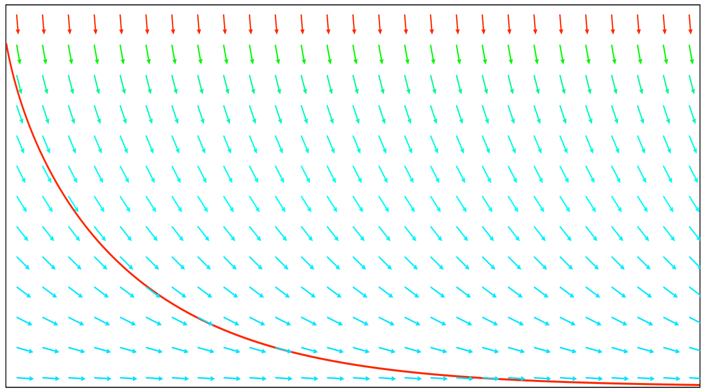
$$\frac{dy}{dx} = \frac{-y}{\sqrt{\ell^2 - y^2}}$$

for some initial condition $y(x_0) = y_0$ with $0 < y_0 < \ell$. It gives the trajectory of an object which is pulled on a plane by a rod of lenght ℓ when the free end of the rod moves along the x-axis. Separating variables we get

$$\int_{y_0}^{y} \frac{\sqrt{\ell^2 - z^2}}{z} dz = x_0 - x \,.$$

To compute the left integral, change variable $z = \ell \sin\theta$, and get, after some computations, the *tractrix* in implicit form

$$x - x_0 = \sqrt{\ell^2 - y_0^2} - \sqrt{\ell^2 - y^2} - \ell \log\left(y/y_0\right) + \ell \log\left(\frac{\ell + \sqrt{\ell^2 - y^2}}{\ell + \sqrt{\ell^2 - y_0^2}}\right) \,.$$



Leibniz's tractrix

The flow generated by an autonomous first order ODE. Assume that an autonomous first order ODE $\dot{x} = v(x)$ admits unique solutions $t \mapsto \varphi(t)$ starting at every point $\varphi(0) = x$ of the phase space X, and that all such solutions are defined for all times $t \in \mathbb{R}$ (such velocity fields are then called *complete*). Then we can define a family of maps $\Phi_t : X \to X$, depending on time $t \in \mathbb{R}$, as follows: the value of $\Phi_t(x)$ is equal to the value $\varphi(t)$ of the solution of the Cauchy problem with initial condition $\varphi(0) = x$. Clearly Φ_0 is the identity map, and

$$\Phi_t \circ \Phi_s = \Phi_{t+s}$$

for any $t, s \in \mathbb{R}$ (why?). Mathematicians say that such family of transformations $\{\Phi_t\}_{t\in\mathbb{R}}$ form a "group acting" on X, and call it the *flow* of the autonomous first order differential equation. Physically, $\Phi_t(x)$ is the state where the system will be after time t if it is observed in the state x at time 0. The group property above is essentially what physicists call "determinism": present uniquely determines past and future of the system.

Given the flow Ψ_t, we recover the velocity field as

$$v(x) = \frac{d}{dt}\Phi_t(x)\bigg|_{t=0} .$$

Hence, the flow may be seen as an alternative way to define a dynamics.

e.g. Gradient flows. Consider the problem to find minima of a real valued function $U(x)$ defined in an interval $X \subset \mathbb{R}$. Let $v(x) := -U'(x)$, and consider the flow Φ_t of the vector field $v(x)$. We compute, using the chain rule,

$$\frac{d}{dt}U(\Phi_t(x))\bigg|_{t=0} = U'(x)v(x) = -(U'(x))^2 .$$

It is clear that stationary solutions of $\dot{x} = v(x)$ are critical points of U. It follows from the above computation that the value of U decreases if we are not at a critical point. Therefore, if p is a strict minimum of U, an initial guess x_0 sufficiently near p have a trajectory $t \mapsto x(t)$ asymptotic to p.

ex: Training. Consider the following autonomous first order ODEs

$$\dot{x} = x - 1 \quad \dot{x} = (x-1)(x-2) \quad \dot{x} = \sqrt{x} \quad \dot{x} = 1 + \sqrt{|x|} \quad \dot{x} = \sqrt{1 - x^2} ,$$

where $\dot{x}$ denotes first derivative of x w.r.t. time t.

- Find, if any, equilibrium solutions.
- Draw the direction fields and conjecture the behavior of solutions.
- Integrate, find solutions, and draw some representative graphs of the solutions you have found.
- Find formulas for the solutions of the Cauchy problem with initial condition $x(a) = b$.

ex: Logistic equation. A more realistic model of population dynamics is the *logistic equation*

$$\dot{N} = \lambda N(1 - N/N_\infty)\,,$$

where the positive constant N_∞ is the asymptotic stationary population in a given limited environment. Observe that $\dot{N} \simeq \lambda N$, as in the exponential model, for N much smaller than N_∞, and that the rate of growth decreases to zero when N approaches N_∞ from below.

- Call $x = N/N_\infty$ the relative population, and show that the function $x(t)$ satisfies

  $$\dot{x} = \lambda x(1 - x)\,,$$

  a dimensionless form of the logistic equation.

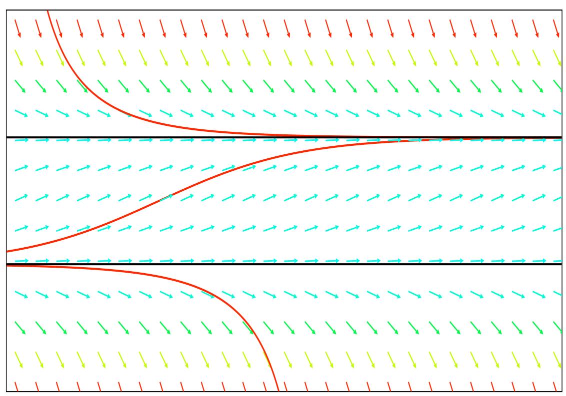
- Find the equilibrium solutions of the logistic equation.

- Show that the solution with initial condition $x(0) = x_0$, with $0 < x_0 < 1$, is

  $$x(t) = \frac{1}{1 + \left(\frac{1}{x_0} - 1\right) e^{-\lambda t}}\,,$$

- Find a formula for the solution of the Cauchy problem with initial condition $x(0) = x_0$, with $x_0 > 1$, and observe that the past history is not defined for any time t.

- Draw graphs of some solutions and say what happens to solutions for large times.



Equilibrium solutions, and three different solutions of the logistic equation.

ex: Super-exponential growth. Another model of population dynamics in a unlimited environment is

$$\dot{N} = \alpha N^2\,,$$

where α is a positive constant.

- Find, if any, equilibrium solutions.

- Write the equation as $dN/N^2 = \alpha dt$, integrate both sides of the equality and find the other solutions.

- Find a formula for the solution of the Cauchy problem with initial data $N(0) = N_0$, with $N_0 > 0$.

- Observe that the solutions you have just found are not defined for all times t: this model predict a catastrophe (infinite population) after a finite time!

ex: Draining a tank. Some liquid is contained in a tank which has section $S(h)$ in correspondence with height h. A hole of section s is opened at the base of the tank, and liquid start to drain. Torricelli's law says that the velocity of the dropping liquid at time t should be $v = -\sqrt{2gh}$, where $h(t)$ is the height of the liquid at time t (since the potential energy mgh gained by liquid particles falling from the liquid surface down to the hole will be transformed into a kinetic energy $mv^2/2$). Actually, due to some friction around the hole, the observed velocity is $-\gamma\sqrt{2gh}$ for some dimensionless coefficient $\gamma < 1$ (which is experimentally seen to be of order 0.6 for usual liquids in usual conditions). There follows that the flow of dropping liquid is $\gamma s\sqrt{2gh}$, hence the volume $V(t)$ of liquid in the tank at time t decreases as

$$\dot{V} = -\gamma s\sqrt{2gh}\,.$$

- Write the volume as $V(t) = \int_0^{h(t)} S(x)dx$ and show that $h(t)$ satisfies the autonomous first order ODE
$$S(h)\dot{h} = -\gamma s\sqrt{2gh}\,.$$
- Solve the equation for a cylindrical tank with constant section $S(h) = S$, and say what time does it take to drain a tank filled up to a height h_0.
- Solve the equation for a funnel, a conical tank having section $S(h) = s + kh$ for some positive k, and answer the same question as above.

ex: Real gravity and second cosmic velocity. The distance r of a particle of mass m from the center of the Earth satisfies the Newton equation

$$m\ddot{r} = -mg\frac{r_0^2}{r^2}\,,$$

where r_0 is the radius of the Earth (and, of course, $r > r_0$). Here we are considering the real gravitational force produced by the Earth, but we are disregarding the gravitational influence of the Sun and other celestial bodies.

- Find the potential $U(r)$ of the gravitational field and write the expression for the total energy of the system.
- Write the integral that represents the time needed to send a particle from the Earth surface r_0 up to a height $r - r_0$ from the Earth surface, given an initial energy $E > gr_0$.
- Find the minimum upward velocity necessary to escape from the Earth gravitational field, i.e. to reach an infinite distance.

e.g. Helmoltz's theorem, or "thermodynamics" of monocyclic motions. Consider a one-dimensional Newtonian system whose orbits are all closed, hence *monocyclic*. This is the case when the potential $U(x)$ is strictly convex (i.e. has positive second derivative) and grows to infinity for large displacements (i.e. $U(x) \to \infty$ for $x \to \pm\infty$). Changing the origin we can assume that the potential is everywhere positive. Moreover, we let the potential to depend smoothly on a parameter V. For any given value E of the total energy. the orbit takes place in a finite interval $[x_-, x_+]$ and has period

$$\sqrt{m/2}\int_{x_-}^{x_+}\frac{dx}{\sqrt{K}}$$

if m is the mass of the particle. Call P as "pressure", the time average of

$$-\frac{\partial U}{\partial V}\,.$$

T, as "temperature", the time average of the kinetic energy $K = E - U$, and define the "infinitesimal work" and "heath" as

$$dL = -PdV \quad dQ = dE + PdV\,.$$

Helmoltz's theorem says that then

$$dQ/T$$

is an exact differential. This means that there exists a function $S(E, V)$, such that $dS = (dE + PdV)/T$. Indeed, define

$$S(E, V) = 2 \log \int_{x_-}^{x_+} \sqrt{E - U(x)} dx\,.$$

Its differential is

$$dS = \frac{\int_{x_-}^{x_+} (dE - (\partial U/\partial V)dV)\, \frac{dx}{\sqrt{K}}}{\int_{x_-}^{x_+} K \frac{dx}{\sqrt{K}}}$$

ex: **Modeling.** Write down differential equations that model each of the following situations, then try to say as much as you can about the solutions and answer the questions posed at the end.

- The rate of change of the temperature of a cup of tea at time t is proportional to the difference between the air temperature, assumed constant, and the tea temperature at time t. Will the cup of tea reach the air temperature in finite time?
- The rate of growth of a population of mushrooms at time t is proportional to the square root of the population at time t. Could you infer the age of a colony of such mushroom from its actual population?
- The upward velocity of a rocket at time t is inverse proportional to the height reached at time t. Will the rocket reach an infinite height?
- The rate of growth of the mass of a cubic crystal at time t is proportional to the crystal's surface at time t. At what rate does the radius of the crystal grow?

3.3 One-dimensional conservative systems

One-dimensional Newtonian motion in a time independent force field. The one-dimensional motion of a particle of mass m subject to a force $F(x)$ that does not depend on time is described by the Newton equation

$$m\ddot{x} = -\frac{dU}{dx}(x)\,,$$

where the potential $U(x) = -\int F(x)dx$ is some primitive of the force. The total energy

$$E\,(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + U(x)$$

(which of course is defined up to an arbitrary additive constant) of the system is a constant of the motion, i.e. is constant along solutions of the Newton equation. In particular, once a value E of the energy is given (depending on the initial conditions), the motion takes place in the region where $U(x) \leqslant E$, since the kinetic energy $\frac{1}{2}m\dot{x}^2$ is non-negative. Conservation of energy allows to reduce the problem to the first order ODE

$$\dot{x}^2 = \frac{2}{m}\left(E - U(x)\right)\,,$$

which has the unpleasant feature to be quadratic in the velocity $\dot{x}$. Meanwhile, if we are interested in a one-way trajectory going from some x_0 to x, say with $x > x_0$, we may solve for $\dot{x}$ and find the first order autonomous ODE

$$\dot{x} = \sqrt{\frac{2}{m}\left(E - U(x)\right)}\,.$$

There follows that the time needed to go from x_0 to x is

$$t(x) = \int_{x_0}^{x} \frac{dy}{\sqrt{\frac{2}{m}\left(E - U(y)\right)}}\,.$$

The inverse function of the above $t(x)$ will give the trajectory $x(t)$ with initial position $x(0) = x_0$ and initial positive velocity $\dot{x}(0) = \sqrt{\frac{2}{m}\left(E - U(x_0)\right)}$, at least for sufficiently small times t.

ex: Harmonic oscillator /particle in a potential well. Consider the motion of a particle of mass m inside a potential well $U(x) = \frac{1}{2}k^2x^2$. The corresponding Newton equation is Hooke's law

$$m\ddot{x} = -k^2 x\,,$$

which can be rewritten in the more familiar form $\ddot{x} = -\omega^2 x$, where $\omega = k/\sqrt{m}$ is the "resonant frequency".

- Show that the energy
$$E(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k^2x^2$$
is a constant of the motion.

- Fixed a positive energy E, the motion takes place in the interval (x_-, x_+) with $x_\pm = \pm\sqrt{2E}/k$, and the velocity $\dot{x}$ satisfies the quadratic equation
$$\dot{x}^2 = \omega\sqrt{(|x_\pm|^2 - x^2)}\,.$$
Find the trajectory from x_- to any $x \leqslant x_+$.

- Compute the time needed to go from x_- to x_+, and show that it does not depend on E.

ex: Mathematical pendulum. Consider now the "real" pendulum, with Hamiltonian

$$H(\theta, p) = \frac{1}{2}p^2 - \cos(\theta)$$

- Show that the motion with energy E is given by
$$t = \int \frac{d\theta}{\sqrt{2(E - \cos(\theta))}}$$

- Define $x = \sqrt{\frac{2}{E+1}}\sin\theta/2$ and $k = \sqrt{\frac{E+1}{2}}$, and show that the motion reads
$$\dot{x} = \sqrt{(1 - x^2)(1 - k^2x^2)}$$
Deduce that time is given by the so called *Jacobi's elliptic integral of the first kind*
$$t = \int \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}$$
whose solution is "defined" as the *elliptic function* $x = \mathrm{sn}(t, k)$.

- Replace $t \mapsto it$ and see what happens.

3.4 Separable first order ODEs

Separable ODEs. A first order ODE $\dot{x} = v(t, x)$ is said *separable* when the velocity field v is a product of a function which only depends on t and another function which only depends on x. So it has the form

$$\dot{x} = g(t)f(x)$$

for some known functions f and g. We assume that both f and g are continuous functions on some intervals of the phase space and the real line, respectively. Observe that both simple ODEs like $\dot{x} = v(t)$ and autonomous ODEs like $\dot{x} = v(x)$ fall in this class.

If x_0 is a zero of f, hence a singular point of the vector field v, then $x(t) = x_0$ is an equilibrium solution.

The recipe to find other solutions is known as "separation of variables". Take a non-singular point x_0, that is a point where $f(x_0) \neq 0$. Choose a maximal interval J containing x_0 where f is different from zero, rewrite the equation formally as "$dx/f(x) = g(t)dt$", and then integrate from x_0 to $x \in J$ the r.h.s. and from t_0 to t the l.h.s. You'll get

$$\int_{x_0}^{x} \frac{dy}{f(y)} = \int_{t_0}^{t} g(s)ds\,.$$

As we did for autonomous equations, we can see that any continuously differentiable solution $t \mapsto x(t)$ of the equation passing through the non-singular point (t_0, x_0) must satisfy the above relation, as long as x is sufficiently near to x_0.

If F is a primitive of $1/f$ and G is a primitive of g, this gives the relation

$$F(x) - F(x_0) = G(t) - G(t_0)\,.$$

There follows that, if you are able to explicitly invert the function F, you'll get the explicit solution as

$$x(t) = F^{-1}\left(G(t) - G(t_0) + F(x_0)\right)\,.$$

e.g. Solve $\dot{x} = tx^3$.

An obvious solution is the equilibrium solution $x(t) = 0$. For a positive initial condition $x(t_0) = x_0 > 0$, rewrite the equation as $dx/x^3 = tdt$ and integrate

$$\int_{x_0}^{x} \frac{dy}{y^3} = \int_{t_0}^{t} sds$$

for $x > 0$. You'll find

$$1/x^2 - 1/x_0^2 = t^2 - t_0^2\,,$$

and, solving for x, the solution

$$x(t) = \frac{1}{\sqrt{t_0^2 + 1/x_0^2 - t^2}}\,.$$

defined for times t in the interval $|t| < \sqrt{t_0^2 + 1/x_0^2}$. In the same way you'll find solutions with negative initial condition $x_0 < 0$.

ex: Training. Solve (i.e. find all solutions of) the following separable ODEs

$$\dot{x} = tx^3 \qquad t\dot{x} + t = t^2 \qquad \dot{x} = t^3/x^2 \qquad x\dot{x} = e^{x+3t^2}t$$

$$\dot{x} = \frac{t-1}{x^2} \qquad \frac{x-1}{t}\dot{x} + \frac{x-x^2}{t^2} = 0 \qquad \frac{dy}{dx} = -\frac{x}{y}$$

$$\left(t^2+1\right)\dot{x} = 2tx \qquad \dot{x} = t\left(x^2 - x\right) \qquad \dot{x} = e^{t-x}$$

defined in appropriate domains of the extended phase space.

ex: Exponential growth in variable environment. The growth of a population in a variable environment may be modeled by

$$\dot{N} = \lambda(t)N$$

where $\lambda(t)$ is a variable growth rate.

- Write the solution $N(t)$ with $N(t_0) = N_0$ as a function of $\lambda(s)$ for $t_0 \leqslant s \leqslant t$.
- Solve the problem with $\lambda(t) = \lambda_0 \sin(\omega t)$.

ex: Inseguimento. Una lepre scappa con moto rettilineo uniforme in un piano. Una volpe la vede e la insegue puntando sempre nella direzione della lepre (e viaggiando a velocità costante). Determinare la traiettoria della volpe.

3.5 Linear first order ODEs

Linear first order ODEs. A *first order linear differential equation* is a differential equation which can be written in the "canonical form"

$$\dot{x} + p(t)x = q(t)\,, \tag{3.1}$$

where p and q are (known) functions of the real variable t in some interval I of the real line, called *coefficients*. We assume that both p and q are continuous functions, and we look for solutions $t \mapsto x(t)$ defined on I. Eventually we will want to solve the Cauchy problem with some initial condition $x(t_0) = x_0$.

The equation

$$\dot{y} + p(t)y = 0 \tag{3.2}$$

is said the *homogeneous* equation associated with the general, hence *non-homogeneous*, equation (3.1) above.

Observations about linearity. The word "homogeneous" is due to the fact that any constant multiple $\lambda y(t)$ of a solution y of the homogeneous equation (3.2) is again a solution. Also, any linear combination (with real coefficients) $ay_1(t)+by_2(t)$ of solutions $y_1(t)$ and $y_2(t)$ of the homogeneous equation (3.2) is still a solution of the homogeneous equation. This means that the space of solutions of the homogeneous equation is a linear space $\mathcal{H}$.

Also interesting is that the difference $y(t) = x_1(t) - x_2(t)$ of any two solutions $x_1(t)$ and $x_2(t)$ of the non-homogeneous equation (3.1) is a solution of the associated homogeneous equation (3.2), hence belongs to the linear space $\mathcal{H}$. Therefore, the space of solutions of the non-homogeneous equation (3.1) is an affine space $x + \mathcal{H}$, where $x(t)$ is any (particular) solution of (3.1).

All this suggests a strategy to solve the existence and uniqueness problem for both equations. We start with

Solving the homogeneous equation. A trivial solution of the homogeneous equation is the equilibrium solution $x(t) = 0$.

Now we look for others. Assume for the moment that the solution $x(t)$ is positive on I. The equation is equivalent to $\dot{x}/x = -p(t)$. The chain rule says that $\dot{x}/x$ is the derivative of $\log x$, hence $\log x$ must be a primitive of $-p(t)$. There follows that

$$\log x(t) - \log x_0 = -\int_{t_0}^{t} p(s)ds\,,$$

hence

$$x(t) = x_0 e^{-\int_{t_0}^{t} p(s)ds}\,.$$

You may want to check that the above formula solves the Cauchy problem with initial condition $x(t_0) = x_0$.

Of course, this makes sense provided that the function $p(t)$ is continuous. Now we claim that the above formula (which includes the equilibrium solution if $x_0 = 0$ as well as the negative solutions if $x_0 < 0$) gives the unique solution of the Cauchy problem.

Proposition 3.2. (Existence and uniqueness theorem for homogeneous first order linear ODEs) *Let p be a continuous function on some interval I. Then the unique solution of the hmogeneous equation $\dot{x} + p(t)x = 0$ with initial condition $x(t_0) = x_0$ is given by*

$$x(t) = x_0 e^{-\int_{t_0}^{t} p(s)ds}\,.$$

Proof. Let $y(t)$ be a second solution of the Cauchy problem above, and define

$$h(t) = y(t) e^{\int_{t_0}^{t} p(s)ds}\,.$$

Its value for t_0 is x_0. Its derivative is

$$\dot{h}(t) = e^{\int_{t_0}^{t} p(s)ds}\left(\dot{y}(t) + p(t)y(t)\right)\,.$$

Since y is supposed to solve the equation, the derivative of h is equal to zero for any t in the chosen interval, and the mean value theorem says that then $h(t)$ is constant and equal to x_0. There follows that $y(t)$ is indeed equal to our solution $x(t)$. □

e.g. Solve $t\dot{x} - 2x = 0$ for $t \in (0, \infty)$ with initial condition $x(t_0) = x_0$.

If $x_0 = 0$, the solution is the equilibrium solution $x(t) = 0$. If $x_0 > 0$, write the equation as $dx/x = 2dt/t$, integrate

$$\int_{x_0}^{x} dy/y = \int_{t_0}^{t} 2ds/s\,,$$

for positive x, obtain

$$\log x - \log x_0 = \log(t^2) - \log\left(t_0^2\right)\,,$$

and solve it for x, the solution being

$$x(t) = \left(x_0/t_0^2\right) t^2\,.$$

Finally observe that this formula gives the solutions for any initial condition x_0.

Back to the non-homogeneous equation. To solve the non-homogeneous equation

$$\dot{x} + p(t)x = q(t)\,,$$

we use the following trick, a first and elementary instance of a much more general method named "variation of parameters" (or, sometimes, with the oxymoron "variation of constants"). We already know that any function proportional to $e^{-\int_a^t p(s)ds}$ solves the homogeneous equation. We look for a solution of the non-homogeneous equation having the form

$$x(t) = \lambda e^{-\int_{t_0}^t p(s)ds}\,,$$

but, instead of treating the parameter λ as a constant, we allow it to depend on t. Putting our guess into the non-homogeneous equation, we get

$$\frac{d}{dt}\left(\lambda(t)e^{-\int_{t_0}^t p(s)ds}\right) + p(t)\lambda(t)e^{-\int_{t_0}^t p(s)ds} = q(t)\,.$$

Computing the derivative, we get

$$\dot{\lambda}(t)e^{-\int_{t_0}^t p(s)ds} - \cancel{p(t)\lambda(t)e^{-\int_{t_0}^t p(s)ds}} + \cancel{p(t)\lambda(t)e^{-\int_{t_0}^t p(s)ds}} = q(t)\,,$$

the two terms containing $p(t)$ do cancel, and we are left with

$$\dot{\lambda}(t)e^{-\int_{t_0}^t p(s)ds} = q(t)\,.$$

This can be solved for $\dot{\lambda}$, and integration gives

$$\lambda(t) = x_0 + \int_{t_0}^t e^{\int_{t_0}^s p(u)du} q(s)ds$$

for some constant x_0 equal to the value of $\lambda(t_0)$. Finally, we get a solution

$$x(t) = \lambda(t)e^{-\int_{t_0}^t p(s)ds}\,,$$

and you may check that it has initial value $x(t_0) = x_0$. Since the difference of any two solutions of the general equation is a solution of the associated homogeneous equation, and since (as follows from the uniqueness theorem above) the only solution of the homogeneous equation with initial condition $x(t_0) = 0$ is the zero solution, we just proved the following

Proposition 3.3. (Existence and uniqueness theorem for first order linear ODEs) *Let p and q be continuous functions in some interval I. Then the unique solution of the linear differential equation $\dot{x} + p(t)x = q(t)$ with initial condition $x(t_0) = x_0$ for $t_0 \in I$ is given by*

$$x(t) = e^{-\int_{t_0}^t p(u)du}\left(x_0 + \int_{t_0}^t e^{\int_{t_0}^s p(u)du} q(s)ds\right)\,.$$

Suggestion. Perhaps, instead of fixing the unpleasant formula in the above theorem, you could simply remember the strategy used to derive it: find one non-trivial solution $y(t)$ of the associated homogeneous equation (which is separable!), and then make the conjecture $x(t) = \lambda(t)y(t)$ for some other unknown function $\lambda(t)$. You'll get a simple differential equation for λ, and integration gives you the solution.

e.g. Solve $t\dot{x} - 2x = t$ for $t \in (0,\infty)$ with initial condition $x(t_0) = x_0$.

You already know that the solution of the associated homogeneous equation $ty' - 2y = 0$ with initial condition $y(t_0) = 1$ is $y(t) = t^2/t_0^2$. Make the conjecture $x(t) = \lambda(t)t^2/t_0^2$, insert your guess into the non-homogeneous equation, and get

$$\dot{\lambda} = t_0^2/t^2\,.$$

Integrate and find

$$\lambda(t) - \lambda(t_0) = t_0 - t_0^2/t\,,$$

and, since $\lambda(t_0) = x(t_0)$, finally find the solution

$$x(t) = \frac{x_0 + t_0}{t_0^2}t^2 - t\,.$$

ex: **Training.**

- Find the general solutions of the following linear first order ODEs

$$2\dot{x} - 6x = e^{2t} \qquad \dot{x} + 2x = t \qquad \dot{x} + x/t^2 = 1/t^2 \qquad \dot{x} + tx = t^2$$

  for t in appropriate intervals of the real line.

- Solve the following initial value problems:

$$\begin{aligned} 2\dot{x} - 3x = e^{2t} &\qquad \text{for } t \in (-\infty, \infty) \quad \text{with } x(0) = 1 \\ \dot{x} + x = e^{3t} &\qquad \text{for } t \in (-\infty, \infty) \quad \text{with } x(1) = 2 \\ t\dot{x} - x = t^3 &\qquad \text{for } t \in (0, \infty) \quad \text{with } x(0) = 1 \\ \dot{x} + tx = t^3 &\qquad \text{for } t \in (0, \infty) \quad \text{with } x(0) = 0 \\ dr/d\theta + r\tan\theta = \cos\theta &\qquad \text{for } \theta \in (-\pi/2, \pi/2) \quad \text{with } r(0) = 1 \end{aligned}$$

ex: **Free fall with friction.** A more realistic model of free fall of a point-like particle near the Earth surface must take into account the air resistance. The latter is assumed to be a force $F_{\text{friction}} = -k\dot{r}$, proportional and opposed to the velocity of the particle, for some positive constant k (observe that, in absence of other forces, the velocity $v = \dot{r}$ would satisfy the equation $\dot{v} = -\frac{1}{\tau}v$, hence decay exponentially with characteristic time $\tau = m/k$). The resulting Newton equation for the free fall is

$$m\ddot{r} = -k\dot{r} - mg\,.$$

This can be thought as a first order ODE for the velocity $v = \dot{r}$ of the particle, namely

$$m\dot{v} = -kv - mg\,.$$

- Find equilibrium solutions for the velocity v, and give a physical interpretation.
- Solve the Cauchy problem with initial velocity $v(0) = 0$.
- Show that the velocity goes to a definite value as time tends to infinity, independently on its initial value.
- Use the above solution to find the trajectory $r(t)$, given an initial position $r(0) = s$.

ex: **Kirchoff's law for a LR circuit.** The electric current $I(t)$ flowing in an electric circuit with resistance R and inductance L driven by a tension $V(t)$ satisfies the first order ODE

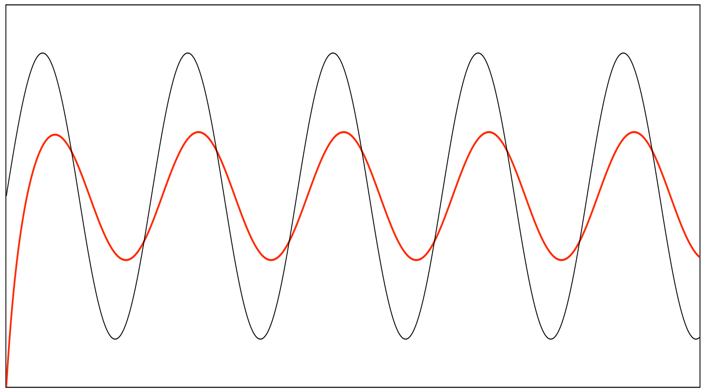
$$L\dot{I}(t) + RI(t) = V(t)\,.$$

- Write the general solution as a function of the tension $V(t)$ and the initial current $I(0)$.
- Solve the equation for a constant tension $V(t) = V_0$. Draw graphs of some solutions for different values of $I(0)$ and say what happen for large times.
- Solve the equation for a circuit driven by an alternate tension $V(t) = V_0\sin(\omega t)$. Show that the solution with initial current $I(0) = 0$ has the form

$$I(t) = \frac{V_0}{\sqrt{R^2 + \omega^2 L^2}}\sin(\omega t - \varphi) + \frac{E\omega L}{R^2 + \omega^2 L^2}e^{-Rt/L}\,,$$

  where φ is a phase (or delay) which depends on ω, L and R.

- Compare with the free fall with friction, and give "mechanical" interpretations of the resistance R and inductance L of an electric circuit.



Response (red) to an alternate tension (black).

ex: **Newton's law of cooling.** The temperature $T(t)$ at time t of a body in contact with a thermostat, maintained at temperature $M(t)$, is assumed to follow the *Newton's law of cooling*

$$\dot{T} = -k\left(T - M(t)\right) ,$$

for some positive constant k.

- Find the formula which solves the Cauchy problem for $T(a) = b$ as a function of $M(t)$.
- Solve the Cauchy problem when the thermostat (supposed much bigger than the body) is maintained at constant temperature $M(t) = M$ and discuss the solutions (observe that $T(t) = M$ is an equilibrium solution, and consider the substitution $x(t) = T(t) - M$).
- Solve the Cauchy problem when the thermostat has temperature $T(t) = T_0 \sin(\omega t)$.
- A cup of coffee, initially at the temperature of 100^oC is left in a room at constant temperature 20^oC. Observing that the coffee reaches a temperature of 60^oC in 15 minutes, compute the value of k for coffee and the time needed for the coffee to reach a temperature of 40^oC..

ex: **Bernoulli equations.** A first order ODE of the form

$$\dot{x} + p(t)x = q(t)x^n ,$$

where p and q are continuous functions in some interval I and $n \neq 0$ or 1 (otherwise it's just a linear ODE!), is called *Bernoulli equation*.

- Show that $x(t) = 0$ is a solution.
- Let $k = 1 - n$. Show that $x(t)$ is a positive solution of the Bernoulli equation with initial condition $x(t_0)^k = x_0$ iff $y = x^k$ is a solution of the linear ODE

  $$\dot{y} + kp(t)y = kq(t)$$

  with initial condition $y(t_0) = x_0$.
- Conjecture and prove an analogous result for negative solutions of the Bernoulli equation, given some appropriate conditions on the exponent n (there is no way to give a useful meaning to an expression like $(-3)^{\sqrt{2}}$!).
- Solve the following initial value problems for Bernoulli equations:

$$\dot{x} - x/t = t\sqrt{x} \quad \text{for } t \in (0, \infty) \quad \text{com } x(0) = 1$$

$$\dot{x} + x = x^2\left(\cos t - \sin t\right) \quad \text{for } t \in (-\infty, \infty) \quad \text{com } x(1) = 2$$

$$t\dot{x} + e^{t^2}x = x^2 \log t \quad \text{for } t \in (0, \infty) \quad \text{com } x(3) = 0$$

4 Existence, uniqueness and stability results for ODEs

Many (not to say almost all!) interesting ODEs of physics are not linear and don't belong to any class of easily integrable differential equations.

For example, analytical solutions of the Lane-Emden equation are known for only a few values of the parameter p. Solutions of the three-body problem in celestial mechanics are known only for a very few symmetrical initial conditions.

How do we attach the problem? The first thing to do is to prove existence and uniqueness theorems that tell you that the equation you wrote does make sense. This done, you may solve "numerically" the equation, that is find approximate solutions and hope that they are not too different from the real ones. Later on, following an idea of Henri Poincaré, you may try to guess the qualitative of solutions without solving the equation.

Here we sketch just the first two steps.

4.1 Existence and uniqueness theorems

Here we consider a generic first order ODE of the form

$$\dot{x} = v(t, x)$$

where the velocity field v is a (continuous) function defined in some extended phase space $\mathbb{R} \times X$. Here X may be some interval of the real line as well as an open subset of some Euclidean $\mathbb{R}^n$. Since we'll prove a local result, everything we'll say will be valid when X is any differentiable manifold.

The problem we address is the existence and uniqueness of solutions of the initial value problem. A *local solution* passing through the point $(t_0, x_0) \in \mathbb{R} \times X$ is a solution $t \mapsto \varphi(t)$, defined in some neighborhood I of t_0, such that $\varphi(t_0) = x_0$. Eventually, we'll be interested also in the possibility of extending such local solutions to larger intervals of times.

Existence. The basic existence theorem is due to Giuseppe Peano[4].

Theorem 4.1. (Peano existence theorem) *Let $v(t, x)$ be a continuous velocity field in some domain D of the extended phase space $\mathbb{R} \times X$. Then for any point $(t_0 x_0) \in D$ passes at least one integral curve of the differential equation $\dot{x} = v(x, t)$.*

Idea of the proof. Natural guesses for the solutions are Euler lines starting through (x_0, t_0). If we restrict to a sufficiently small neighborhood of (t_0, x_0), we can assume that the velocity field is bounded, say $|v(t, x)| \leqslant K$, and that all such Euler lines lies in the "papillon" made of two triangles touching at (t_0, x_0) with slopes $\pm 1/K$. Construct a family of Euler lines, graphs of $\varphi_n(t)$, such that the maximal step ε_n of the n-th line goes to 0 as $n \to \infty$. One easily sees that the family (φ_n) is bounded and equicontinuous. By the Ascoli-Arzelá theorem it admits a (uniformly) convergent subsequence. Finally, we claim that the sublimit $\varphi_{n_i} \to \varphi$ solves the differential equation. □

Both existence and uniqueness may fail. The Hamilton-Jacobi equation

$$(\dot{x})^2 - xt + 1 = 0$$

cannot have solutions satisfying the initial condition $x(0) = 0$, for otherwise we would have a negative "kinetic energy" $(\dot{x})^2 = -1$ at that point!

Some regularity of the functions involved in a differential equation is also needed to ensure the uniqueness of solutions. For example, both curves $t \mapsto 0$ and $t \mapsto t^3$ solve the equation

$$\dot{x} = 3x^{2/3}$$

with initial condition $x(0) = 0$. The problem here is that the velocity field $v(t, x) = 3x^{2/3}$, although continuous, is not differentiable and not even Lipschitz at the origin. Indeed, a sufficient condition for uniqueness is

[4] G. Peano, Sull'integrabilità delle equazioni differenziali del primo ordine, *Atti Accad. Sci. Torino* **21** (1886), 677-685. G. Peano, Demonstration de l'intégrabilité des équations différentielles ordinaires, *Mathematische Annalen* **37** (1890) 182-228.

The Lipschitz condition. A velocity field $v(t,x)$, defined in a domain $I \times D$ of the extended phase space $\mathbb{R} \times \mathbb{R}^n$, is *locally Lipschitz* w.r.t. to the variable x if for any $(t_0, x_0) \in I \times D$ there is a neighborhood $J \times U \ni (t_0, x_0)$ and a constant $L \geqslant 0$ such that

$$\|v(t,x) - v(t,y)\| \leqslant L \cdot \|x - y\| \quad \forall\, (t,x),(t,y) \in J \times U$$

If $v(t,x)$ has continuous derivative w.r.t. x, i.e. if the Jacobian

$$D_x v(t,x) = \left(\frac{\partial v_i}{\partial x_j}(t,x) \right)$$

exists and is continuous, then $v(t,x)$ is locally Lipschitz in any compact convex domain $I \times K \subset \mathbb{R} \times \mathbb{R}^n$.

Uniqueness. Here is the uniqueness theorem, due to Émile Picard and Ernst Lindelöf [5].

Theorem 4.2. (Picard-Lindelöf uniqueness theorem) *Let $v(t,x)$ be a continuous velocity field defined in some domain D of the extended phase space $\mathbb{R} \times X$. If v is locally Lipschitz (for example continuously differentiable) w.r.t. the second variable x, then there exist one and only one local solution of $\dot{x} = v(t,x)$ passing through any point $(t_0, x_0) \in D$.*

Geometrically, the uniqueness theorem says that through any point (t_0, x_0) of the domain D there pass one and only one solution. Hence solutions, considered as curves in the extended phase space, cannot intersect each other.

In a domain where Picard's theorem applies, if two local solutions agree in a common interval of times then they are indeed restrictions of a unique solution defined in the union of the respective domains. There follows that solutions are always extendible to a maximum domain. Such solutions are called *maximal solutions*.

Strategy of the proof of the Picard's theorem. The first observation is that a function $\varphi(t)$ is a solution of the Cauchy problem for $\dot{x} = v(t,x)$ with initial condition $\varphi(t_0) = x_0$ iff

$$\varphi(t) = x_0 + \int_{t_0}^{t} v\left(s, \varphi(s)\right) ds$$

Now, we notice that the above identity is equivalent to the statement that φ is a fixed point of the so called *Picard's map* $\phi \mapsto \mathcal{P}\phi$, sending a function $t \mapsto \phi(t)$ into the function

$$\left(\mathcal{P}\phi\right)(t) = x_0 + \int_{t_0}^{t} v\left(s, \phi(s)\right) ds$$

At this point, one must chose cleverly the domain of the Picard's map, which is the space of functions where we think a solution should be. It will be a certain space $\mathcal{C}$ of continuous functions, defined in an appropriate neighborhood I of t_0, equipped with a norm that makes it a complete metric space (hence a Banach space). The Lipschitz condition, together with continuity, satisfied by the velocity field will imply that if the interval I is sufficiently small then the Picard's map $\mathcal{P} : \mathcal{C} \to \mathcal{C}$ is a contraction. The contraction principle (a.k.a. Banach fixed point theorem) finally guarantees the existence and uniqueness of the fixed point of $\mathcal{P}$ in $\mathcal{C}$.

Picard's iterations. The contraction principle actually says that the fixed point, i.e. the solution we are looking for, is the limit of any sequence ϕ, $\mathcal{P}\phi$, ..., $\mathcal{P}^n\phi$, ... of iterates of the Picard map starting with any initial guess $\phi \in \mathcal{C}$. In other words, the existence part of the theorem is "constructive", it gives us a procedure to find out the solution, or at least a sequence of functions which approximate the solution.

e.g. **Simple ODEs.** Consider the simple ODE $\dot{x} = v(t)$ with initial condition $x(t_0) = x_0$. Picard's recipe, starting from the initial guess $\phi(t) = x_0$ gives, already at the first step,

$$\left(\mathcal{P}\phi\right)(t) = x_0 + \int_{t_0}^{t} v(s) ds$$

which is the solution we know.

[5] M. E. Lindelöf, Sur l'application de la méthode des approximations successives aux équations différentielles ordinaires du premier ordre, *Comptes rendus hebdomadaires des séances de l'Académie des sciences* **114** (1894), 454-457. Digitized version online via http://gallica.bnf.fr/ark:/12148/bpt6k3074

e.g. The exponential. Suppose you want to solve $\dot{x} = x$ with initial condition $x(0) = 1$. You start with the guess $\phi(t) = 1$, and then compute

$$(\mathcal{P}\phi)(t) = 1 + t \quad \left(\mathcal{P}^2\phi\right)(t) = 1 + t + \frac{1}{2}t^2 \quad \dots \quad (\mathcal{P}^n\phi)(t) = 1 + t + \frac{1}{2}t^2 + \dots + \frac{1}{n!}t^n$$

Hence the sequence converges (uniformly on bounded intervals) to the Taylor series of the exponential function

$$(\mathcal{P}^n\phi)(t) \to 1 + t + \frac{1}{2}t^2 + \dots + \frac{1}{n!}t^n + \dots = e^t\,,$$

which is the solution we already knew.

Details of the proof of the Picard's theorem. Choose a sufficiently small rectangular neighborhood

$$I \times B = [t_0 - \varepsilon, t_0 + \varepsilon] \times \overline{B}_\delta(x_0)$$

around (t_0, x_0), where $B = \overline{B}_\delta(x_0)$ denotes the closed ball with center x_0 and radius δ in X. There follows from continuity of v that there exists K such that

$$|v(t,x)| \leqslant K$$

for any $(t,x) \in I \times B$. There follows from the local Lipschitz condition for v that there exists M such that

$$|v(t,x) - v(t,y)| \leqslant M|x-y|$$

for any $t \in I$ and any $x, y \in B$. Now restrict, if needed, the (radius of the) interval I in such a way to get both the inequalities $K\varepsilon \leqslant \delta$ and $M\varepsilon < 1$. Let $\mathcal{C}$ be the space of continuous functions $t \mapsto \phi(t)$ sending I into B. Equipped with the sup norm

$$\|\phi - \varphi\| = \sup_{t \in I} |\phi(t) - \varphi(t)|$$

this is a complete space. One verifies that the Picard's map sends $\mathcal{C}$ into $\mathcal{C}$, since

$$|(\mathcal{P}\phi)(t) - x_0| \leqslant \int_{t_0}^{t} |v(s, \phi(s))|\, ds \leqslant K\varepsilon \leqslant \delta.$$

Finally, given two functions $\phi, \varphi \in \mathcal{C}$, one sees that

$$|(\mathcal{P}\phi)(t) - (\mathcal{P}\varphi)(t)| \leqslant \int_{t_0}^{t} |v(s, \phi(s)) - v(s, \varphi(s))|\, ds \leqslant M\varepsilon \sup_{t \in I} |\phi(t) - \varphi(t)|$$

hence $\|\mathcal{P}\phi - \mathcal{P}\varphi\| < M\varepsilon \|\phi - \varphi\|$. Since $M\varepsilon < 1$, this proves that the Picard's map is a contraction and the fixed point theorem allows to conclude. □

We may not be able to solve them! Last but not least, we must keep in mind that we are not able to solve all equations. Actually, although we may prove the existence and the uniqueness for large classes of equations, we are simply not able to explicitly integrate the really interesting differential equations...

Ultimately we must recur to numerical methods to find approximate solutions and to qualitative analysis

4.2 Dependence on initial data and parameters

Consider a family of ODEs

$$\dot{x} = v(t, x, \lambda)$$

where λ is a real parameter. We want to understand how solutions depend on the parameter λ. A basic instrument is the

Theorem 4.3. (Grönwall's lemma[6]) *Let $\phi(t)$ and $\psi(t)$ be two non-negative real valued functions defined in the interval $[a, b]$, and assume that*

$$\phi(t) \leqslant K + \int_a^t \psi(s)\phi(s)\, ds$$

for any $a \leqslant t \leqslant b$ and some constant $K \geqslant 0$. Then

$$\phi(t) \leqslant K e^{\int_a^t \psi(s)\, ds}\,.$$

[6]T. H. Gronwall, Note on the derivative with respect to a parameter of the solutions of a system of differential equations, *Ann. of Math* **20** (1919), 292-296.

Proof. First, assume $K > 0$. Define

$$\Phi(t) := K + \int_a^t \psi(s)\phi(s)\, ds$$

and observe that $\Phi(a) = K > 0$, hence $\Phi(t) > 0$ for all $a \leqslant t \leqslant b$. The logarithmic derivative is

$$\frac{d}{dt} \log \Phi(t) = \frac{\psi(t)\phi(t)}{\Phi(t)} \leqslant \psi(t)$$

where we used the hypothesis $\phi(t) \leqslant \Phi(t)$. Integrating the inequality we get, for $a \leqslant t \leqslant b$,

$$\log \Phi(t) \leqslant \Phi(a) + \int_a^t \psi(s)\, ds\,.$$

Exponentiation gives the result, since

$$\phi(t) \leqslant \Phi(t) \leqslant K \cdot e^{\int_a^t \psi(s)\, ds}\,.$$

The case $K = 0$ follows taking the limit of the above inequalities along a sequence of $K_n > 0$ decreasing to zero. □

Continuous dependence on initial conditions. If $x(t)$ and $y(t)$ are two solutions of

$$\dot{x} = v(t, x)$$

then

$$x(t) - y(t) = x(0) - y(0) + \int_{t_0}^t (v(s, x(s)) - v(s, y(s)))\, ds$$

If $L(s)$ denotes the Lipschitz constant of $v(s, \cdot)$, we get

$$\|x(t) - y(t)\| \leqslant \|x(0) - y(0)\| + \int_{t_0}^t L(s)\|x(s) - y(s)\| ds$$

The Gronwall's lemma gives the control

$$\|x(t) - y(t)\| \leqslant e^{\int_{t_0}^t L(s) ds} \|x(0) - y(0)\|$$

Observe that the above control also gives an alternative proof of uniqueness of solutions given a Lipschitz condition on the vector field.

Theorem 4.4. (Smooth dependence on parameters) *Let $v(t, x, \lambda)$ be a family of vector fields defined on some domain $D \subset \mathbb{R} \times X$ of the extended phase space, depending on a parameter $\lambda \in \Lambda \subset \mathbb{R}$. If v is of class C^k with $k \geqslant 1$, then in some neighborhood of any $(t_0, x_0, \lambda_0) \in D \times \Lambda$ the local solutions of*

$$\dot{x} = v(t, x, \lambda)$$

with initial condition $x(t_0) = x_0$ are differentiable (indeed C^k) functions of (t, x, λ).

Proof. see [?] □

Warning. Continuous dependence does not exclude sensitive dependence on both initial conditions and parameters, even in the linear case! For example, the distance between solutions of $\dot{x} = \mu x$ with different $x(0)$ and/or μ may diverge for large time ...

4.3 Autonomous systems and flows

The flow generated by an autonomous first order ODE. Let $v(x)$ be a vector field defined on some domain $X \subset \mathbb{R}^n$, or on a manifold. Assume that an autonomous first order ODE

$$\dot{x} = v(x)$$

admits unique solutions $t \mapsto \varphi(t)$ starting at every point $\varphi(0) = x \in X$, and that all such solutions are defined for all times $t \in \mathbb{R}$. Such vector fields are then called *complete*. Then we can define a family of maps $\Phi_t : X \to X$, depending on time $t \in \mathbb{R}$, as follows: $\Phi_t(x)$ is equal to the value $\varphi(t)$ of the solution of $\dot{x} = v(x)$ with initial condition $\varphi(0) = x$. Clearly Φ_0 is the identity map, and

$$\Phi_t \circ \Phi_s = \Phi_{t+s}$$

for any $t, s \in \mathbb{R}$ (why?). Mathematicians say that such a family of transformations $\{\Phi_t\}_{t\in\mathbb{R}}$ form a "group acting" on X, and call it the *flow* of the vector field v. Physically, $\Phi_t(x)$ is the state where the system will be after time t if it is observed in the state x at time 0. The group property above is essentially what physicists call "determinism": present uniquely determines past and future of the system.

Given the flow we recover the vector field as

$$v(x) = \frac{d}{dt}\Phi_t(x)\bigg|_{t=0} .$$

Hence, the flow may be seen as an alternative way to define the dynamics.

Observables.

Derivative along the flow. The *Lie derivative* of a differentiable function $f : X \to \mathbb{R}$ along the vector field v is the function $\pounds_v f$ defined by

$$(\pounds_v f)(x) := \frac{d}{dt} f(\Phi_t(x))\bigg|_{t=0} .$$

Therefore, an observable is constant of the motion, or *invariant*, if $\pounds_v f = 0$, i.e. if $f(\Phi_t x) = f(x)$ for any time $t \in \mathbb{R}$.

Reparametrizations.

Lyapunov functions.

e.g. **Gradient flow.** Suppose you have a "potential" $U : X \subset \mathbb{R}^n \to \mathbb{R}$, and you want to find its minima. You could make a first guess x_0, and then follow the flow of $v = -\nabla U$, namely solve $\dot{x} = -\nabla U(x)$ with $x(0) = x_0$. Indeed, equilibrium points are critical points of U, and minus the gradient of U is the direction where U decreases the fastest, and computation gives $\pounds_v U = ...$

All this seems quite trivial, but it becomes extremely powerful in the infinite dimensional case of the Laplace equation. The gradient flow of the Dirichlet integral $\int_M |\nabla u|^2 dx$ in a Riemannian manifold, minimized by harmonic functions (those such that $\Delta u = 0$), is the heat equation $u_t = \Delta u$.

Rectifiability

Proposition 4.1. (Flow box theorem.) *A differentiable vector field near a nonsingular point is rectifiable, i.e. diffeomorphic to a constant vector field. Explicitly, a nonsigular point $p \in X$ of the vector field $v(x)$ admits a neighborhood U with local coordinates $(x_1, x_2, \ldots, x_n)$ such that the vector field is the constant vector field $v = (1, 0, \ldots, 0)$.*

5 Some geometrical considerations on ODEs

5.1 Homogeneity and other dimensional considerations

You may have noticed that the only non-linear first order ODEs which we are able to integrate by "quadratures" (i.e. computing integrals) are the separable ones. Moreover, simple equations like $\dot{x} = v(t)$ and autonomous equations like $\dot{x} = v(x)$ have symmetries, since their directions field is constant along vertical and horizontal lines in the extended phase space, respectively. Here we show that also other less trivial symmetries implies separability, hence integrability by quadratures.

Homotheties and homogeneous functions. *Homotheties* (with center 0) of the Euclidean space $\mathbb{R}^n$ are the transformations $x \mapsto e^{\lambda} x$, for $\lambda \in \mathbb{R}$ (thus $e^{\lambda} \in \mathbb{R}_+$). Observe that homotheties form a group, parametrized by the multiplicative group $\exp(\mathbb{R}) \simeq \mathbb{R}_+$.

Let $f : D \to \mathbb{R}$ be a function defined in a domain $D \subset \mathbb{R}^n \backslash \{0\}$ which is invariant under homotheties (i.e. if D contains a point p different from the origin then it contains the whole semirect $\mathbb{R}_+ p = \{e^t p \, , \, t \in \mathbb{R}\}$). A function $f : D \to \mathbb{R}$ is called *(positively) homogeneous of degree* k if

$$f(e^{\lambda} x) = e^{\lambda k} f(x) \tag{5.1}$$

for any point $x \in D$ and for all $\lambda \in \mathbb{R}$. In particular, a function $f : D \to \mathbb{R}$ is called *(positively) homogeneous (of degree* 0*)* if it is invariant under homotheties, i.e. of

$$f\left(e^{\lambda} x\right) = f(x) \, ,$$

for any point $x \in D$ and any $\lambda \in \mathbb{R}$. In other words, an homogeneous function f is constant on rays coming out from the origin, hence it is defined by its values on the unit sphere $S^{n-1} := \{x \in \mathbb{R}^n \text{ s.t. } |x| = 1\}$.

According to *Euler's homogeneous function theorem*, a differentiable function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is (positively) homogeneous of degree k if and only if

$$\langle x, \nabla f \rangle = k \, f(x) \tag{5.2}$$

ex: Differentiate both sides of (5.1) at $\lambda = 0$ and prove Euler's formula (5.2). Integrate (5.2) along rays from the origin, and check that it implies that the function f is positively homogeneous.

Homogeneous first order ODEs. A first order ODE

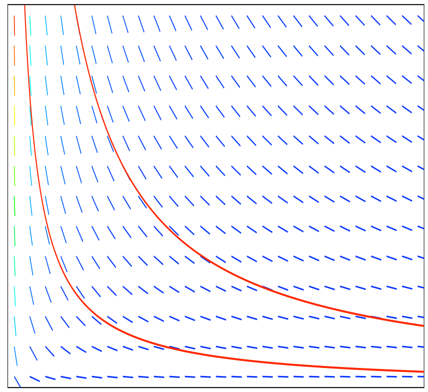
$$\dot{x} = v(t, x)$$

is said *homogeneous* if the velocity field v is an homogeneous function of t and x, thought as points $(t, x) \in \mathbb{R}^2$.

A first observation is that homotheties $(t, x) \mapsto (e^{\lambda} t, e^{\lambda} x)$ send integral curves into integral curves. Indeed, if $t \mapsto \varphi(t)$ is a solution of $\dot{x} = v(t, x)$, then also $t \mapsto \phi(t) := e^{\lambda} \varphi\left(e^{-\lambda} t\right)$, for $\lambda \in \mathbb{R}$, is a solution, because

$$\begin{aligned}
\dot{\phi}(t) &= \frac{d}{dt} e^{\lambda} \varphi\left(e^{-\lambda} t\right) = \dot{\varphi}\left(e^{-\lambda} t\right) \\
&= v\left(e^{-\lambda} t, \varphi\left(e^{-\lambda} t\right)\right) = v\left(t, e^{\lambda} \varphi\left(e^{-\lambda} t\right)\right) \qquad \text{(using homogeneity)} \\
&= v\left(t, \phi(t)\right) \, .
\end{aligned}$$

This means that if we could find just one solution, we'll have indeed a whole family of homothetic solutions, depending on a parameter $\lambda \in \mathbb{R}$.



Directions field and two homothetic solutions of the homogeneous equation $\dot{y} = -y/x$.

Homogeneity amounts to say that v is actually a function of the ratio x/t, since

$$v(t, x) = v(1, x/t)$$

as long as $t > 0$ (and the case $t < 0$ can be treated in a similar way). This suggests that we may try and see what the equation implies for the new unknown function $y(t) = x(t)/t$. Indeed, in a domain where $t > 0$, the guess $x(t) = ty(t)$ gives

$$y + t\dot{y} = v(1, y)\,,$$

hence the separable ODE

$$\dot{y} = (v(1, y) - y)\,/t$$

for y. Once you have $y(t)$, you'll have the solution $x(t) = t\,y(t)$ as well as the whole family of solutions $t \mapsto t\,y\left(e^{-\lambda}t\right)$ for $\lambda \in \mathbb{R}$.

e.g. **Example.** Solve

$$\dot{x} = \frac{x^2 + t^2}{tx}$$

in the first quadrant, i.e. for $t > 0$ and $x > 0$.

Make the conjecture $x(t) = ty(t)$, compute $\dot{x} = y + ty'$, and substitute this expression into the equation. This gives

$$t\dot{y} + y = \frac{y^2 + 1}{y}\,,$$

hence the separable equation

$$y\dot{y} = 1/t\,.$$

A positive solution is $y(t) = \sqrt{2\log t}$, defined for times $t > 1$. Back to the original variable, you find the solution $x(t) = t\sqrt{2\log t}$. Finally, use homotheties to find the whole family of solutions

$$x(t) = t\sqrt{2\log(e^{-\lambda}t)}\,,$$

depending on the parameter $\lambda \in \mathbb{R}$, defined for times $t > e^{\lambda}$, in the first quadrant.

ex: **Training.** Solve the following homogeneous ODEs

$$\dot{x} = -t/x \qquad \dot{x} = \frac{x - t}{x + t} \qquad \dot{x} = 1 + x/t \qquad v^3 + \left(u^3 - uv^2\right)\frac{dv}{du} = 0$$

$$\dot{x} = x/t \qquad \dot{x} = 2\frac{t}{x}e^{x/t} + \frac{x}{t} \qquad \frac{dy}{dx} = y/x + \sin(y/x)$$

in appropriate domains of the extended phase space, and draw some integral curves.

ex: **Exercise.** Show that an homogeneous ODE

$$\frac{dy}{dx} = v(x,y)$$

in the x-y plane can be transformed into a separable ODE in polar coordinates ρ-θ, i.e. setting $\rho = \sqrt{x^2+y^2}$ and $\theta = \arctan(y/x)$.

Quasi-homogeneous functions and ODEs. Here we consider the *quasi-homotheties* $g_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ defined

$$(x,y) \mapsto \left(e^{\lambda\alpha}x, e^{\lambda\beta}y\right)$$

for some (possibly different) *weights* α and β.

A function $f(x,y)$ is said *quasi-homogeneous* of *degree* k and *weights* α and β if

$$f\left(e^{\lambda\alpha}x, e^{\lambda\beta}y\right) = e^{\lambda k} f(x,y)$$

for any $\lambda \in \mathbb{R}$.

Example (Kepler 3rd law)

5.2 Newton equation in homogeneous potentials

see [Ar85, Ar89, LL78]

5.3 Exact differentials and conservative fields

Here we describe a sequence of observations which, once followed in the reversed order, will suggest a method to deal with some first order ODEs. More interesting are the physical and the geometrical interpretations.

Level sets of smooth functions on the plane. Let $U : D \to \mathbb{R}$ be a twice continuously differentiable real valued function defined in some domain $D \subset \mathbb{R}^2$. *Level sets* of U are the sets

$$\Sigma_c = \left\{(x,y) \in \mathbb{R}^2 \text{ s.t. } U(x,y) = c\right\} ,$$

for $c \in \mathbb{R}$. If c is a *regular value* of U, i.e. if $\nabla U \neq 0$ at the points of Σ_c, then the level set Σ_c is a differentiable curve.

If (x_0, y_0) is a point in Σ_c where $\partial U/\partial y(x_0,y_0) \neq 0$, the implicit function theorem tells us that Σ_c is locally (in a neighborhood of (x_0,y_0)) the graph of a differentiable function $x \mapsto y(x)$. Such a function satisfies the constraint $U(x, y(x)) = c$, hence deriving w.r.t. x we get

$$\frac{d}{dx} U(x, y(x)) = 0 , \qquad \text{so that} \qquad \frac{\partial U}{\partial x}(x,y(x)) + \frac{\partial U}{\partial y}(x,y(x))\,\frac{dy}{dx} = 0 .$$

If we define the functions $p := \partial U/\partial x$ and $q := \partial U/\partial y$, this means that the function $x \mapsto y(x)$ is a local solution of the differential equation

$$p(x,y) + q(x,y)\frac{dy}{dx} = 0 \tag{5.3}$$

which satisfies the initial condition $y(x_0) = y_0$.

The very same reasoning, near a point (which could be the same) where $\partial U/\partial x \neq 0$, gives a local solution $y \mapsto x(y)$ of the differential equation

$$p(x,y)\frac{dx}{dy} + q(x,y) = 0 . \tag{5.4}$$

For this reason, we'd better write both the differential equations (5.3) and (5.4) in the suggestive single form

$$p(x,y)\,dx + q(x,y)\,dy = 0 \tag{5.5}$$

(called "Pfaffian equation" by mathematicians), to be solved for dy/dx or for dx/dy, and say that the curve Σ_c contains the graphs of the local solutions of (5.5).

e.g. . Let $U(x,y) = x^2 + y^2$. Level sets Σ_c are the family of circles $x^2 + y^2 = c$, and they are regular as long as $c > 0$, since $\nabla U = (2x, 2y)$. Near the point $\left(1/\sqrt{2}, 1/\sqrt{2}\right)$, the curve Σ_1 is the graph of both functions $y(x) = \sqrt{1-x^2}$ and $x(y) = \sqrt{1-y^2}$, which are local solutions of the differential equation

$$x dx + y dy = 0 .$$

Exact differentials and exact differential equations. Now, we reverse the reasoning, and give the following definitions.

Let $p(x,y)$ and $q(x,y)$ be continuous functions defined in some domain D of the plane. A "differential" $pdx + qdy$, or a differential equation $pdx + qdy = 0$, is called *exact* (in the domain D) if there exists a continuously differentiable function $U : D \to \mathbb{R}$ such that $dU = pdx + qdy$, namely

$$\frac{\partial U}{\partial x} = p \quad \text{and} \quad \frac{\partial U}{\partial y} = q .$$

Such U, if it exists, is called *primitive* of the exact differential $pdx + qdy$. Observe that this is equivalent to the statement that $\nabla U = (p, q)$, hence level curves of U are orthogonal to the vectors (p, q) at every points. The solutions of the exact differential equation are then implicitly given by

$$U(x,y) = c ,$$

where the constants c are the regular values of U. Level sets of U are called *integral curves* of the equation. Explicit local solutions, whose existence is guaranteed by the implicit function theorem, may be obtained solving the equation $U(x,y) = c$ for x or y, depending the case.

To decide whether a differential or a differential equation is exact or not is an easy task, thanks to the following

Theorem 5.1. (Euler-Poincaré) *Let p and q be continuously differentiable functions in some "simply connected" (for example a "convex" domain, or simply a "rectangle") domain $D \subset \mathbb{R}^2$. Then the differential $pdx + qdy$ is exact iff*

$$\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x} .$$

The condition is certainly necessary, since it amounts to exchanging the order in the mixed second partial derivatives of U. So, we must prove the reverse implication, namely the existence of U. We start with the simple case, considered by Euler, in which the domain D is a rectangle. In this case we'll get a very simple recipe to compute U that, modulo integrations, explicitly solves the problem.

Before, we give one of the possible physical interpretations.

Conservative fields. Instead of looking at the differential $p(x,y)dx + q(x,y)dy$, look at the vector field

$$F(x,y) = -\left(p(x,y), q(x,y)\right) .$$

which you may think as a "force field". Finding U such that $dU = pdx + qdy$ amounts to finding a "potential" for the force field, a function U such that $F = -\nabla U$. The level sets of U are then equipotential lines, and physicists know that they must be curves orthogonal to the force field F at every point.

Since potentials are defined modulo constant additive terms, you may fix any value of $U(x_0, y_0)$. To find the value of U at a generic point (x,y) you choose a path γ from (x_0, y_0) to (x,y) and compute the "work"

$$W_\gamma = \int_\gamma F d\ell$$

done by the force field along the path. Force fields which are gradients are called "conservative" by physicists, meaning that the work done to displace a particle from one position to another position does not depend on the chosen path, but only on the initial and final points. This work must be equal to $U(x_0, y_0) - U(x,y)$. Now, the work does not depend on the chosen path exactly when the force field is "irrotational", namely $\mathrm{rot} F := \partial p/\partial y - \partial q/\partial x = 0$, provided there are "no holes" in the domain where paths are chosen.

Euler's constructive proof. Fix any starting point $(x_0, y_0) \in D$ and set $U(x_0, y_0) = 0$. The recipe to get other values of U at points $(x, y) \in D$ is

$$U(x,y) = \int_{x_0}^{x} p(s, y_0) ds + \int_{y_0}^{y} q(x, s) ds .$$

If D is a rectangle the above integrations are well defined. Now we show that U is a primitive of $pdx + qdy$. The identity

$$\frac{\partial U}{\partial y}(x,y) = q(x,y)$$

is obvious. Computing the other partial derivative, using differentiation under the integral and the hypothesis, we see that

$$\begin{aligned} \frac{\partial U}{\partial x}(x,y) &= p(x,y_0) + \int_{y_0}^{y} \frac{\partial q}{\partial x}(x,s) ds \\ &= p(x,y_0) + \int_{y_0}^{y} \frac{\partial p}{\partial y}(x,s) ds \\ &= p(x,y_0) + p(x,y) - p(x,y_0) , \end{aligned}$$

so that also $\partial U/\partial x(x,y) = p(x,y)$. □

Modern/abstract proof. Observe that the above recipe amounts to define U integrating the differential $pdx + qdy$ (or the vector field $F = (p, q)$ if you want to think about forces and work) along a particular path going from (x_0, y_0) to (x, y) (go from (x_0, y_0) to (x, y_0) along a horizontal segment and then from (x, y_0) to (x, y) along a vertical segment). But we could as well define $U(x, y)$ as being the integral of $pdx + qdy$ along any piecewise smooth path γ' going from (x_0, y_0) to (x, y) and lying inside D, namely

$$U(x,y) = \int_{\gamma'} (pdx + qdy)$$

and still get $\partial U/\partial x = p$ and $\partial U/\partial y = q$. The only problem here is that the value of $U(x, y)$ may depend on the chosen path. To see that this is not the case, take any other path γ'' going from (x_0, y_0) to (x, y). If you follow γ' in the right direction and then γ'' in the reverse direction, you'll get a closed path γ going from (x_0, y_0) back to (x_0, y_0) passing through (x, y). If the domain D is simply connected, γ is the boundary $\partial\Omega$ of some domain Ω contained inside D (you may think that this is a definition of "simply connectedness"). But then the Stokes-Green theorem says that

$$\begin{aligned} \int_{\gamma'} (pdx + qdy) - \int_{\gamma''} (pdx + qdy) &= \textstyle\int_{\gamma} (pdx + qdy) \\ &= \textstyle\int_{\partial\Omega} (pdx + qdy) \\ &= \textstyle\int_{\Omega} \left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} \right) dxdy , \end{aligned}$$

and the last integral is equal to zero due to the hypothesis of the theorem. □

e.g. Decide if the differential $(2xy + 1)\, dx + x^2 dy$ is exact, find a primitive and solve the differential equation $(2xy + 1)\, dx + x^2 dy = 0$.

Compute partial derivatives and check that $\partial(2xy + 1)/\partial y = \partial \left(x^2\right)/\partial x$. Then set $U(0, 0) = 0$ and integrate

$$U(x,y) = \int_0^x (2s0 + 1)\, ds + \int_0^y x^2 ds .$$

A primitive is $U(x, y) = x^2 y + x$. The curves $x^2 y + x = c$ are the integral curves of the differential equation $(2xy + 1)\, dx + x^2 dy = 0$.

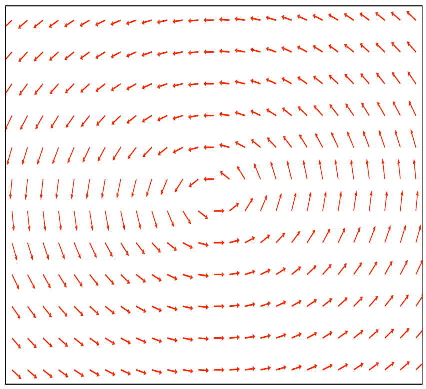
e.g. Magnetic field in the plane. Consider the "magnetic field"

$$F(x,y) := \left(\frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \right)$$

generated by an electric current flowing along the z-axis of the 3-dimensional space. This field is defined in the domain $\mathbb{R}^2 \setminus \{0\}$, which is not simply connected. The rotational is zero, but if you integrate the field along the unit circle, a closed curve around the "hole" at the origin, you get the value

$$\begin{aligned}\oint_{x^2+y^2=1} \left(\frac{-y}{\sqrt{x^2+y^2}} dx + \frac{x}{\sqrt{x^2+y^2}} dy \right) &= \int_0^{2\pi} d\theta \\ &= 2\pi\end{aligned}$$

for the work done. This implies that it is impossible to find a globally defined potential. On the other end, if you restrict the domain of the field to a half-space as $\{(x, y) \in \mathbb{R}^2 \text{ s.t. } x > 0\}$, you do get single-valued potentials as $\arctan(y/x)$.



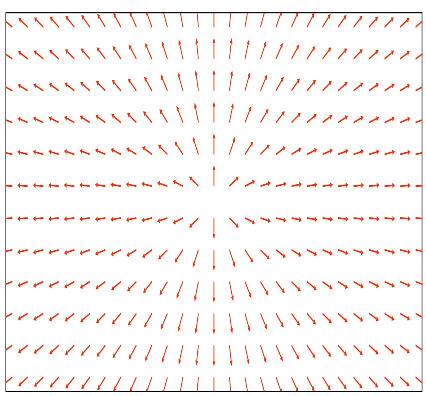
Magnetic field in the plane.

e.g. Electric field in the plane. A force field may be conservative without being defined in a simply connected domain! For example, the "electric field" $F = -\nabla U$, with

$$U(x, y) = \log \sqrt{x^2 + y^2}\,,$$

generated by a point-like charge at the origin of the plane has a singularity at the origin, hence is defined in the punctured plane $\mathbb{R}^2 \setminus \{0\}$.

Similarly, the electric potential $U(r) = 1/\|r\|$ generated by a point-like charge in the 3-dimensional physical space is also singular at the origin.



Electric field generated by a charge at the origin.

ex: Training. Tell which of the following differentials are exact

$$\begin{array}{ll} dx + dy & (t + 2x)\, dt + (2t + 3x)\, dx \\ e^{xy} dx + e^{xy} dy & \dfrac{x}{y} dy + (1 + \log y)\, dx \end{array}$$

and draw level sets of their primitives.

ex: Training. Tell which of the following differential equations are exact

$$5 + 3\frac{dx}{dt} = 0 \qquad (x-t)\,\frac{dx}{dt} + e^x = 0 \qquad \frac{1}{x} + t - \frac{t}{x^2}\frac{dx}{dt} = 0$$

$$\left(4x + 3y^2\right) + 2xy\frac{dy}{dx} = 0 \qquad 2x^2 + 4t^3 + (4tx+1)\,\frac{dx}{dt} = 0$$

$$(t+2x)\,dt + (2t+3x)\,dx = 0 \qquad \left(r^2+1\right)\cos\theta d\theta + 2r\sin\theta dr = 0$$

and solve them.

Orthogonal trajectories. If a family $\mathcal{C}$ of curves in the plane is given as the integral lines of a differential equation

$$p(x,y)dx + q(x,y)dy = 0\,,$$

then the family $\mathcal{C}^\perp$ of orthogonal trajectories (those lines which meet orthogonally the curves of $\mathcal{C}$ at every point of mutual intersection) are the integral curves of the differential equation

$$p(x,y)dy - q(x,y)dx = 0\,.$$

(the operator sending the differential $\omega = pdx + qdx$ into $*\omega = pdy - qdx$ is known as "Hodge star operator" in the Euclidean plane). Indeed, at a point where the first ODE can be solved for $y(x)$, the curves $\mathcal{C}$ have slope $\frac{dy}{dx} = v(x,y)$, with $v = -p/q$, so that orthogonal lines must have slope $\frac{dy}{dx} = -1/v(x,y)$.

Observe that if $\mathcal{C}$ are the level set of a differentiable function U, which you may think as a potential, then orthogonal trajectories are "force lines", since are everywhere tangent to the force field $F = -\nabla U$. The differential equation for such orthogonal trajectories become

$$\frac{\partial U}{\partial x}dy - \frac{\partial U}{\partial y}dx = 0\,.$$

e.g. Find the orthogonal trajectories to the family of circles $x^2 + y^2 = c$.

Call $U(x,y) = x^2 + y^2$. The family of circles $U(x,y) = c$ solve the differential equation

$$xdx + ydy = 0\,.$$

Orthogonal trajectories are integral lines of

$$xdy - ydx = 0\,.$$

Solutions are the lines $y = kx$, for real k, and the vertical line $x = 0$.

ex: Find the family of curves orthogonal to the family of ellipses $x^2 + \lambda^2 y^2 = c$, the family of hyperbolas $xy = c$, and the family of parabolas $y^2 = cx$.

ex: Find equipotential lines of the following force fields:

$$F(x,y) = (3,2) \qquad F(x,y) = (x,y) \qquad F(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

Integrating factors. It may happen that, although the differential $pdx + qdy$ is not exact, we may find a positive function $\lambda(x,y)$ such that the differential

$$\lambda pdx + \lambda qdy$$

became exact.

The differential equations $pdx + qdy = 0$ and $\lambda pdx + \lambda qdy = 0$ are "equivalent", since they have the same integral lines. This means that if you can find a primitive U of $\lambda pdx + \lambda qdy$, you can integrate the differential equation $pdx + qdy = 0$. For this reason, such a function λ is called *integrating factor* for $pdx + qdy = 0$.

Physicists' example. Physicists already know an example in thermodynamics: the heat $\delta Q = dU - PdV$ exchanged in a infinitesimal reversible transformation is not an exact differential, meanwhile the inverse temperature $\beta = 1/T$ is an integrating factor for δQ since $\delta Q/T$ is the differential of a state function $S = \int \delta Q/T$ called "entropy".

ex: Consider the following differential equations:

$$\left(4x + 3y^2\right) + 2xy\frac{dy}{dx} = 0 \qquad \left(2x^2 + y\right) + \left(x^2 y - x\right)\frac{dy}{dx} = 0\,.$$

- Show that they are not exact.
- Find integrating factor of the form x^n for some integer n, and solve the resulting exact equations.

Exercise. Show that an integration factor for the linear first order ODE

$$y' + p(x)y = q(x) \qquad \text{i.e.} \qquad dy + (p(x)y - q(x))\,dx = 0\,,$$

is $\lambda(x) = e^{\int p(x)dx}$, and use this observation to solve the equation.

ex: Let $\lambda(x, y)$ be an integrating factor for the differential equation

$$p(x, y)dx + q(x, y)dy = 0\,.$$

- Show that
$$\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} = q\frac{\partial}{\partial x}\log|\lambda| - p\frac{\partial}{\partial y}\log|\lambda|\,.$$
- Deduce the following recipes to find integrating factors:
  - if $\left(\partial p/\partial y - \partial q/\partial x\right)/q$ is a function of x alone, say $f(x)$, then an integrating factor is $\lambda(x) = e^{\int f(x)dx}$,
  - if $\left(\partial p/\partial y - \partial q/\partial x\right)/p$ is a function of y alone, say $g(y)$, then an integrating factor is $\lambda(y) = e^{\int g(y)dy}$.

6 Second order linear ODEs on the line

Many interesting models in physics, engineering and other natural sciences, arise naturally as differential equations of order two or, occasionally, more. One possibility to deal with them is transforming the equation into a system of first order ODEs. Meanwhile, in some special cases may be useful to rest with the high order equation. This is the case of some *linear* ODEs, those that may be written as

$$a_k(t)x^{(k)} + a_{k-1}(t)x^{(k-1)} + \ldots + a_2(t)\ddot{x} + a_1(t)\dot{x} + a_0(t)x = f(t)$$

for some (continuous) functions $a_i(t)$, called *coefficients*, and $f(t)$, called *r.h.s. term*, defined in some interval I of the real line. Although it is possible to prove existence and uniqueness theorems for this class, there is no "formula" giving the general solution by integration, as for the first order case. It happens that physically interesting equations of this kind (like Legendre's equation $\left(1-t^2\right)\ddot{x} - 2t\dot{x} + \alpha(\alpha+1)x = 0$, Bessel's equation $t^2\ddot{x} + t\dot{x} + \left(t^2-\alpha^2\right)x = 0$ or Hermite's equation $\ddot{x} - 2t\dot{x} + 2\alpha x = 0$) require a case by case investigation, and their solutions even deserve special names.

It turns out that a satisfactory theory (I mean general strategies to solve the equation!) can only be given when the coefficients do not depend on time, i.e. for the class of *linear ODEs with constant coefficients*.

Since most linear ODEs of physics are of order two, and since the general theory is actually constructed starting with order two "differential operators", we start with

6.1 General considerations on second order linear ODEs

A *second order linear differential equation* is a differential equation of the form

$$a(t)\ddot{x} + b(t)\dot{x} + c(t)x = f(t) \tag{6.1}$$

where the *coefficients* $a(t)$, $b(t)$ and $c(t)$, and the *r.h.s. term* ("segundo membro") $f(t)$ are continuous functions defined in some interval $I \subset \mathbb{R}$ of the real line.

The strategy to solve such equations passes through the solution of the *associated homogeneous equation*

$$a(t)\ddot{y} + b(t)\dot{y} + c(t)y = 0\,. \tag{6.2}$$

The initial problem for both differential equations is: find the solution with initial condition $x(t_0) = x_0$ and $\dot{x}(t_0) = v_0$.

We will be mainly interested in linear ODEs with constant coefficients, so we give a

Mechanical interpretation of second order linear ODEs with constant coefficients. It may be useful to keep in mind the "mechanical" interpretation of the ingredients of a generic second order linear ODE with constant coefficients like

$$a\ddot{x} + b\dot{x} + cx = f(t)\,.$$

To understand it, consider the one-dimensional motion of a particle of mass m. The trivial Newton equation

$$m\ddot{q} = 0$$

describes free motion in an inertial frame. It says $\frac{d}{dt}(m\dot{q}) = 0$, so that the linear momentum $p = m\dot{q}$ is a constant of the motion. The kinetic energy $K = p^2/2m$ of the particle is also a constant of the motion. The plane with coordinate (q,p) is called "phase space" of the system, and images of trajectories $t \mapsto (q(t),p(t))$ in the phase space are called "phase curves".

Newton equation

$$m\ddot{q} = F(t)$$

describes the motion driven by an external time-dependent force $F(t)$. So, the first coefficient $a = m$ is the inertia of the system, and the r.h.s. is an external force.

Newton equation

$$m\ddot{q} = -\alpha\dot{q}$$

describes free motion with friction, provided that α is positive. Indeed, the kinetic energy decreases exponentially as $\frac{d}{dt}K = -\alpha K$. So, the second coefficient $b = \alpha$ represent dissipation, when positive, or energy production, when negative (a rather unphysical situation!)

Newton equations

$$m\ddot{q} = -\omega^2 q \qquad \text{and} \qquad m\ddot{q} = k^2 q$$

describe a particle in a potential well $U(q) = \frac{1}{2}\omega^2 q^2$ and in a potential hill $U(q) = -\frac{1}{2}k^2 q^2$, respectively. In both cases the total energy

$$E := p^2/2m + U(q)$$

is a constant of the motion. The trajectory $q(t) = 0$ is an equilibrium solution, and other phase curves are contained in the level sets of the energy. Draw pictures of these level sets, and convince yourself that the equilibrium solution is stable in the first case (the force tends to push the particle towards the equilibrium position), and unstable in the second case (the force tends to fasten the particle from the equilibrium position). So, the third coefficient $c = \omega^2$ or $-k^2$ is the stiffness of an attracting or repelling force.

Newton equations

$$m\ddot{q} = -\alpha\dot{q} - \omega^2 q \quad \text{and} \quad m\ddot{q} = -\alpha\dot{q} + k^2 q$$

describe damped systems, as long as α is positive.

Finally, Newton equations

$$m\ddot{q} = -\alpha\dot{q} - \omega^2 q + F(t) \quad \text{and} \quad mq\ddot{q} = -\alpha\dot{q} + k^2 q + F(t) \quad .$$

describe damped systems forced by a time-dependent external force $F(t)$.

Observations on linearity and strategy. The non-homogeneous equation (6.1) and the associated homogeneous equation (6.2) can be written as

$$Lx = r \quad \text{and} \quad Ly = 0\,,$$

respectively, if we define the "differential operator" L according to

$$L := a(t)\frac{d^2}{dt^2} + b(t)\frac{d}{dt} + c(t)$$

sending a twice differentiable functions $x(t)$ into the function $(Lx)(t) = a(t)\ddot{x}(t) + b(t)\dot{x}(t) + c(t)x(t)$.

The operator L is linear, meaning that $L(x + y) = Lx + Ly$ for any two functions x and y, and $L(\lambda x) = \lambda Lx$ for any function x and real number $\lambda \in \mathbb{R}$. Therefore, the space of solutions of the homogeneous equation $Ly = 0$ is a linear space $\mathcal{H}$.

The difference $y = x_1 - x_2$ between any two solutions x_1 and x_2 of the non-homogeneous equation $Lx = r$ is a solution of the homogeneous equation $Ly = 0$ Therefore, the space of solutions of the non-homogeneous equation (6.1) is an affine space modeled on the linear space H of solutions of the associated homogeneous equation (6.2), where x is any solution of $Lx = r$. Hence, once you have just one solution z of the non-homogeneous equation, you recover the whole space of solutions as $z + \mathcal{H}$.

This suggests the following strategy to solve the initial value problem for $Lx = r$.

- First, solve the homogeneous equation $Ly = 0$, and show that it has enough solutions, actually a two dimensional space of solutions, which can be written as $y(t) = c_+\phi_+(t) + c_-\phi_-(t)$, where ϕ_+ and ϕ_- form a "basis" of the space $\mathcal{H}$ of solutions and c_+ and c_- are arbitrary constants.
- Find one "particular solution" z of the non-homogeneous equation $Lz = r$.
- Try a solution of the non-homogeneous equation $Lx = r$ having the form $x = z + y$, where z is the particular solution of the non-homogeneous equation and $y = c_+\phi_+ + c_-\phi_-$ is the general solution of the homogeneous equation. Since y depends on two free parameters, $c_\pm$, choose them to match your initial conditions $x(t_0) = x_0$ and $\dot{x}(t_0) = v_0$.
- Finally, prove an existence and uniqueness theorem for the homogeneous equation. This will imply an existence and uniqueness theorem for the non-homogeneous equation as well.

Superposition principle. If x_1 and x_2 are solutions of the non-homogeneous equations $Lx = f_1$ and $Lx = f_2$, respectively, then the linear combination $c_1x_1 + c_2x_2$ is a solution of the non-homogeneous equation $Lx = c_1f_1 + c_2f_2$. The same holds, of course, for any finite number of r.h.s. terms. This observation is known as *superposition principle*. If you think at the solution x of the differential equation $Lx = f$ as the system's response to the external input f, the principle just says that the system responds linearly. So, the strategy to solve an equation with a complicated r.h.s. could be: try to write $f(t)$ as a sum $\sum_i f_i(t)$ of simpler functions, solve separately each $Lx_i = f_i$, and finally sum the solutions $\sum_i x_i(t)$.

This said, we start with solving

6.2 Second order homogeneous ODEs with constant coefficients

Here we consider a second order homogeneous equation with constant coefficients as

$$\ddot{x} + 2\alpha\dot{x} + \beta x = 0\,.$$

(note the factor 2 before the second/friction coefficient: it will simplify all formulas below!) Observe that the homogeneous equation is autonomous (nothing depends on time explicitly), hence if $\varphi(t)$ is a solution, also any $\varphi(s+t)$ is a solution, for any time s. This implies that we may only consider the initial value problem for initial time $t_0 = 0$.

An obvious solution is the equilibrium solution $x(t) = 0$.

Suppose that $x(t)$ is a solution of the above equation, and make the conjecture $x(t) = e^{-\alpha t}y(t)$ for some other function $y(t)$. Computation shows that then $y(t)$ must be a solution of

$$\ddot{y} = \delta y$$

where $\delta = \alpha^2 - \beta$ (the parameter 4δ is called *discriminant* of the linear ODE).

Now, solving $\ddot{y} = \delta y$ is quite simple. Three different cases are possible, depending on the sign of δ, and a couple of solutions for each case are obvious.

If δ is positive, hence equal to k^2, two solutions of the equation $\ddot{y} = k^2 y$ are

$$\varphi_+(t) = e^{kt} \qquad \text{and} \qquad \varphi_-(t) = e^{-kt}\,.$$

If δ is negative, hence equal to $-\omega^2$, two solutions of the equation $\ddot{y} = -\omega^2 y$ are

$$\varphi_+(t) = \cos(\omega t) \qquad \text{and} \qquad \varphi_-(t) = \sin(\omega t)\,.$$

If $\delta = 0$, and this is a degenerate case, two solutions of the equation $\ddot{y} = 0$ are

$$\varphi_+(t) = 1 \qquad \text{and} \qquad \varphi_-(t) = t\,.$$

Now we claim that the three couples of solutions above are linearly independent (this simply means that their quotient is not constant, which is the case). Since, as shown by the following existence and uniqueness theorem, they form a basis of the space of solutions of the respective equations $y'' = \delta y$, these (like any other independent) couples are called *fundamental solutions* of the corresponding homogeneous equation. Going back to the original equation $\ddot{x} + 2\alpha\dot{x} + \beta x = 0$, we get the couples of fundamental solutions

$$\phi_+(t) = e^{-\alpha t}\varphi_+(t) \qquad \text{and} \qquad \phi_-(t) = e^{-\alpha t}\varphi_-(t)\,.$$

There follows that the formula

$$x(t) = c_+\phi_+(t) + c_-\phi_-(t)\,,$$

where c_+ and c_- are arbitrary real numbers, gives solutions of the homogeneous equation.

The free parameters c_+ and c_- may be chosen to match any initial condition $x(0) = x_0$ and $\dot{x}(0) = v_0$. Indeed, this amount to solve the system

$$\begin{cases} c_+\phi_+(0) + c_-\phi_-(0) &= x_0 \\ c_+\dot{\phi}_+(0) + c_-\dot{\phi}_-(0) &= v_0\,, \end{cases}$$

and the vectors $(\phi_+(0), \dot{\phi}_+(0))$ and $(\phi_-(0), \dot{\phi}_-(0))$ are linearly independent (check this!). Now, we claim that solutions of the Cauchy problem are unique.

Proposition 6.1. (Existence and uniqueness theorem for homogeneous second order linear ODEs with constant coefficients) *The Cauchy problem for $\ddot{x}+2\alpha\dot{x}+\beta x = 0$ with any initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$ has one and only one solution. The unique solution may be written as*

$$x(t) = c_+e^{-\alpha t}\varphi_+(t) + c_-e^{-\alpha t}\varphi_-(t)\,,$$

where c_+ and c_- are constant coefficients, and φ_+ and φ_- are a pair of fundamental solutions of $\ddot{y} = \delta y$ with $\delta = \alpha^2 - \beta$, for example:

$$\begin{aligned} \varphi_+(t) = 1 \quad &\textit{and} \quad \varphi_-(t) = t\,, \quad &\textit{if} \quad \delta = 0\,, \\ \varphi_+(t) = e^{kt} \quad &\textit{and} \quad \varphi_-(t) = e^{-kt}\,, \quad &\textit{if} \quad \delta = k^2 > 0\,, \\ \varphi_+(t) = \cos(\omega t) \quad &\textit{and} \quad \varphi_-(t) = \sin(\omega t)\,, \quad &\textit{if} \quad \delta = -\omega^2 < 0\,. \end{aligned}$$

Proof. It is sufficient to prove the result for the linear ODE $\ddot{x} = \delta x$. Also, by linearity, it is sufficient to show that the only solution of $\ddot{x} = \delta x$ with zero initial data $x(0) = 0$ and $\dot{x}(0) = 0$ is the trivial solution.

The starting observation is that solutions of $\ddot{x} = \delta x$ are analytic functions (their Taylor series converges to the function). The way you prove it is an elementary instance of a strategy, called "bootstrap", which works for eigenfunctions of a Laplacian or more generally of any elliptic differential operator in any dimension. The equation $\ddot{x} = \delta x$ implies that x admits derivatives all orders, and we can actually compute them. Indeed, $\ddot{x}' = (\ddot{x})' = \delta\dot{x}$, $x^{(4)} = (\ddot{x}')' = (\delta\dot{x})' = \delta\ddot{x} = \delta^2 x$, ..., and by induction you see that

$$x^{(2n)} = \delta^n x \quad \text{and} \quad x^{(2n+1)} = \delta^n \dot{x}\,.$$

Since x are $\dot{x}$ are bounded on a bounded interval (because they are continuous), the derivatives of x grow at most polynomially, say $x^{(k)}(t) \leqslant CK^k$ for some constants C and K and any t in a fixed bounded interval. Now you use the fact that a polynomial bound for the derivatives of a function in some bounded interval implies (by the Taylor formula with error, or, if you want, because the series is bounded by the Taylor series of an exponential) absolute convergence of the Taylor series.

Now, assume that $x(t)$ is a solution of $\ddot{x} = \delta x$ with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$. The above formulas show that all the derivatives of x at the origin are zero. There follows from analyticity that x is identically equal to zero on any bounded interval around the origin. □

Characteristic equation and fundamental solutions. Here we provide a unifying picture of the above apparently different cases.

Consider the second order differential operator with constant coefficients

$$L = \frac{d^2}{dt^2} + 2\alpha\frac{d}{dt} + \beta$$

which defines the second order homogeneous ODE $Lx = 0$. We make the conjecture that the solution is an exponential, say $x(t) = e^{zt}$, for some "frequency" $z \in \mathbb{C}$ to be determined. Computation shows that

$$(Lx)(t) = \left(z^2 + 2\alpha z + \beta\right) x(t)\,.$$

The quadratic polynomial $p(z) = z^2 + 2\alpha z + \beta$ is called *characteristic polynomial* of the second order differential operator L. The above computation shows that $x(t) = e^{zt}$ is a solution of the homogeneous equation $Lx = 0$ provided that the "frequency" z is a zero of $p(z)$. The equation

$$z^2 + 2\alpha z + \beta = 0$$

is called *characteristic equation* associated with the homogeneous second order ODE $\ddot{x} + 2\alpha\dot{x} + \beta x = 0$.

The resolvent formula says that the zeros of the quadratic polynomial $z^2 + 2\alpha z + \beta$ are given by

$$z_{\pm} = -\alpha \pm \sqrt{\delta}\,,$$

where $\delta = \alpha^2 - \beta$. Depending on the sign of δ, the characteristic polynomial may have two distinct real roots, no real root but two complex conjugate roots, or one real root of multiplicity two. Here we give the recipe to find a couple of fundamental solutions of the homogeneous ODE for each of these cases.

- If the discriminant is positive, say $\delta = k^2 > 0$, the characteristic polynomial has two distinct real roots $z_{\pm} = -\alpha \pm k$. But this means that the polynomial factorizes like $(z - z_+)(z - z_-)$. The corresponding factorization for the differential operator

  $$\frac{d^2}{dt^2} + 2\alpha\frac{d}{dt} + \beta = \left(\frac{d}{dt} - z_+\right)\left(\frac{d}{dt} - z_-\right)$$

  shows that any function in the kernel of one of the two first order operators above, hence solution of the first order ODEs $\dot{x} = z_{\pm}x$, is a solution of the homogeneous equation. This gives the known fundamental solutions

  $$e^{z_{\pm}t} = e^{-\alpha t}e^{\pm kt}\,.$$

- If the discriminant is negative, say $\delta = -\omega^2 < 0$, the characteristic polynomial has no real roots. Meanwhile, it has two conjugate complex roots $z_{\pm} = -\alpha \pm i\omega$, where i denotes $\sqrt{-1}$. The corresponding factorization

  $$\frac{d^2}{dt^2} + 2\alpha\frac{d}{dt} + \beta = \left(\frac{d}{dt} - z_+\right)\left(\frac{d}{dt} - z_-\right)$$

shows that solutions of the first order ODEs $\dot{x} = z_\pm x$ are also solutions of the homogeneous equation. We get the two linearly independent complex valued solutions as

$$e^{z_\pm t} = e^{-\alpha t} e^{\pm i\omega t}\,.$$

If we don't like complex valued solutions, we may finally take suitable linear combinations, use Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, and get our familiar fundamental solutions

$$\frac{e^{z_+ t} + e^{z_- t}}{2} = e^{-\alpha t}\cos(\omega t) \qquad \text{and} \qquad \frac{e^{z_+ t} - e^{z_- t}}{2i} = e^{-\alpha t}\sin(\omega t)\,.$$

- If the discriminant is zero, i.e. $\delta = 0$, the polynomial has just one real root $z_0 = -\alpha$ of multiplicity two. The factorization now is

$$\frac{d^2}{dt^2} + 2\alpha\frac{d}{dt} + \alpha^2 = \left(\frac{d}{dt} + \alpha\right)^2\,.$$

The kernel of $\frac{d}{dt} + \alpha$ gives the first solution $e^{-\alpha t}$. But the conjecture $x(t) = e^{-\alpha t}y(t)$ says that y must be a solution of the trivial equation $\ddot{y} = 0$, hence a polynomial of degree one. There follows that a set of fundamental solutions is

$$e^{-\alpha t} \qquad \text{and} \qquad te^{-\alpha t}\,.$$

Linear independence and Wronskian. We claimed that the two fundamental solutions ϕ_+ and ϕ_- of the homogeneous equation $\ddot{x} + 2\alpha\dot{x} + \beta x$ were linearly independent, namely that there exist no constants c_+ and c_-, apart for the trivial case $c_+ = c_- = 0$, such that

$$c_+\phi_+(t) + c_-\phi_-(t) = 0$$

for any t. This is the same as saying that the quotient ϕ_+/ϕ_- (whenever defined) is not constant. Here we provide a sophisticated tool to check linear independence.

Let $f(t)$ and $g(t)$ be two differentiable functions defined in some interval of the real line. The *Wronskian* between f and g is defined as

$$W_{f,g}(t) := f(t)\dot{g}(t) - \dot{f}(t)g(t)\,.$$

Observe that this is nothing but the derivative of the ratio g/f multiplied by f^2, as well as minus the derivative of the ratio f/g multiplied by g^2. If $W_{f,g}(t) = 0$ for any t in some interval, then the quotient g/f (or f/g) is constant on that interval. There follows that if the quotient between f and g is not constant, that is if f and g are linearly independent, then the Wronskian $W_{f,g}$ must be different from zero somewhere in the interval.

If ϕ_+ and ϕ_- are two solution of the same linear homogeneous second order ODE (not necessarily with constant coefficients), say

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0\,,$$

then their Wronskian is either everywhere zero or everywhere different from zero. Indeed, deriving one gets

$$\begin{aligned} \dot{W}_{\phi_+,\phi_-}(t) &= \phi_+(t)\ddot{\phi}_-(t) - \ddot{\phi}_+(t)\phi_-(t) \\ &= -p(t)W_{\phi_+,\phi_-}(t)\,, \end{aligned}$$

and integration gives *Abel's identity*

$$W_{\phi_+,\phi_-}(t) = W_{\phi_+,\phi_-}(0)e^{-\int_0^t p(s)ds}\,.$$

Since the exponential is never zero, there follows that

Proposition 6.2. *Two solutions of the same homogeneous second order ODE are linearly independent iff their Wronskian is different from zero in at least (hence in any!) one point.*

Observe that, taking $t = 0$, the condition $W_{\phi_+,\phi_-}(0) \neq 0$ amounts to say that $\phi_+(0)\dot{\phi}_-(0) - \dot{\phi}_+(0)\phi_-(0) \neq 0$. But this is the determinant of the two-by-two matrix

$$\begin{pmatrix} \phi_+(0) & \dot{\phi}_+(0) \\ \phi_-(0) & \dot{\phi}_-(0) \end{pmatrix}\,,$$

so that this is the same as saying that the two vectors $(\phi_+(0), \phi_-(0))$ and $\left(\dot{\phi}_+(0), \dot{\phi}_-(0)\right)$ are independent. This last statement is precisely the statement that, given any initial conditions x_0 and v_0, we can unique constants c_+ and c_+ such that the solution $x(t) = c_+\phi_+(t) + c_+\phi_-(t)$ satisfies $x(0) = x_0$ and $\dot{x}(0) = v_0$.

ex: Finally, you may want to check that the Wronskian between the couples of fundamental solutions of the homogeneous equation $\ddot{x} - 2\alpha\dot{x} + \beta x = 0$ are different from zero. Compute

$$W_{e^{-\alpha t}, te^{-\alpha t}}(t)\,, \quad W_{e^{-\alpha t}e^{kt}, e^{-\alpha t}e^{-kt}}(t) \quad \text{and} \quad W_{e^{-\alpha t}\sin(\omega t), e^{-\alpha t}\cos(\omega t)}(t)\,.$$

e.g. Free motion. Free motion of a particle in a inertial frame and without forces is governed by the trivial Newton equation $m\ddot{q} = 0$. Solutions are $q(t) = s + vt$, where s is the initial position $q(0)$ and v is the initial velocity $\dot{q}(0)$.

e.g. Example. Solve $\ddot{x} + 2\dot{x} + 5x = 0$ with initial conditions $x_(0) = 3$ and $\dot{x}(0) = -2$.

The characteristic polynomial $z^2 + 2z + 5$ has complex conjugate roots $z_\pm = -1 \pm i2$, hence a couple of fundamental solutions are $e^{-t}\cos(2t)$ and $e^{-t}\sin(2t)$. The general solution is

$$x(t) = c_+ e^{-t}\cos(2t) + c_- e^{-t}\sin(2t)\,.$$

To determine the value of the constants, you solve the system

$$\begin{cases} x(0) &= 3 \\ \dot{x}(0) &= -2 \end{cases} \quad \Rightarrow \quad \begin{cases} c_+ &= 3 \\ -c_+ + c_- &= -2 \end{cases}$$

given by the initial conditions, and get the solution $x(t) = 3e^{-t}\cos(2t) + e^{-t}\sin(2t)$.

ex: Training.

- Find the general solution of the following EDOs:

$$\ddot{x} - 2x = 0 \quad \ddot{x} + \pi^2 x = 0 \quad 3\ddot{x} + \dot{x} = 0 \quad \ddot{x} - \dot{x} = 0$$

$$\ddot{x} + 2\dot{x} - x = 0 \quad \ddot{x} + 2\dot{x} + x = 0 \quad \ddot{x} + 4\dot{x} + 5x = 0 \quad \ddot{x} - 4\dot{x} + x = 0$$

- Solve the following initial values problems:

$$\begin{aligned}
\ddot{x} + 2x = 0 &\quad \text{with } x(0) = 0 \text{ and } \dot{x}(0) = 2 \\
\ddot{x} + \dot{x} = 0 &\quad \text{with } x(0) = 1 \text{ and } \dot{x}(0) = 0 \\
\ddot{x} + 4\dot{x} + 5x = 0 &\quad \text{with } x(1) = 2 \text{ and } \dot{x}(1) = -1 \\
\ddot{x} - 17\dot{x} + 13x = 0 &\quad \text{with } x(3) = 0 \text{ and } \dot{x}(3) = 0 \\
\ddot{x} - 2\dot{x} - 2x = 0 &\quad \text{with } x(0) = 0 \text{ and } \dot{x}(0) = 9 \\
\ddot{x} + 2\dot{x} - 2x = 0 &\quad \text{with } x(3) = 0 \text{ and } \dot{x}(3) = 9 \\
\ddot{x} - 4\dot{x} - x = 0 &\quad \text{com } x(1) = 2 \text{ e } \dot{x}(1) = 1\,.
\end{aligned}$$

- Find second order ODEs which admit the following pairs of independent solutions:

$$e^{2t} \text{ and } e^{-2t}\,, \qquad e^{-t}\sin(2\pi t) \text{ and } e^{-t}\cos(2\pi t)\,, \qquad \sinh(t) \text{ and } \cosh(t)\,,$$

$$e^{-3t} \text{ and } te^{-3t}\,, \qquad \sin(2\pi t) \text{ and } \sin(2\pi t + \pi/2)\,, \qquad 3 \text{ and } 5t\,.$$

- Find for which values of λ there exist non-trivial solutions of

$$\frac{d^2y}{dx^2} = \lambda y$$

in the segment $[0, \ell]$ with boundary conditions $y(0) = 0$ and $y(\ell) = 0$.

6.3 Oscillations

Here we study in details the most important second order linear differential equations, describing oscillations of a system near its equilibrium positions[7].

[7] "The harmonic oscillator, which we are about to study, has close analogs in many other fields; although we start with a mechanical example of a weight on a spring, or a pendulum with a small swing, or certain other mechanical devices, we are really studying a certain *differential equation*. This equation appears again and again in physics and other sciences, and in fact is a part of so many phenomena that its close study is well worth our while. Some of the phenomena involving this equation are the oscillations os a mass on a spring; the oscillations of charge flowing back and forth in an electrical circuit; the vibrations of a tuning fork which is generating sound waves; the analogous vibrations of the electrons in an atom, which generate light waves; the equations for the operation of a servosystem, such as a thermostat trying to adjust a temperature; complicated interactions in chemical reactions; the growth of a colony of bacteria in interaction with the food supply and the poison the bacteria produce; foxes eating rabbits eating grass, and so on; ..."

Richard P. Feynman [Fe63]

From the mathematical pendulum to the harmonic oscillator. Oscillations of a "mathematical pendulum" (a point-like mass attached to a wire of negligible weight, under a constant gravitational force) are modeled by the Newton equation

$$I\ddot{\theta} = -mg\ell \sin\theta \,,$$

where θ is the angle formed by the wire with the vertical line (hence $\theta = 0$ is the equilibrium position), m is the mass, ℓ is the length, g is the gravitational acceleration and $I = m\ell^2$ is the momentum of inertia of the pendulum. The conserved energy of the system is

$$E\left(\theta, \dot{\theta}\right) = \frac{1}{2} I\dot{\theta}^2 - mg\ell\cos\theta \,.$$

Introducing the "resonant frequency" $\omega = \sqrt{g/\ell}$, the Newton equation may be written as

$$\ddot{\theta} = -\omega^2 \sin\theta \,.$$

Small oscillations of a pendulum are modeled after the approximation $\sin\theta \simeq \theta$, hence by the Newton equation

$$\ddot{\theta} = -\omega^2\theta \,,$$

called *harmonic oscillator*.

The harmonic oscillator is a quite universal equation, since it describes small oscillations around a "generic" stable equilibrium of any one-dimensional Newtonian system. Indeed, take a Newton equation $m\ddot{x} = -dU/dx$ of a particle in a potential field U. An equilibrium position is a zero of the force, i.e. a point x_0 where $dU/dx(x_0) = 0$. It is "stable" if x_0 is a local minimum of the potential, so that the Taylor expansion of a generic potential around x_0 starts with

$$U(x) = \alpha + \frac{1}{2}\beta\,(x - x_0)^2 + \frac{1}{6}\gamma\,(x - x_0)^3 + \dots \,,$$

for some positive second derivative $d^2U/dx^2(x_0) = \beta$. If we are only interested in small displacements of x around x_0, we can safely disregard high order terms and approximate the Newton equation with the Hooke's law

$$m\frac{d^2}{dt^2}\,(x - x_0) \simeq -\beta\,(x - x_0) \,,$$

which is an harmonic oscillator with resonant frequency $\omega = \sqrt{\beta/m}$.

Harmonic oscillator. Consider the harmonic oscillator

$$\ddot{q} = -\omega^2 q \,.$$

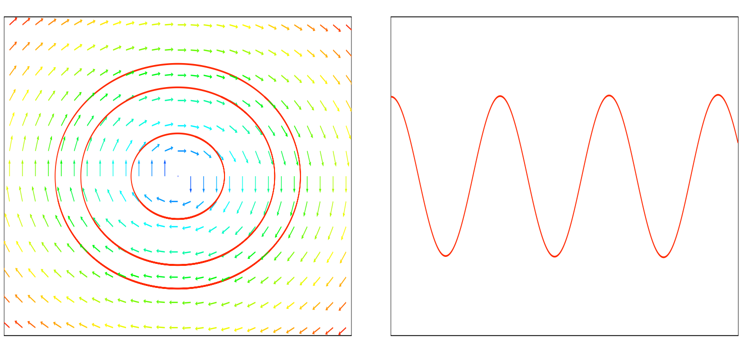
The solution with initial data $q(0) = q_0$ and $\dot{q}(0) = v_0$ is

$$q(t) = q_0\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t) \,,$$

representing oscillations with period $2\pi/\omega$. The above solution can be written as

$$q(t) = A\sin(\omega t + \varphi) \quad \text{as well as} \quad A\cos(\omega t + \phi)$$

for some "amplitude" $A = \sqrt{q_0^2 + (v_0/\omega)^2}$ and "phases" φ and ϕ, which depend on the initial data.



Harmonic oscillator, phase curves and time series.

The energy

$$E(q,\dot{q}) = \frac{1}{2}\dot{q}^2 + \frac{1}{2}\omega^2 q^2$$

is a constant of the motion. As a function of the amplitude and the resonant frequency, the energy is $E = \omega^2 A^2$.

Call $p = \dot{q}$ the momentum. Level sets of E in the q-p plane (the phase-space of the system) are ellipses, and argue that that they are the phase curves (i.e. the images of the trajectories of the harmonic oscillator in the phase space). The Newton equation $\ddot{q} = -\omega^2 q$ is equivalent to Hamilton's first order equations

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -\omega^2 q\,. \end{aligned}$$

Eliminate dt, and show that phase curves are solutions of the first order (homogeneous and exact) ODE

$$pdp + \omega^2 q dq = 0\,,$$

which is nothing but $dE = 0$.

ex: **Particle in a potential hill.** Solve and discuss the Newton equation

$$m\ddot{q} = k^2 q\,,$$

of a particle of mass m in a potential $U(q) = -\frac{1}{2}k^2 q^2$.

Does it admit equilibrium solutions? Does it admit periodic orbits? Does it admit bounded orbits?

Damped oscillations. Adding friction to an harmonic oscillator we get

$$\ddot{q} = -2\alpha\dot{q} - \omega^2 q\,,$$

where α is some positive constant.

- Find the general solution, draw pictures and discuss the cases
  $\alpha^2 < \omega^2$ (under-critical damping),
  $\alpha^2 = \omega^2$ (critical damping),
  and $\alpha^2 > \omega^2$ (overcritical damping). .
- Show that the energy

  $$E(q,\dot{q}) = \frac{1}{2}\dot{q}^2 + \frac{1}{2}\omega^2 q^2$$

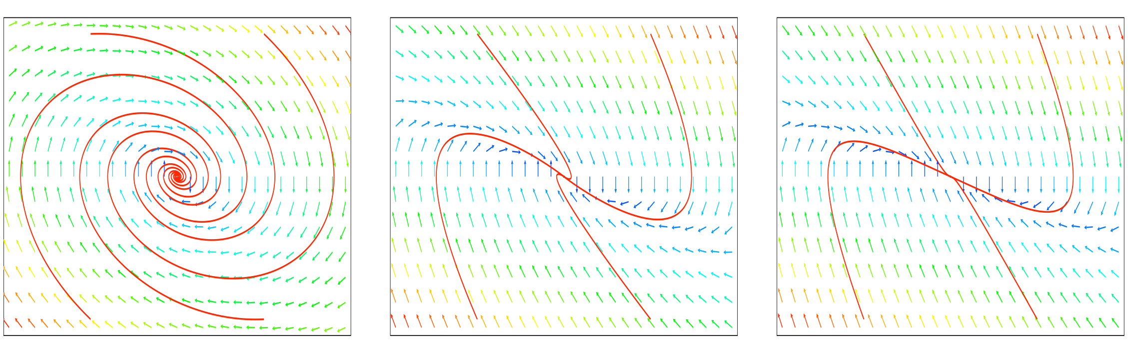
  decreases with time outside equilibrium points.
- Call $p = \dot{q}$ the momentum. The Newton equation $\ddot{q} = -2\alpha\dot{q} - \omega^2 q$ is equivalent to Hamilton's first order equations

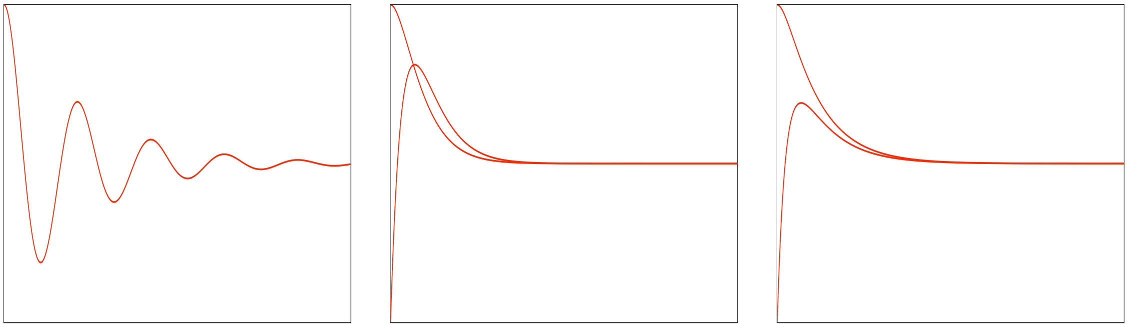
  $$\begin{aligned} \dot{q} &= \quad p \\ \dot{p} &= \quad -2\alpha p - \omega^2 q \end{aligned}$$

  Eliminate dt, and show that phase curves are solutions of the homogeneous first order ODE

  $$pdp + \left(2\alpha p + \omega^2 q\right) dq = 0\,.$$

  Solve the equation, or try to understand the qualitative behavior of its solutions, depending on the ratio α^2/ω^2, and draw phase curves in the phase space q-p.
- What does it change if α is supposed to be negative?





Underdamped, critical and overdamped oscillations (phase portrait and time series).

ex: **Equidimensional equations.** An ODE like

$$ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy=0$$

is called *equidimensional*, or *Cauchy-Euler*, equation.

- Show that the substitution $x=e^t$ transform an equidimensional equation for $y(x)$ in an equation with constant coefficients for $z(t)=y(x(t))$.
- Solve

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y=0\,,$$

for $x>0$.

Harmonic oscillator in complex coordinates. Consider the harmonic oscillator

$$\ddot{q}=-\omega^2 q\,.$$

Define the complex variable $z=\omega q+i\dot{q}$. Newton equation then takes the form of a first order linear equation in the complex line, namely

$$\dot{z}=-i\omega z$$

and the solution can be written as

$$z\left(t\right)=e^{-i\omega t}z\left(0\right)\,.$$

Going back to your original variables, and using Euler's formula, you get the familiar "sin and cos" solution.

e.g. **Stationary Schrödinger equation on the line.** The stationary Schrödinger equation for the complex valued wave-function $\Psi(x)$ of a particle of energy E in a potential $U(x)$ is

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2}=\left(E-U(x)\right)\Psi\,,$$

where m is the mass of the particle and $\hbar=h/2\pi$ is the reduced Planck constant ($h\simeq 6.6262\times 10^{-34}$ J s).

The equation for the free particle, the one with $U(x)=0$, has the couple of independent solutions

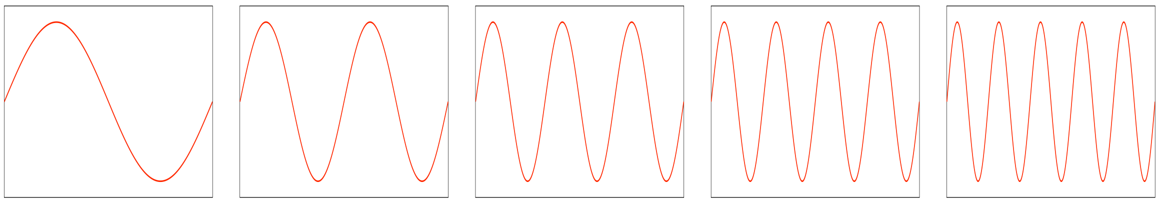
$$\Psi(x)=e^{\pm ipx/\hbar}$$

corresponding to the value $E=p^2/2m$ of the energy. The parameter p is then interpreted to be the momentum of the particle. The modulus $|\Psi(x)|$ of the wave-function is interpreted to be the probability density to find the particle in the position x. The fact that the solutions $e^{\pm ipx/\hbar}$ have unit modulus at all points (and are not square integrable!) is a manifestation of Heisenberg's uncertainty principle $\Delta x\Delta p\geqslant\hbar/2$: fixing the value of the momentum produces an infinite uncertainty for the position!

A free particle confined in a box, here simply a segment $[0, \ell]$ since we are in dimension one, is modeled taking (the limit of a sequence of smooth potentials that tends to) a potential which is zero inside the interval and infinite outside. This produces natural boundary conditions $\Psi(0) = 0$ and $\Psi(\ell) = 0$ for the wave function. The only non-trivial solutions have then energy levels

$$E_n = \frac{h^2}{8m\ell^2} n^2 \,,$$

for $n = 1, 2, 3, \ldots$.



Wave functions of the first 5 energy levels.

6.4 Second order non-homogeneous ODEs with constant coefficients

Consider the non-homogeneous equation

$$\ddot{x} + 2\alpha\dot{x} + \beta x = f(t) \,,$$

where the "external force" $f(t)$ is a continuous function defined in some interval $I \subset \mathbb{R}$ of the time axis. The difference between any two solutions must be a solution of the associated homogeneous equation

$$\ddot{y} + 2\alpha\dot{y} + \beta y = 0 \,.$$

But we already know how to solve it: its general solution is a linear combination

$$y(t) = c_1\phi_+(t) + c_2\phi_-(t) \,,$$

where ϕ_+ and ϕ_- are a pair of fundamental solutions. This implies that if we could find just one solution $z(t)$ of the non-homogeneous equation, any other solution will be a sum

$$x(t) = z(t) + c_1\phi_+(t) + c_2\phi_-(t)$$

of this particular solution and a solution of the homogeneous equation. In other words,

Proposition 6.3. *The general solution of the non-homogeneous equation is given by the sum of a particular solution of the non-homogeneous equation and the general solution of the associated homogeneous equation.*

So, we are left with the problem to determine one particular solution of the non-homogeneous equation. We first show a general method, working in any case.

Method of variation of parameters to find a particular solution. Let $\phi_+(t)$ and $\phi_-(t)$ be two independent solutions of the associated homogeneous equation. We try a solution of the non-homogeneous equation having the form

$$z(t) = \lambda_+(t)\phi_+(t) + \lambda_-(t)\phi_-(t)$$

where λ_+ and λ_- are two functions to be determined. Inserting our guess into the non-homogeneous equation

$$\ddot{x} + \alpha\dot{x} + \beta x = r(t) \,,$$

we get, after some computations, the result that $z(t)$ satisfies the non-homogeneous equation if (but not only if!) λ'_+ and λ'_- solve the system of algebraic equations

$$\begin{aligned} \dot{\lambda}_+(t)\phi_+(t) + \dot{\lambda}_-(t)\phi_-(t) &= 0 \\ \dot{\lambda}_+(t)\dot{\phi}_+(t) + \dot{\lambda}_-(t)\dot{\phi}_-(t) &= f(t) \,. \end{aligned}$$

The determinant of the system is the Wronskian $W_{\phi_+,\phi_-}(t)$, hence is everywhere different from zero. The solution of the system is

$$\begin{aligned}
\dot{\lambda_+}(t) &= -\phi_-(t)\frac{f(t)}{W_{\phi_+,\phi_-}(t)} \\
\dot{\lambda_-}(t) &= \phi_+(t)\frac{f(t)}{W_{\phi_+,\phi_-}(t)} .
\end{aligned}$$

Integrating the above ODEs for λ_+ and λ_-, we get finally the following recipe

Proposition 6.4. (variation of parameters) *Let $\phi_+(t)$ and $\phi_-(t)$ be two independent solutions of the homogeneous equation $\ddot{y} + 2\alpha\dot{y} + \beta y = 0$. A (particular) solution of the non-homogeneous equation $\ddot{x} + 2\alpha\dot{x} + \beta x = f(t)$ is given by*

$$z(t) = \lambda_+(t)\phi_+(t) + \lambda_-(t)\phi_-(t) ,$$

where

$$\lambda_+(t) = -\int \phi_-(t)\frac{f(t)}{W_{\phi_+,\phi_-}(t)}\, dt \qquad \text{and} \qquad \lambda_-(t) = \int \phi_+(t)\frac{f(t)}{W_{\phi_+,\phi_-}(t)}\, dt .$$

e.g. Solve $\ddot{x} + x = 1/\sin(t)$ for $t \in (0, \pi)$.

A set of fundamental solutions of the homogeneous equation $\ddot{y} + y = 0$ is $\phi_+(t) = \cos(t)$ and $\phi_- = \sin(t)$, and we may compute $W_{\phi_+,\phi_-} = 1$. The recipe above gives a particular solution

$$z(t) = \lambda_+ \cos(t) + \lambda_- \sin(t)$$

where

$$\lambda_+(t) = -\int dt = t \quad \text{and} \quad \lambda_-(t) = \int \frac{\cos(t)}{\sin(t)} dt = \log(\sin(t)) .$$

The general solution is finally

$$x(t) = -t\cos(t) + \log(\sin(t))\sin(t) + c_+ \cos(t) + c_- \sin(t) .$$

Method of undetermined coefficients to find a particular solution. When the r.h.s.'s $f(t)$ is particularly simple, we have at our disposal a less painful method to find particular solutions, which involves no integrations at all.

We are looking for just one solution of the non-homogeneous equation $Lx = f$, where L is the differential operator $L = d^2/dt^2 + \alpha d/dt + \beta$. Assume that the r.h.s. $f(t)$ belongs to a finite dimensional space of functions $\mathcal{F}$ which is left invariant by L (i.e., if $\varphi \in \mathcal{F}$, then also $L\varphi \in \mathcal{F}$). If $\varphi_1(t), \varphi_2(t), ..., \varphi_n(t)$ is a basis of $\mathcal{F}$, we may try a particular solution having the form

$$z(t) = z_1\varphi_1(t) + z_2\varphi_2(t) + ... + z_n\varphi_n(t)$$

for some coefficients $z_1, z_2, ..., z_n$ to be determined. Indeed, since Lz is again an element of $\mathcal{F}$, and since also the r.h.s. admits a (unique) expression as $f(t) = f_1\varphi_1(t) + f_2\varphi_2(t) + ... + f_n\varphi_n(t)$, we may adjust the coefficients in order to have $Lz = f$. This method works whenever the space $\mathcal{F}$ is not too small, namely when it is not contained in the kernel of L.

Here are the recipes.

- **Polynomials.** The space of polynomials of given finite degree is left invariant by any differential operator with constant coefficients. So, if the r.h.s.is a polynomial of degree n, say $f(t) = \sum_{i=0}^{n} r_i t^i$, a particular solution may be found between polynomials of the same degree, if $\beta \neq 0$, of degree $n+1$ if $\beta = 0$ but $\alpha \neq 0$, or of degree $n+2$ for the trivial equation $\ddot{x} = f(t)$.

  To do this, you just try a solution $z(t) = \sum_{i=0}^{n+2} z_i t^i$, substitute it into the equation, and equal the coefficients of the same powers of t.

- **Exponentials.** If the r.h.s. is an exponential, say $f(t) = e^{\gamma t}$, try a particular solution of the form $z(t) = p(t)e^{\gamma t}$, where p is a polynomial of degree less then or equal to two.

  If the r.h.s. is an polynomial times an exponential, say $f(t) = e^{\gamma t}\sum_{i=0}^{n} r_i t^i$, try a particular solution of the form $z(t) = e^{\gamma t}\sum_{i=0}^{n+2} z_i t^i$.

- **Trigonometric functions.** If the r.h.s. is a trigonometric function as $f(t) = \sin(\gamma t + \varphi)$ or $\cos(\gamma t + \varphi)$, try a particular solution as $z(t) = p(t)\sin(\gamma t + \varphi) + q(t)\cos(\gamma t + \varphi)$, where p and q are polynomial of degree less than or equal to two.
- **General case.** If, finally, the r.h.s. is a product of a polynomial times an exponential times sin's and cos's, try a linear combination of polynomials times exponentials times sin's and cos's.

Example. Find a particular solution of $\ddot{x} + \dot{x} = te^{2t}$.

Example. Find a particular solution of $\ddot{x} + x = \cos(2t)$.

Example. Find a particular solution of $\ddot{x} + x = \cos(t)$.

Example. Find a particular solution of $\ddot{x} - 2\dot{x} + 2x = e^t \cos(t)$.

Training. Find the general solution of the following non-homogeneous equations:

$$\ddot{x} + x = t \quad \ddot{x} - \dot{x} = t^2 \quad \ddot{x} - 4x = e^{-2t} \quad \ddot{x} + 2\dot{x} + x = e^{-t} \quad \ddot{x} + 4\dot{x} + 3x = t^2 - 1$$

Integral representation of the response. Show that a particular integral of the equation $\ddot{x} + \omega^2 x = r(t)$ is given by the formula

$$\varphi(t) = \frac{1}{\omega}\int_0^t r(s)\sin\left(\omega(t-s)\right) ds\,,$$

and that a particular integral of the equation $\ddot{x} - k^2 x = r(t)$ is given by the formula

$$\phi(t) = \frac{1}{k}\int_0^t r(s)\sinh\left(k(t-s)\right) ds\,.$$

Training. Find the general solution of the following non-homogeneous equations:

$$\ddot{x} + 9x = \sin(t) \quad \ddot{x} + 4x = \cos(2t) \quad \ddot{x} - 4x = e^{-2t} \quad \ddot{x} - 4x = e^{-t}$$

Exercise (forced particle). Consider the Newton equation

$$m\ddot{q} = -2\alpha\dot{q} + F(t)$$

of a particle of mass m subject to a time dependent force $F(t)$, with some positive friction coefficient α. Given initial data $q(0) = q_0$ and $\dot{q}(0) = v_0$, find the trajectory when the force is

- constant, i.e. $F(t) = g$,
- a polynomial like $F(t) = 3 - t^2$,
- an exponential like $F(t) = e^{-3t}$,
- a periodic function like $F(t) = F_0\cos(\gamma t)$,
- a superposition of periodic functions like $F(t) = \sum_{i=1}^n F_i \cos(\gamma_i t)$.
- Find the corresponding trajectories when there is no friction, and compare with the limits of your solutions above for $\alpha \to 0$.

6.5 Driven oscillations

Driven oscillations. If an external periodic force $F(t) = F_0 \cos(\gamma t)$ acts on a harmonic oscillator we get the Newton equation

$$\ddot{q} = -\omega^2 q + F(t)\,.$$

The homogeneous equation $\ddot{y} = -\omega^2 y$ has solution

$$y(t) = A\sin(\omega t + \phi)$$

for some amplitude A and some phase ϕ. A particular solution of the non-homogeneous equation may be found using the guess $z(t) = a\cos(\gamma t) + \sin(\gamma t)$. This gives (inserting the guess into the differential equation and equating the coefficients of cos and sin) the linear system

$$\begin{aligned} a(\omega^2 - \gamma^2) &= F_0 \\ b(\omega^2 - \gamma^2) &= 0 \end{aligned}$$

for the undetermined coefficients a and b. When $\gamma^2 \neq \omega^2$ it can be solved, and gives the response

$$z(t) = \frac{F_0}{\omega^2 - \gamma^2}\cos(\gamma t)$$

so that the general solution is

$$q(t) = A\sin(\omega t + \phi) + \frac{F_0}{\omega^2 - \gamma^2}\cos(\gamma t)$$

For $\omega^2 = \gamma^2$, i.e. the force has the same frequency of the free oscillator, we must modify our guess. The right guess is $z(t) = t(a\cos(\omega t) + b\sin(\omega t))$. We find the linear system

$$\begin{aligned} 2\omega a &= 0 \\ 2\omega b &= F_0 \end{aligned}$$

The solution is now

$$q(t) = A\sin(\omega t + \varphi) + \frac{F_0}{2\omega} t \sin(\omega t)\ .$$

and, as expected, is an oscillation with amplitude increasing with time. This phenomena is known as *resonance.*

We can have idea of what happens near resonance observing the solution for $\gamma \simeq \omega$. Indeed, the solution with trivial initial conditions (both position and velocity) is

$$q(t) = \frac{F_0}{\omega^2 - \gamma^2}\left(\cos(\gamma t) - \cos(\omega t)\right)$$

and addiction formulas give

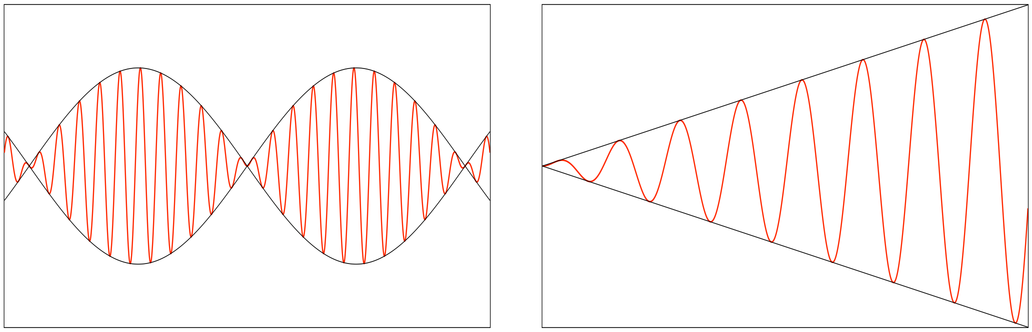
$$q(t) = \frac{F_0}{\omega^2 - \gamma^2} 2\sin\left(\frac{\omega - \gamma}{2} t\right) \cdot \sin\left(\frac{\omega + \gamma}{2} t\right)$$

When the difference $\omega - \gamma = 2\varepsilon$ is small, hence $\frac{\omega+\gamma}{2} \simeq \omega$, the above is

$$q(t) = \frac{F_0}{2\omega\varepsilon}\sin(\varepsilon t) \cdot \sin((\omega + \varepsilon)t)$$

i.e. the product of a factor $\sin((\omega+\varepsilon)t) \simeq \sin(\omega t)$, oscillating with a frequency near the proper frequency of the oscillator, times a slowly oscillating factor $\sin(\varepsilon t)$. This phenomena is known as *beats*, due to the strange effect that it produces in human hears (like a single note with slowly oscillating amplitude). Taking the limit $\varepsilon \to 0$ we recover the resonant solution

$$\frac{F_0}{2\omega\varepsilon}\sin(\varepsilon t) \cdot \sin((\omega + \varepsilon)t) \ \to\ \frac{F_0}{2\omega} t \cdot \sin(\omega t)$$



Beats and resonance.

We may use the complex variable $z := p + i\omega q$, which satisfies the first order ODE

$$\dot{z} - i\omega z = F(t)\,.$$

If we look for a solution $z(t) = \lambda(t)e^{i\omega t}$ (where $e^{i\omega t}$ solves the homogeneous equation), we get for λ the simple ODE $\dot{\lambda} = F(t)e^{-i\omega t}$. There follows that we may write the system response as

$$z(t) = e^{i\omega t}\left(z(t_0) + \int_{t_0}^{t} F(s)e^{-i\omega s}ds\right)\,.$$

In particular, the energy that a force $F(t)$, acting in an infinite interval $(-\infty, \infty)$ of time, transfers to the system is given by

$$E = \frac{1}{2}\,|z(\infty)|^2 = \frac{1}{2}\left|\int_{-\infty}^{\infty} F(t)e^{-i\omega t}dt\right|^2\,.$$

i.e., the square modulus $|\widehat{F}(\omega)|^2$ of the Fourier transform (if you already know what it is) of the force computed at the frequency ω of the oscillator.

ex: **Driven and damped oscillations.** If an external periodic force $F(t) = F_0\cos(\gamma t)$ acts on a damped oscillator we get the Newton equation

$$\ddot{q} = -2\alpha\dot{q} - \omega^2 x + F(t)\,.$$

- Show that solutions have the form

  $$q(t) = Ae^{-\alpha t}\sin\left(\sqrt{\omega^2 - \alpha^2}t + \varphi\right) + \frac{F_0}{\sqrt{\left(\omega^2 - \gamma^2\right)^2 + 4\alpha^2\gamma^2}}\sin\left(\gamma t + \phi\right)\,,$$

  where φ and ϕ are two phases. The first term in the above solution is a transient term, which vanishes for large times. The second term, called steady-state solution, is a (out of phase) synchronous response to the force. The function

  $$R(\gamma) = \frac{1}{\sqrt{\left(\omega^2 - \gamma^2\right)^2 + 4\alpha^2\gamma^2}}$$

  is called resonant curve, or frequency response curve, since it represent the proportionality factor between the input magnitude and the asymptotic response of the system.

- Show that, if $\alpha^2 < \omega^2$ (i.e. if the non forced system is under-critical), the resonant curve $R(\gamma)$ reaches a maximum for the value

  $$\gamma_r = \sqrt{\omega^2 - 2\alpha^2}$$

  of the forcing frequency, called *resonant frequency*.

- Discuss the behavior of the resonant curve for different values of the damping coefficient α, and the limit of the resonant response $R(\gamma_r)$ for small values of α.

- Use the superposition principle to show that if the force acting on a damped oscillator is a superposition like

$$F(t) = \sum_{i=1}^{n} F_i \cos(\gamma_i t) ,$$

then the steady-state solution will be

$$q(t) = \sum_{i=1}^{n} R(\gamma_i) F_i \sin(\gamma_i t + \phi_i) .$$

- Discuss what happens in the critical ($\alpha^2 = \omega^2$) and overcritical ($\alpha^2 > \omega^2$) cases.

ex: **Kirchoff's law for a LRC circuit.** The electric current $I(t)$ flowing in an electric circuit with resistance R, inductance L, capacitance C and driven by a tension $V(t)$ satisfies the second order linear ODE

$$L\ddot{I} + R\dot{I} + \frac{1}{C} I = \dot{V}(t) .$$

- Compute the current flowing in a circuit driven by a constant tension $V(t) = V_0$, and discuss its behavior (compare with damped oscillations).
- Compute the current flowing in a circuit driven by an alternate tension $V(t) = V_0 \sin(\gamma t)$ (compare with forced and damped oscillations).
- Find the resonant frequency of a LRC circuit.

6.6 Central forces and Kepler problem

Motion in a central force. Consider the Newton equation

$$M_{♂}\ddot{r} = F(|r|)\,\hat{r}$$

describing the motion of a particle, say planet Mars, of mass $M_{♂}$ in a central force field F. Conservation of angular momentum implies that the motion is planar, hence we may take $r \in \mathbb{R}^2$. In polar coordinates $r = \rho e^{i\theta}$, the equations reed

$$\begin{aligned} \ddot{\rho} - \rho\dot{\theta}^2 &= F(\rho)/M_{♂} \\ \rho\ddot{\theta} + 2\dot{\rho}\dot{\theta} &= 0 . \end{aligned}$$

The second equation says that the "areal velocity" ("velocidade areal") $\ell = \rho^2\dot{\theta}$ is a constant of the motion (Kepler's second law).

Planetary motion. Taking Newton's gravitational force $F(\rho) = -\frac{GM_{♂}M_{\odot}}{\rho^2}$, where $M_{\odot}$ is the mass of the Sun and G is the gravitational constant, the first equation may be written as

$$M_{♂}\ddot{\rho} = -\frac{\partial}{\partial\rho} V_{\ell}(\rho) ,$$

where we defined the "effective potential energy" as

$$V_{\ell}(\rho) := \frac{1}{2} M_{♂} \frac{\ell^2}{\rho^2} - G\frac{M_{♂}M_{\odot}}{\rho} .$$

The conserved energy is

$$E = \frac{1}{2} M_{♂}\dot{\rho}^2 + \frac{1}{2} M_{♂} \frac{\ell^2}{\rho^2} - G\frac{M_{♂}M_{\odot}}{\rho} .$$

Now we set $\rho = 1/x$ and look for a differential equation for x as a function of θ. Computation shows that $dx/d\theta = -\dot{\rho}/\ell$, and, using conservation of ℓ, that $d^2x/d\theta^2 = -\rho^2\ddot{\rho}/\ell^2$. There follows that the first Newton equation reads

$$\frac{d^2x}{d\theta^2} + x = -\frac{1}{\ell^2 x^2 M_{♂}} F(1/x) .$$

we get

$$\frac{d^2x}{d\theta^2} + x = -\frac{GM_\odot}{\ell^2} \, .$$

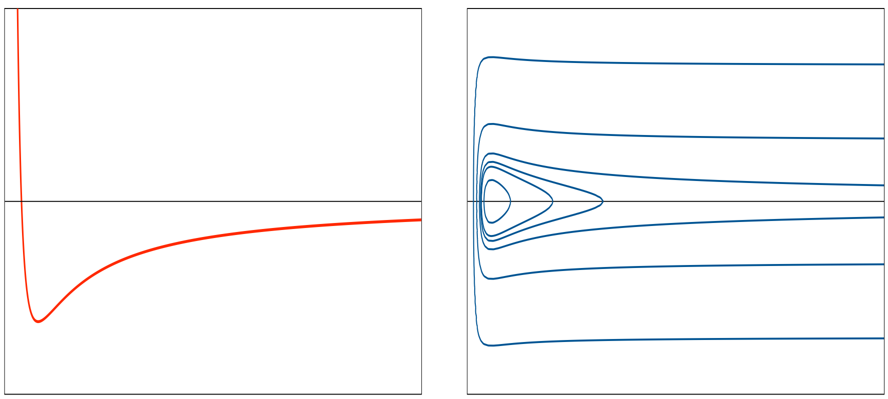
The general solution of this second order linear differential equation is

$$x\left(\theta\right) = \frac{GM_\odot}{\ell^2}\left(1 + e\cos\left(\theta - \theta_0\right)\right) \, ,$$

for some constants e and θ_0. Back to the original radial variable we get the solution

$$\rho\left(\theta\right) = \frac{\ell^2/GM_\odot}{1 + e\cos\left(\theta - \theta_0\right)} \, ,$$

Hence, orbits are conic sections with eccentricity ("excentricidade") e and focus at the origin: an ellipse for $0 \leqslant e < 1$ (corresponding to negative energy, hence to planets, and this is Kepler's first law), a parabola for $e = 1$ (corresponding to zero energy), an hyperbola for $e > 1$ (corresponding to positive energy). .



Kepler's effective potential and some energy level sets.

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