Lecture notes on "Análise Matemática 3"
Fourier series and transform

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Abstract

This is not a book! These are personal notes written while preparing lectures on "Análise Matemática 3" for students of FIS in the a.y. 2007/08 and then 2009/10. They are based on previous notes on "Complementos de Análise Matemática" for students of ENGSI, FIS, FQ(E), QP and QT. They are rather informal and may even contain mistakes. I tried to be as synthetic as I could, without missing the observations that I consider important.

I probably will not lecture all I wrote, and did not write all I plan to lecture. So, I included empty or sketched paragraphs, about material that I think should/could be lectured within the same course.

References contain some introductory manuals, some classics, and other books where I have learnt things in the past century. Besides, good material and further references can easily be found on the web, for example in Wikipedia.

Pictures were made with "Grapher" on my MacBook.

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1 Heat and diffusion

The simples model for propagation of heath and diffusion are

$$\frac{\partial u}{\partial t} - \beta \frac{\partial^2 u}{\partial x^2} = 0 \qquad \dots \qquad \frac{\partial u}{\partial t} - \beta \triangle u = 0\,,$$

where β is a positive parameter.

Example (heat propagation/conduction). Consider a thin wire of length ℓ, section s and density ρ, so thin that its temperature profile at time t may be considered a function of the length only, say $u(x,t)$ with $0 \le x \le \ell$. If the two ends are in thermal contact with two thermostats maintained at constant temperatures a and b, experiments show that the temperature profile stabilizes at the stationary linear profile

$$a + \frac{b-a}{\ell} x\,.$$

The heath flowing along the wire's section in unit time is seen to be $\delta Q = -ks\frac{b-a}{\ell}$, where k is a coefficient of thermal conduction which characterize the wire's material.

Assume now that we have a non-stationary temperature profile $u(x,t)$ at time t. The heath flow across the x-cross-section between the times t_1 and t_2 is then

$$\delta Q = -\int_{t_1}^{t_2} ks \frac{\partial u}{\partial x}(x,t)dt\,.$$

But the amount of heat necessary to increase the temperature of a conductor by δT is

$$\delta Q = cv\delta T\,,$$

where c is the specific heat of the material and v is the volume. Hence the heat balance for the piece of wire between x_1 and x_2, and between times t_1 and t_2 is

$$\int_{x_1}^{x_2} c\rho s\left(u(x,t_2) - u(x,t_1)\right)dx = \int_{t_1}^{t_2} ks\left(\frac{\partial u}{\partial x}(x_2,t) - \frac{\partial u}{\partial x}(x_1,t)\right)dt + \int_{x_1}^{x_2}\int_{t_1}^{t_2} F(x,t)dtdx\,,$$

where $F(x,t)$ represents the contribution of some heat source. Using the mean values theorem we find, in the limit $x_2 \to x_1$ and $t_2 \to t_1$, the equation

$$c\rho\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + F(x,t)\,,$$

that we may write as

$$\frac{\partial u}{\partial t} = \beta\frac{\partial^2 u}{\partial x^2} + f(x,t)\,,$$

having defined the coefficient of thermal conduction $\beta = k/(c\rho)$, and where $f(x,t) = F(x,t)/(c\rho)$ is a heat source density.

Example (diffusion).

Example (Brownian motion). The first satisfactory theory of Brownian motion (the erratic movements of particles suspended in a liquid, observed by the botanist Robert Brown in 1827) is due to Albert Einstein [1]. With clever use of ideal experiments, mechanical and thermodynamic ideas, he was able to show that the probability density $P(x_0|x,t)$ to find a Brownian particle in x at time t provided it were at x_0 at time 0 is the non-negative solution of the diffusion equation

$$\frac{\partial P}{\partial t} - \beta\frac{\partial^2 P}{\partial x^2} = 0\,,$$

such that $\lim_{t\to 0} P(x_0|x,t) = 0$ for any $x \neq x_0$, and $\int P(x_0|x,t)dx = 1$. Above, the "diffusion constant" is

$$\beta = \frac{RT}{N\alpha}\,,$$

[1] A. Einstein, Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen, *Ann. Phys.* **17**, 549, 1905. Translated and reprinted in A. Einstein, *Investigations on the Theory of Brownian Movement*, Dover, New York, 1956.

where $R \simeq 8314.51$J/kmol·K is the perfect gas constant, T the absolute temperature, $N \simeq 6.00221 \times 10^{23}$mol^{-1} the Avogadro number, and $\alpha = 6\pi\eta a$ a friction coefficient (depending on the dynamic viscosity η of the liquid/gas and the radius a of the Brownian particle). One can check that the solution of this problem is given by the (shifted and scaled) Gaussian

$$P(x_0|x,t) = \frac{1}{2\sqrt{\pi\beta t}} e^{-(x-x_0)^2/4\beta t} .$$

The mean square displacement in time t is

$$\begin{aligned} \langle x(t)^2 \rangle &= \int_{-\infty}^{\infty} x^2 P(x,t) dx \\ &= 2\beta t , \end{aligned}$$

so that the diffusion constant may be measured in experiments (and indeed in this way Perrin estimated a value of the Avogadro number, winning a Nobel prize few years after Einstein proposal). You may also see that the Brownian particle is displaced of an average amount

$$|x(t-\delta t) - x(t)| \simeq \sqrt{2\beta t}$$

during a small interval of time with length δt, a fact which explains why its trajectories do not look like the familiar differentiable curves of Newtonian mechanics. Of course, you may say that Einstein's model does not work for short intervals of time, and indeed improved models for the Brownian motion were proposed later by L.S. Ornstein and G.E. Uhlenbeck.

1.1 Maximum value principle and uniqueness theorem

The uniqueness and stability theorems for the one-dimensional heat equation come from the obvious physical principle saying that, in absence of internal heat sources, any interior point of a conductor at time $t > 0$ cannot be hotter or colder than it were a time $t = 0$ or than the hotter or colder boundary points (where energy is coming in or out). Technically, the principle is stated as

Maximum value principle. *Let $u(x,t)$ be a smooth solution of the heat equation in a finite rectangle $R = [a,b] \times [0,T]$. Then u attains its maximum and its minimum for $t = 0$ or at the boundary $x = a$ or b.*

Proof. First, call $B = \{(x,t) \in R \text{ s.t. } t = 0 \text{ or } x = a \text{ or } x = b\}$ the set where the maximum and minimum of u will eventually attained. Let $M = \max_{(x,t)\in B} u(x,t)$, and assume that there is some point $(x_0,t_0) \in R \backslash B$ where $u(x_0,t_0) = M + \varepsilon$ for some positive ε. Define an auxiliary function v as

$$v(x,t) = u(x,t) - \frac{\varepsilon}{2T}(t - t_0) ,$$

and observe that it is bounded by $v(x,t) \leq M + \varepsilon/2$ on B. Since $v(x_0,t_0) = M + \varepsilon$, v attains its maximum in some point $(x_1,t_1) \in R \backslash B$. Computing derivatives, we must have

$$\frac{\partial^2 v}{\partial x^2}(x_1,t_1) = \frac{\partial^2 u}{\partial x^2}(x_1,t_1) \leq 0$$

and

$$\frac{\partial v}{\partial t}(x_1,t_1) = \frac{\partial u}{\partial t}(x_1,t_1) - \frac{\varepsilon}{2T} \geq 0 ,$$

which is impossible since u satisfies the heat equation. To prove the analogous statement for the minimum, just repeat the argument for the function $-u(x,t)$. **q.e.d.**

Applying the maximum principle to the difference of any two solutions of the heat equation we get the

Uniqueness theorem. *There exists at most one smooth solution of the heat equation in a bounded interval, given any initial and boundary conditions.*

Another interesting consequence of the maximum principle is the

Stability theorem. *If $u(x,t)$ and $v(x,t)$ are two solutions of the heat equation on a finite interval I such that $|u(x,t)-v(x,t)| \leq \varepsilon$ for $t=0$ and at the boundary points of I, then $|u(x,t)-v(x,t)| \leq \varepsilon$ for any $x \in I$ and any time $t \geq 0$.*

It may be rephrased saying that the Cauchy problem for the heat equation is "well posed", a small uncertainty on the initial and boundary conditions does not grow with time.

The uniqueness theorem in a infinite domain needs a separate proof.

1.2 Diffusion on the line and heat kernel

Direct computation shows that the "gaussian"

$$P(x,t) = \frac{1}{2\sqrt{\pi\beta t}} e^{-x^2/4\beta t} .$$

is a solution of the heat equation

$$\frac{\partial P}{\partial t} - \beta \frac{\partial^2 P}{\partial x^2} = 0$$

on the line $x \in \mathbf{R}$. Moreover, it has constant integral

$$\int_{-\infty}^{\infty} P(x,t) dx = 1$$

and satisfies

$$\lim_{t\to 0} P(x,t) = 0$$

for any $x \neq 0$. Observe that, for any $\varepsilon > 0$,

$$\int_{|\xi - x| \leq \varepsilon} P(x-\xi,t) d\xi \to 1 \qquad \text{and} \qquad \int_{|\xi - x| > \varepsilon} P(x-\xi,t) d\xi \to 0$$

as $t \to 0^+$. This implies that, if $\varphi(x)$ is a continuous and bounded function, then

$$\lim_{t\to 0^+} \int_{-\infty}^{\infty} \varphi(\xi) P(x-\xi,t) d\xi = \varphi(x) ,$$

so that we can interpret $P(x,t)$ as being the solution of the heat equation with initial condition $P(x,0) = \delta(x)$, and as such it is called *fundamental solution* of the heat equation, or *heat kernel*. But then we can write other solutions as "superpositions" of the fundamental solutions, namely as integrals

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \varphi(\xi) P(x-\xi,t) d\xi \\ &= \frac{1}{2\sqrt{\pi\beta t}} \int_{-\infty}^{\infty} \varphi(\xi) e^{-(x-\xi)^2/4\beta t} d\xi . \end{aligned}$$

You may check that the above formula, called *Poisson's formula*, is the solution of the heat equation with initial condition $u(x,0) = \varphi(x)$, provided that $\varphi(x)$ is continuous and bounded on the real line.

1.3 Separation of variables and Fourier series solutions

Here we pose the problem to find solutions, as many as possible, of the heat equation

$$\frac{\partial v}{\partial t} - \beta \frac{\partial^2 v}{\partial x^2} = 0$$

in some interval $0 \leq x \leq \ell$.

Constant boundary conditions. We start with the problem with constant boundary conditions, say $v(0,t) = a$ and $v(\ell,t) = b$. Observe that, if we set $v(x,t) = u(x,t) + a + (b-a)x/\ell$, then the new function $u(x,t)$ also satisfies the same heat equation

$$\frac{\partial u}{\partial t} - \beta \frac{\partial^2 u}{\partial x^2} = 0$$

with zero boundary conditions $u(0, t) = 0$ and $u(\ell, t) = 0$. So, we may restrict to this last problem, describing heat propagation in a thin conductor whose ends are in thermal contact with two thermostats at fixed zero temperature.

Separation of variables. An obvious "stationary solution" is the trivial solution $u(x, t) = 0$, the conductor in in thermal equilibrium with the two thermostats.

We try non-trivial solutions having the form $u(x, t) = f(x)g(t)$, for some functions $f(x)$, which only depends on the position, and $g(t)$, which only depends on time, to be determined. Substituting the guess into the heat equation, we get

$$\frac{f''(x)}{f(x)} = \frac{1}{\beta}\frac{g'(t)}{g(t)},$$

at least at those points where f and g are different from zero. Now we notice that the l.h.s. only depends on the position x and the r.h.s. only depends on the time t. This may only happens when they are both constant, say equal to μ. But then we are left with the two second order ODEs

$$f'' = \mu f \qquad \text{and} \qquad g' = \beta\mu g$$

for f and g. The only non trivial solutions of $f'' = \mu f$ with zero boundary conditions $f(0) = 0$ and $f(\ell) = 0$ occur when the "eigenvalue" μ is equal to

$$\mu_n = -\left(\frac{\pi n}{\ell}\right)^2$$

for $n = 1, 2, 3, \ldots$, and they are proportional to

$$\sin\left(\frac{\pi n}{\ell}x\right).$$

For any given n, we then solve $g' = -\beta\left(\frac{\pi n}{\ell}\right)^2 g$. The result is that $g(t)$ is proportional to

$$e^{-\beta\left(\frac{\pi n}{\ell}\right)^2 t}.$$

Hence, we have found solutions of the heath equation as "modes"

$$u_n(x, t) = b_n e^{-\beta(\pi n/\ell)^2 t}\sin\left(\frac{\pi n}{\ell}x\right) \qquad \text{for } n = 1, 2, 3, \ldots,$$

where b_n are arbitrary constants.

Isolated boundaries. If the conductor is isolated, hence there is no heat flow at the boundaries, we must solve the heat equation

$$\frac{\partial v}{\partial t} - \beta\frac{\partial^2 v}{\partial x^2} = 0$$

with boundary conditions $\frac{\partial v}{\partial x}(0, t) = 0$ e $\frac{\partial v}{\partial x}(\ell, t) = 0$. The conjecture $v(x, t) = f(x)g(t)$, for some functions $f(x)$, which only depends on the position, and $g(t)$, which only depends on time, still lead to

$$\frac{f''(x)}{f(x)} = \frac{1}{\beta}\frac{g'(t)}{g(t)},$$

hence to the two second order ODEs

$$f'' = \mu f \qquad \text{and} \qquad g' = \beta\mu g$$

for f and g. This time, we must find non-trivial solutions of $f'' = \mu f$ with zero derivatives $f'(0) = 0$ and $f'(\ell) = 0$ at the boundary points. These exist provided the eigenvalue is equal to

$$\mu_n = -\left(\frac{\pi n}{\ell}\right)^2$$

for $n = 0, 1, 2, 3, \ldots$, and they are proportional to

$$\cos\left(\frac{\pi n}{\ell}x\right).$$

For any given n, we then solve $g' = -\beta\left(\frac{\pi n}{\ell}\right)^2 g$. The result is that $g(t)$ is proportional to

$$e^{-\beta\left(\frac{\pi n}{\ell}\right)^2 t}.$$

Hence, we have solutions of the heath equation as "modes"

$$v_n(x,t) = a_n e^{-\beta(\pi n/\ell)^2 t}\cos\left(\frac{\pi n}{\ell}x\right) \qquad \text{for } n = 0,1,2,3,\dots,$$

where a_n are arbitrary constants.

Superpositions and Fourier's idea. Since the heat equation is linear, any finite superposition of modes, say

$$u(x,t) = \sum_{n=1}^{N} b_n e^{-\beta(\pi n/\ell)^2 t}\sin\left(\frac{\pi n}{\ell}x\right)$$

for zero boundary conditions, or

$$v(x,t) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n e^{-\beta(\pi n/\ell)^2 t}\cos\left(\frac{\pi n}{\ell}x\right)$$

for zero derivative boundary conditions, is again a solution of the heat equation. We note that their initial values are

$$u(x,0) = \sum_{n=1}^{N} b_n \sin\left(\frac{\pi n}{\ell}x\right) \qquad \text{and} \qquad v(x,0) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{\pi n}{\ell}x\right),$$

respectively. This says that every time we are able to write the initial condition as a "trigonometric polynomial", the formulas above (multiplication of each coefficient by the exponentially decaying factor $e^{-\beta(\pi n/\ell)^2 t}$) solves the heat equation.

We may also observe that the amplitude of each mode decreases exponentially in time, with a speed that depends on the frequency number n. In particular, asymptotically the solution tends to the stationary solution $u(x,t) = 0$ or $v(x,t) = a_0/2$, in accordance with our physical intuition.

It was Fourier [2] who first conjectured the possibility to express an "arbitrary" well behaved function $\varphi(x)$, say defined in the interval $-\ell \le x \le \ell$, as a "infinite trigonometric polynomial", i.e. a trigonometric series

$$\varphi(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{\pi n}{\ell}x\right) + b_n \sin\left(\frac{\pi n}{\ell}x\right)\right),$$

for some coefficients a_n's and b_n's. On the right we have a series of functions, and equality should mean that, for any fixed x, the resulting numerical series is summable and has sum equal to $\varphi(x)$. Observe that the series should contain only sin's if the function φ is odd (as the initial value for the heat problem with zero boundary conditions), and only cos's if the function φ is even (as the initial value for the heat problem with zero derivative boundary conditions). Now, if the above expression for φ does make sense, then multiplication of each coefficient by the exponentially decaying factor $e^{-\beta(\pi n/\ell)^2 t}$ should give the solution of the heat equation with initial condition $u(x,0) = \varphi(x)$. Indeed, if we admit that we can differentiate the series term by term, once w.r.t. time t and twice w.r.t. space x, and that the resulting series for $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ are still point-wise absolutely convergent, then the heat equation will be satisfied for trivial arithmetical reasons.

Fourier's trigonometric series. Assume that the function $\varphi(x)$, defined in the interval $-\ell \le x \le \ell$, admits a representation as a trigonometric series

$$\varphi(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{\pi n}{\ell}x\right) + b_n \sin\left(\frac{\pi n}{\ell}x\right)\right).$$

To understand the meaning of the "coefficients" a_n and b_n, we integrate the series against the functions $\sin\left(\frac{\pi n}{\ell}x\right)$ and $\sin\left(\frac{\pi n}{\ell}x\right)$, assuming that we can exchange the integral with the infinite sums. The first observation is that the non-oscillating term $a_0/2$ is the mean value of φ, namely

$$\frac{a_0}{2} = \frac{1}{2\ell}\int_{-\ell}^{\ell}\varphi(x)\,dx.$$

[2] Joseph Fourier, *Théorie Analytique de la Chaleur*, 1822. Translated as *The Analytical Theory of Heat*, Dover, 2003

As for the other terms, we get

$$\begin{aligned}\int_{-\ell}^{\ell} \varphi(x)\cos\left(\frac{\pi n}{\ell}x\right)dx &= \int_{-\ell}^{\ell}\left(\sum_{k=1}^{\infty} a_n\cos\left(\frac{\pi n}{\ell}x\right)+b_n\sin\left(\frac{\pi n}{\ell}x\right)\right)\cos\left(\frac{\pi n}{\ell}x\right)dx\\ &= \int_{-\ell}^{\ell} a_n\left(\cos\left(\frac{\pi n}{\ell}x\right)\right)^2 dx\\ &= \ell a_n \qquad \text{if } n>1\,,\end{aligned}$$

and

$$\begin{aligned}\int_{-\ell}^{\ell} \varphi(x)\sin\left(\frac{\pi n}{\ell}x\right)dx &= \int_{-\ell}^{\ell}\sum_{k=1}^{\infty}\left(a_n\cos\left(\frac{\pi n}{\ell}x\right)+b_n\sin\left(\frac{\pi n}{\ell}x\right)\right)\sin\left(\frac{\pi n}{\ell}x\right)dx\\ &= \int_{-\ell}^{\ell} b_n\left(\sin\left(\frac{\pi n}{\ell}x\right)\right)^2 dx\\ &= \ell b_n\,,\end{aligned}$$

Hence, the *Fourier coefficients* of the function φ are

$$a_n=\frac{1}{\ell}\int_{-\ell}^{\ell}\varphi(x)\cos\left(\frac{\pi n}{\ell}x\right)dx \qquad \text{e} \qquad b_n=\frac{1}{\ell}\int_{-\ell}^{\ell}\varphi(x)\sin\left(\frac{\pi n}{\ell}x\right)dx\,.$$

For the moment, we'll content with the "formal" solution we've found. We'll see later which conditions on the initial value $\varphi(x)$ will guarantee that our formula actually gives a genuine (meaning smooth) solution of the heat equation. Meanwhile, it must be said that real world situations may be, and sometimes must be, also modeled with non-smooth, e.g. discontinuous, functions (for example, when you suddenly put in thermal contact two conductors at different temperatures). In such cases, the formal solution is all we have, and it often does provide the correct answer, once properly interpreted.

2 Waves

The simplest PDEs modeling propagation of "waves" are written, in one and three spatial dimensions, as

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad \ldots \qquad \frac{\partial^2 u}{\partial t^2} - c^2 \triangle u = 0\,.$$

Above, $\triangle$ is the *Laplacian* in Euclidean three dimensional space, the differential operator $\triangle = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. Physicists also use the *D'Alambert's operator* $\Box = \partial^2/\partial t^2 - \triangle$ to write the wave equation in the compact form $\Box u = 0$, in a system of units where the velocity c has been set equal to one (or time has been redefined to be ct).

Our basic example will be

Example (transversal small vibrations of a string). Consider a thin string of length ℓ and constant linear density ρ maintained in equilibrium by a certain tension applied to its ends. Transversal vibrations are described by a displacement field $u(x,t)$, where $x \in [0,\ell]$ and t is time, which represents the (one-dimensional) transversal displacement of the string from its rest position $u(x,t) = 0$. For small vibrations, we will consider $\partial u/\partial x$ small and disregard higher order quantities. In this approximation, there is no stretching of the string, since the length of the piece of string between any two points is

$$\int_{x_1}^{x_2} \sqrt{1 + (\partial u/\partial x)^2} dx \simeq x_2 - x_1$$

This implies that the tension at each point is constant in time, say equal to $k(x)$. Moreover, the longitudinal tension is $k(x)\cos(\arctan u_x) \simeq k(x)$, and the transversal tension is $k(x)\sin(\arctan u_x) \simeq k(x)u_x$. Since the longitudinal tensions between any two points must balance, we see that within this approximation the tension does not depends on the position, it is a constant k. Now we compute the change in moment, for the piece of string between any two nearby points $x_1 < x_2$, in the interval of times from t_1 to t_2,

$$\int_{x_1}^{x_2} \rho(x) \left(\frac{\partial u}{\partial t}(x,t_2) - \frac{\partial u}{\partial t}(x,t_1) \right) dx\,,$$

and equals to the work done by the transversal tension and an external force field $F(x,t)$ in the same time interval,

$$\int_{t_1}^{t_2} k \left(\frac{\partial u}{\partial x}(x_2,t) - \frac{\partial u}{\partial x}(x_1,t) \right) dt + \int_{t_1}^{t_2} F(x,t)dt$$

But then

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2} \rho(x) \frac{\partial^2 u}{\partial t^2} dxdt = \int_{t_1}^{t_2}\int_{x_1}^{x_2} k \frac{\partial^2 u}{\partial x^2} dtdx + \int_{t_1}^{t_2}\int_{x_1}^{x_2} \rho f(x,t) dtdx\,,$$

where $f(x,t) = F(x,t)/\rho$ is the force density, and, since it must hold for any x_1, x_2, t_1, t_2, we finally get

$$\rho(x) \frac{\partial^2 u}{\partial t^2} - k \frac{\partial^2 u}{\partial x^2} = \rho f(x,t)\,.$$

Dividing by the density, we get the equation in the standard form

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f(x,t)\,,$$

where $c = \sqrt{k/\rho}$ has the dimensions of a velocity. If external forces are absent, we are left with the homogeneous equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0\,,$$

describing the free vibrations of the string.

In applications, you must remember that the model was obtained assuming that the string does not stretch, hence that the amplitude of vibrations is small. Moreover, you may have noticed that real strings (as in a piano) do stop vibrating appreciably after a finite time. The simplest way to model this fact is putting a friction term as $-\alpha \partial u/\partial t$ on the r.h.s. of the wave equation.

An heuristic look at the wave equation. Divide the string in a large number of short intervals of length ε centered at evenly spaced points $0 < x_1 < x_2 < x_3 < \ldots < x_n < \ldots < \ell$. If the length ε is not greater than your instrument's resolution, you may imagine that the n-th piece of string is actually a point-like particle of mass $m_n = \rho(x_n)\varepsilon$ with height $q_n(t) = u(x_n, t)$ over x_n. But then

$$\rho\varepsilon \frac{\partial^2 u}{\partial t^2}(x_n, t) = m\ddot{q}_n$$

is equal to the mass times the transversal acceleration of the particle, and the wave equation is just the Newton's equation saying that this quantity is equal to a certain force

$$f_n = k\varepsilon \frac{\partial^2 u}{\partial x^2}(x_n, t)\,.$$

Observe that the force is positive at those points x_n where the shape u is convex, and negative at points where the shape is concave, a fact which is in agreement with your intuition (just imagine pulling up or down the extremes of a jumping' string!). Now, the second derivative of u w.r.t. x is well approximated, within your instrument's resolution, by the quantity

$$\frac{\partial^2 u}{\partial x^2}(x_n, t) \simeq \frac{u(x_{n+1}, t) - 2u(x_n, t) + u(x_{n-1}, t)}{\varepsilon^2}\,,$$

so that the force acting on the n-th particle,

$$f_n \simeq k\frac{q_{n+1} - q_n}{\varepsilon} - k\frac{q_n - q_{n-1}}{\varepsilon}\,,$$

is a superposition of two forces obeying Hooke's law with stiffness k and displacement proportional to the distance between the n-th particle and its two neighbors. The string may be considered a continuous limit of a system of point-like masses coupled with springs. This is one reason to believe that solving a partial differential equation (as this one) is conceptually different from solving an ordinary differential equation: morally, it amounts to solving an uncountable number of ODEs in a time!

Initial and boundary value problems for the wave equation. Since the equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

gives the second derivative of u w.r.t. time as a function of something else, it is natural to pose the problem of solving the equation given *initial conditions*

$$u(x, 0) = \varphi(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x, 0) = \phi(x)$$

for u and its first time derivative at time $t = 0$.

Together with the initial conditions, it is necessary to say what happens to the field u at the boundary of the space domain. Such conditions are called *boundary conditions* ("condições de fronteira"). They may read

$$u(0, t) = \lambda(t) \qquad \text{and} \qquad u(\ell, t) = \mu(t)$$

if the problem is formulated in a bounded space domain $x \in [0, \ell]$, or may just say that $u(x, t) \to 0$ with a certain speed for $|x| \to \infty$, if the problem is formulated in a infinite space domain (an infinite space domain is also a good choice if we are interested in short time phenomena which occur far from the ends of the string). Different kinds of boundary conditions may involve partial derivatives of u at the boundary. For example, saying that $\frac{\partial u}{\partial x}(0, t) = 0$ for any time t means that the 0-end of the string is left loose ...

Example (longitudinal vibrations).

Example (electric oscillations in conductors).

2.1 d'Alembert's traveling waves

Consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

on the line. In the new variables $\xi = x + ct$ and $\eta = x - ct$, it takes the (canonical) form (of hyperbolic second order PDEs)

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0\,.$$

The obvious general solution of this equation is $f(\xi)+g(\eta)$, where f and g are arbitrary twice continuously differentiable functions. Back to the original space and time variables, we get *d'Alembert's solution*

$$u(t, x) = f(x + ct) + g(x - ct)\,,$$

representing a superposition of two waves, with shapes f and g, traveling to the left and to the right with speed c.

The arbitrary shapes f and g are determined by the initial conditions. Assume that

$$u(0, x) = \phi(x) \qquad \text{and} \qquad u_t(0, x) = \varphi(x)\,.$$

Then we get

$$f(x) + g(x) = \phi(x) \qquad \text{and} \qquad cf'(x) - cg'(x) = \varphi(x)\,.$$

Integrating the second equation and substituting the result into the first, we finally get *d'Alembert's formula*

$$u(t, x) = \frac{1}{2}\left(\phi(x + ct) + \phi(x - ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \varphi(y)dy\,,$$

solving the Cauchy problem for the infinite string.

Exercise. Consider small vibrations of an infinite string.

- Show that if the initial conditions $u(x, 0)$ and $u_t(x, 0)$ are zero outside an interval $[-L, L]$ then he solution $u(x, t)$ is zero outside the interval $[-L - ct, L + ct]$. Discuss the physical meaning of this fact.

- Find the solution when the initial conditions are

  $$u(x, 0) = 0 \qquad \text{and} \qquad u_t(x, 0) = \cos(2\pi x)\,,$$

  or

  $$u(x, 0) = e^{-x^2} \qquad \text{and} \qquad u_t(x, 0) = 0\,.$$

- Show that if the initial conditions $\phi(x)$ and $\varphi(x)$ are odd functions, then the solution $u(x, t)$ is an odd function of x for any time t. Use this observation to solve the problem in the semirect $x \geq 0$ with zero boundary condition $u(0, t) = 0$.

- Show that if the initial conditions $\phi(x)$ and $\varphi(x)$ are even functions, then the solution $u(x, t)$ is an even function of x for any time t. Use this observation to solve the problem in the semirect $x \geq 0$ with "loose-end" boundary condition $\frac{\partial u}{\partial x}(0, t) = 0$.

2.2 Energy, uniqueness and stability theorems

The uniqueness theorem for the wave equation is obtained from a physical principle: conservation of energy. The energy of a vibrating string is

$$E = \frac{1}{2}\int_0^\ell \left(\rho\left(\frac{\partial u}{\partial t}\right)^2 + k\left(\frac{\partial u}{\partial x}\right)^2\right) dx$$

Integrating by parts one may check that the energy is a constant of the motion, provided that the extremes are fixed (for otherwise we must take into account the work done by an external force to move them!). Indeed,

$$\begin{aligned}
\frac{dE}{dt} &= \int_0^\ell \left(\rho\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + k\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\right) dx \\
&= \left[k\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right]_0^\ell + \int_0^\ell \left(\rho\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} - k\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}+\right) dx \\
&= \int_0^\ell \rho\frac{\partial u}{\partial t}\left(\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2}\right) dx = 0\,.
\end{aligned}$$

Uniqueness theorem. *There is at most one $\mathcal{C}^2$ solution of the wave equation*

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f(x,t)$$

with given initial conditions

$$u(x,0) = \phi(x) \qquad \text{and} \qquad u_t(x,0) = \varphi(x)\,,$$

and boundary conditions $u(0,t) = \lambda(t)$, $u(\ell,t) = \mu(t)$.

Proof. The difference w between any two solutions is a solution of the same wave equation with trivial initial and boundary conditions. Since its energy is constant, it is equal to initial value zero. But this implies that both $\partial w/\partial x$ and $\partial w/\partial t$ are constant and equal to zero. There follows that w is constant and equal to its initial condition $w(x,t) = 0$ for any $x \in [0,\ell]$ and any time $t \geq 0$. **q.e.d.**

For an infinite string, it is possible to obtain a uniqueness theorem provided that the solution vanishes outside a bounded interval, or that it decreases so rapidly to zero that all the integrals above are absolutely convergent.

Another important issue is that of "stability" of solutions. May small uncertainties in the initial conditions produce large effects as times goes by? If so, how large? The result is that we have some control. More precisely, initial perturbations grow at most linearly in time, as stated in the following

Stability theorem. *For any positive ε and any positive time T there exists a positive $\delta(\varepsilon,T)$ such that if $u(x,t)$ and $v(x,t)$ are two solutions of the wave equation with initial and boundary conditions that differ by no more than $\delta(\varepsilon,T)$ then*

$$|u(x,t) - v(x,t)| \leq \varepsilon$$

for any position x and any time $0 \leq t \leq T$.

Proof. If initial and boundary conditions are bounded by some δ, then d'Alembert formula shows that

$$|u(x,t)| \leq \delta + \delta t\,.$$

But if $0 \leq t \leq T$, then the above is bounded by $\delta(1+T)$. Applying this to the difference of two solutions, we see that given a required precision ε, an initial precision

$$\delta(\varepsilon,T) = \varepsilon/(1+T)$$

will do the job. **q.e.d.**

2.3 Separation of variables and stationary waves

Here we pose the problem to find solutions, as many as possible, of the wave equation

$$\frac{\partial^2 v}{\partial t^2} - c^2 \frac{\partial^2 v}{\partial x^2} = 0$$

in some interval $0 \leq x \leq \ell$, with time invariant boundary conditions, say $v(0,t) = a$ and $v(\ell,t) = b$. Observe that, if we set $v(x,t) = u(x,t) + a + (b-a)x/\ell$, then the new function $u(x,t)$ also satisfies the same wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

with zero boundary conditions $u(0,t) = 0$ and $u(\ell,t) = 0$. So, we may restrict to this last problem, describing small vibrations of a string with fixed ends (as a violin's string).

Separation of variables. An obvious "stationary solution" is the trivial solution $u(x,t) = 0$, the string doesn't vibrate.

We look for non-trivial solutions having the form $u(x,t) = f(x)g(t)$, for some functions $f(x)$, which only depends on the position, and $g(t)$, which only depends on time, to be determined. Substituting the guess into the wave equation, we get

$$\frac{f''(x)}{f(x)} = \frac{1}{c^2}\frac{g''(t)}{g(t)}\,,$$

at least at those points where f and g are different from zero. Now we notice that the l.h.s. only depends on the position x and the r.h.s. only depends on the time t. This may only happens when they are both constant, say equal to μ. But then we are left with the two second order ODEs

$$f'' = \mu f \qquad \text{and} \qquad g'' = c^2 \mu g$$

for f and g. The only non trivial solutions of $f'' = \mu f$ with zero boundary conditions $f(0) = 0$ and $f(\ell) = 0$ occur when the "eigenvalue" μ is equal to

$$\mu_n = -\left(\frac{\pi n}{\ell}\right)^2$$

for $n = 1, 2, 3, ...$, and they are proportional to

$$\sin\left(\frac{\pi n}{\ell} x\right) .$$

For any given n, we then solve $g'' = -c^2 \left(\frac{\pi n}{\ell}\right)^2 g$. The result is that $g(t)$ is a linear combinations of

$$\cos\left(\frac{\pi c n}{\ell} t\right) \qquad \text{and} \qquad \sin\left(\frac{\pi c n}{\ell} t\right) .$$

Stationary waves. Back to the function $u(x,t)$, we have found solutions of the wave equation, with zero boundary conditions at $x = 0$ and $x = \ell$, in the form of *stationary waves*

$$\begin{aligned} u_n(x,t) &= \Big(a_n \cos(2\pi\nu_n t) + b_n \sin(2\pi\nu_n t)\Big) \sin(2\pi x/\lambda_n) \\ &= A_n \sin(2\pi\nu_n t + \tau_n) \sin(2\pi x/\lambda_n) \ , \end{aligned}$$

for $n = 1, 2, 3, ...$, where we have defined the *wavelengths* and the *proper frequencies* as

$$\lambda_n = \frac{2\ell}{n} \qquad \text{and} \qquad \nu_n = \frac{c}{2\ell} n \, , \qquad \text{for } n = 1, 2, 3, \dots ,$$

and where a_n and b_n are arbitrary constants, $A_n = \sqrt{a_n^2 + b_n^2}$ is an amplitude, and $\tau_n = \arctan(a_n/b_n)$ a phase. Sometimes, also the quantities $\omega_n = 2\pi\nu_n = \pi c n/\ell$ are called frequencies (their use allows to forget the ubiquitous factor 2π in all formulas!). The first allowed frequency, $\nu_1 = c/2\ell$, is said *fundamental frequency* of the vibrating string, and ν_2, ν_3, ν_4, ... are called *2nd, 3rd, 4th, ... harmonics.*

The energy of the n-th stationary wave $u_n(x,t)$ is

$$\begin{aligned} E_n &= \frac{1}{2} \int_0^\ell \left(\rho \left(\frac{\partial u_n}{\partial t}(x,t) \right)^2 + k \left(\frac{\partial u_n}{\partial x}(x,t) \right)^2 \right) dx \\ &= \pi^2 M A_n^2 \nu_n^2 \, , \end{aligned}$$

where $M = \ell\rho$ is the mass of the string.

Exercise. The E-string of a violin, which is about 325mm length and use to be tuned with a tension $\simeq$ 70N (i.e. $\simeq$ 7.1Kg$\times$9.8m/s^2), vibrates with frequencies 660Hz, 1320Hz, 1980Hz, ... (corresponding to E5, E6, E7, ...) Find the linear density and the weight of the string.

What should a violinist do in order to obtain the A5 of 880Hz with this string?

Homework. Investigate the ratios between the frequencies of the notes in our western scale C-D-E-F-G-A-B-C. The story starts with Pitagoras ...

Superpositions of stationary waves. Since the wave equation is linear, any superposition

$$u(x,t) = \sum_{n \geq 1} \Big(a_n \cos(2\pi\nu_n t) + b_n \sin(2\pi\nu_n t)\Big) \sin(2\pi x/\lambda_n) \ ,$$

of stationary waves is still a solution, provided that the sum is finite or that is absolutely convergent together with its partial derivatives up to order two. Computation shows that the initial conditions of the above superposition are

$$\begin{aligned} u(x,0) &= \sum_{n \geq 1} a_n \sin(2\pi x/\lambda_n) \\ \frac{\partial u}{\partial t}(x,0) &= \sum_{n \geq 1} \left(\frac{\pi c n}{\ell}\right) b_n \sin(2\pi x/\lambda_n) \ . \end{aligned}$$

But this gives us a recipe to solve the wave equation whenever the initial conditions are given as superpositions of $\sin(2\pi x/\lambda_n)$.

Exercise. Find solutions of the wave equation

$$\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = 0\,, \qquad \text{with } 0 \leq x \leq \pi\,,$$

with zero boundary conditions, $u(0,t) = 0$ and $u(\pi,t) = 0$, and initial conditions

$$u(x,0) = \sin(3x) \qquad \text{e} \qquad \frac{\partial u}{\partial t}(x,0) = 2\sin(4x)\,,$$

or

$$u(x,0) = 3\sin(x) - \sin(2x) \qquad \text{e} \qquad \frac{\partial u}{\partial t}(x,0) = 7\sin(5x) - 2\sin(6x)\,.$$

Example (dumped vibrations). Real strings in real musical instruments do stop vibrating after a finite time. The simplest way to model this fact is introducing a dumping term in the wave equation, like

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = -\alpha\frac{\partial u}{\partial t}\,.$$

The conjecture $u_n(x,t) = q_n(t)\sin\left(\frac{\pi n}{\ell}x\right)$ implies that $q_n(t)$ satisfies the Newton equation of a dumped oscillator, namely

$$\ddot{q}_n + \omega_n^2 q_n = -\dot{q}_n\,,$$

with resonant frequency $\omega_n^2 = (\pi c n/\ell)^2$. ...

2.4 Waves in 2-dimensional Euclidean space

Example (small vibrations of a membrane). (membrane elástica)

2.5 Waves in 3-dimensional Euclidean space

Example (electromagnetic waves). Maxwell's equations for the electric and magnetic fields E and H, in absence of charges and currents, read

$$\frac{\partial^2 E}{\partial t^2} - c^2\triangle E = 0 \qquad \text{and} \qquad \frac{\partial^2 H}{\partial t^2} - c^2\triangle H = 0$$

where $c \simeq 2.998 \times 10^8$m/s is the speed of light in free space.

Example (non-viscous fluids and acoustic waves). The macroscopic motion of a fluid (a collection of a large number of microscopic molecules) can be described by the following macroscopic observables: a density (scalar) field $\rho(r)$, a velocity (vector) field $v(r)$, and a pressure (scalar) field $p(r)$, where $r = (x,y,z) \in \mathbf{R}^3$ is the Euclidean coordinate in the observer's reference systems. If we disregard viscosity (for otherwise we end up with Navier-Stokes equation, a problem of the millennium!), Newton equations of motion are

$$\frac{dv}{dt} = f - \frac{1}{\rho}\text{grad}(p)\,,$$

where $f = F/\rho$ is an external force field per unit mass. Observe that the time derivative of v is actually $dv/dt = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}\dot{x} + \frac{\partial v}{\partial y}\dot{y} + \frac{\partial v}{\partial z}\dot{z}$, or better $\frac{\partial v}{\partial t} + \langle v|\nabla\rangle\, v$. It must be solved given the continuity equation

$$\frac{\partial \rho}{\partial t} + \text{div}\,(\rho v) = 0$$

(saying that no mass is lost), and an equation of state

$$p = f(\rho)$$

giving the pressure as a function of the density (and of the temperature, but we assume it constant).

First, we assume that the process is adiabatic (i.e. there is no heath exchange, so that the entropy is constant). This implies that we may use the Poisson's equation of state $p/p_0 = (\rho/\rho_0)^\gamma$, where p_0 and ρ_0 are the initial equilibrium pressure and density, and the exponent $\gamma = c_p/c_v$ is the ratio between the constant pressure and constant volume specific heaths. Second, we consider small values of the *condensation* $s = (\rho - \rho_0)/\rho_0$. In first approximation we get the equations

$$\frac{\partial^2 s}{\partial t^2} - c^2 \triangle s = 0 \qquad \text{and} \qquad \frac{\partial^2 u}{\partial t^2} - c^2 \triangle u = 0$$

for both the condensation s (or for the density ρ) and the *velocity potential* u (defined modulo a constant by the identity $v = -\mathrm{grad} u$), where the velocity $c = \sqrt{\gamma p_0/\rho_0}$. For air at usual temperature and pressure, reasonable values are $\gamma \simeq 7/5$, $\rho_0 \simeq 0.001293\mathrm{g/cm}^3$ and $p_0 \simeq 1033\mathrm{g/cm}^3$, so that the sound's speed is $c \simeq 336\mathrm{m/s}$.

Spherical waves. Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \triangle u = 0$$

in Euclidean 3-dimensional space. If we look for a solution which only depends on the radial coordinate $r = \sqrt{x^2 + y^2 + z^2}$, we must solve

$$\frac{\partial^2 u}{\partial t^2} - c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} \right) ,$$

which may be rewritten as

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2}{\partial r^2} (ru) \ .$$

The general solution of the above equation is a superposition of two spherical waves

$$u(r,t) = \frac{1}{r} f(r + ct) + \frac{1}{r} g(r - ct)$$

contracting and expanding around the origin with velocity c.

We now look for solutions of the wave equation which are superpositions of such spherical waves, centered at all points of the Euclidean space. Let S^2 be the unit sphere in $\mathbf{R}^3$, with coordinates (ξ, η, ζ) and area form $d\omega$. Given a function $\varphi(x,y,z)$, we define its *mean value* on the sphere of radius ct centered at (x,y,z) as

$$M_{ct}\left[\varphi\right] = \frac{1}{4\pi} \int_{S^2} \varphi\left(x + ct\xi, y + ct\eta, z + ct\zeta\right) d\omega \,.$$

.. *Huygens' principle*

3 Fourier series

3.1 Complex Fourier series

Fourier series of holomorphic functions. Se $f(z)$ é uma função holomorfa num domínio que contém a circunferência unitária, então a sua expansão em série de Laurent pode ser escrita, nos pontos $z = e^{i\theta}$, como

$$f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \qquad \text{onde} \qquad c_n = \frac{1}{2\pi i} \oint_{|z|=1} \frac{f(z)}{z^{n+1}} dz = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta$$

Complex Fourier series. Em geral, se $f(\theta)$ é uma função integrável em $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ (ou seja, $f : \mathbf{R} \to \mathbf{C}$ é uma função periódica com período 2π), a sua *série de Fourier complexa* é

$$\boxed{f(\theta) \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{in\theta}}$$

(o símbolo "$\sim$" é apenas uma notação!), onde os *coeficientes de Fourier complexos* de $f(\theta)$ são

$$\boxed{\widehat{f}(n) = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta}$$

Riemann-Lebesgue lemma. *If $g(x)$ is an integrable function in some bounded interval $[a, b]$, then the oscillatory integrals*

$$\int_a^b g(x) \cos(Nx) dx \qquad \textit{and} \qquad \int_a^b g(x) \sin(Nx) dx$$

tend to zero as $N \to \infty$.

Proof. If g is continuously differentiable, the result follows from integration by parts. In the general case, you should know that any integrable function may be approximated, in the L_1-norm, by continuously differentiable functions. A standard triangular argument then finishes the proof. **q.e.d.**

Se $f(\theta)$ é uma função seccionalmente de classe $\mathcal{C}^1$, então a sua série de Fourier no ponto θ converge uniformemente para o valor médio $(f(\theta_+) + f(\theta_-))/2$. Em particular, a série de Fourier de uma função $f(\theta) \in \mathcal{C}^1(S^1)$ converge para $f(\theta)$ na norma uniforme, ou seja,

$$\sup_{\theta \in S^1} \left| f(\theta) - \sum_{n=-N}^{N} \widehat{f}(n) e^{in\theta} \right| \to 0 \qquad \text{quando } N \to \infty\,.$$

3.2 Fourier series of square integrable functions

O produto interno e a norma L^2 no espaço $L^2(S^1)$ das funções complexas em $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ com quadrado integrável são definidos por

$$(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \overline{g(\theta)} d\theta \qquad \|f\| = \sqrt{(f, f)}$$

A série de Fourier $f(\theta) \mapsto \hat{f}(n)$ define um isomorfismo de $L^2(S^1)$ em ℓ_2, o espaço das sucessões $(x_n)_{n\in\mathbf{Z}}$ tais que $\sum_{n=-\infty}^{\infty} |x_n|^2 < \infty$, munido do produto interno $(x, y) = \sum_{n=-\infty}^{\infty} x_n \overline{y_n}$. De facto, vale

$$\boxed{(f, g) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) \overline{\widehat{g}(n)}}$$

e a *identidade de Parseval*

$$\boxed{\|f\|^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2}$$

A série de Fourier de uma função $f(\theta) \in L^2(S^1)$ converge para $f(\theta)$ na norma L^2, ou seja,

$$\left\| f(\theta) - \sum_{n=-N}^{N} \widehat{f}(n) e^{in\theta} \right\| \to 0 \qquad \text{quando } N \to \infty\,.$$

- Verifique as relações de ortogonalidade

$$(e^{in\theta}, e^{im\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}e^{-im\theta}d\theta = \begin{cases} 1 & \text{se } n = m \\ 0 & \text{se } n \neq m \end{cases}$$

- Mostre que, se $f(\theta)$ é diferenciável e a derivada $f'(\theta)$ é integrável, então

$$\widehat{f'}(n) = in\hat{f}(n)$$

3.3 Fourier series

Let $f(x)$ periodic integrable function with period 2π. Its *Fourier series (expansion)* is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \ ,$$

where the *Fourier coefficients* of f are defined as

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx \qquad \text{and} \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \, .$$

If f is piecewice $\mathcal{C}^1$, the Fourier series converges to $f(x)$ at the points of continuity, uniformly on bounded intervals where f is continuous.

Fourier series of periodic functions with arbitrary periods. If the period is 2ℓ, we may change variable, replacing x with $\pi x/\ell$,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(\omega_n x) + b_n \sin(\omega_n x)) \ ,$$

where $\omega_n = \frac{\pi n}{\ell}$, and the *Fourier coefficients* of f are now

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos(\omega_n x)dx \qquad \text{and} \qquad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin(\omega_n x)dx \, .$$

Comparison with the complex notation. Let $f(x)$ be an integrable periodic function with period 2ℓ. Its complex Fourier series is

$$f \sim \sum_{n=-\infty}^{\infty} \hat{f}_n e^{ik_n x}$$

where the Fourier coefficients are

$$\hat{f}_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} e^{-ik_n x} f(x)dx$$

and $k_n = \frac{\pi}{\ell}n$. If f is piecewice $\mathcal{C}^1$, the Fourier series converges to $f(x)$ at the points of continuity, uniformly on bounded intervals where f is continuous.

3.4 Pointwise convergence of Fourier series

Let $f(x)$ be a 2π-periodic integrable function, and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

its Fourier series. We pose the question whether the series does converge at a given point x, and what the sum eventually is. The partial sums of the Fourier series of f are the trigonometric polynomials

$$S_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N} (a_n \cos(nx) + b_n \sin(nx)) \ ,$$

so the problem is to understand if, and if so what, limit does $S_N(x)$ have for $N \to \infty$.

Point-wise convergence theorem. *If $f(x)$ is integrable and sectionally $\mathcal{C}^1$, then the partial sums of its Fourier series converge at every point x to the arithmetic mean*

$$\frac{f(x-0)+f(x+0)}{2}$$

of the left and right limits of f at x. In particular, the Fourier series of f converges to $f(x)$ at continuity points of f.

Uniform convergence theorem. *If $f(x)$ is absolutely continuous and its derivative f' is square integrable, then the its Fourier series converges uniformly to f.*

Proof of the point-wise convergence theorem. Using the integral formulas for the coefficients, we get

$$\begin{aligned} S_N(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\left(\frac{1}{2}+\sum_{k=1}^{N} a_n\cos(nx)\cos(ny)+b_n\sin(nx)\sin(ny)\right)dy \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\left(\frac{1}{2}+\sum_{k=1}^{N}\cos(n(y-x))dy\right). \end{aligned}$$

The trigonometric identity

$$\frac{1}{2}+\cos(x)+\cos(2x)+...+\cos(Nx)=\frac{\sin\left(\frac{2N+1}{2}x\right)}{2\sin(x/2)}$$

(which you can prove writing the sum in complex notation and using the formula for the partial sums of a geometric series) implies that we can represent the partial sum of the Fourier series of f as

$$S_n(x)=\int_{-\pi}^{\pi} f(x+y)D_N(y)dy\,,$$

where the *Dirichlet kernel* is defined as

$$D_N(y)=\frac{\sin\left(\frac{2N+1}{2}x\right)}{2\pi\sin(x/2)}\,.$$

Observe that from the above trigonometric identity follows that

$$\int_{-\pi}^{\pi} D_N(y)dy=1$$

for any N. Hence, we may finally write the difference between $f(x)$ and the N-th partial sum of its Fourier series as

$$f(x)-S_N(x)=\int_{-\pi}^{\pi}\left(f(x)-f(x+y)\right)D_N(y)dy\,.$$

Fixed x, we break the integral into two parts,

$$\int_{-\pi}^{0}\left(f(x-0)-f(x+y)\right)D_N(y)dy+\int_{0}^{\pi}\left(f(x+0)-f(x+y)\right)D_N(y)dy$$

If f admits left and right derivatives at the point x, then the functions

$$\frac{f(x-0)-f(x+y)}{y} \qquad \text{and} \qquad \frac{f(x+0)-f(x+y)}{y}$$

are integrable at in their respective domains (this integrability condition, called *Dini's condition*, is the real sufficient condition for the point-wise convergence theorem!). But then both integrals

$$\int_{-\pi}^{0}\frac{f(x-0)-f(x+y)}{y}\,\frac{y}{2\pi\sin(y/2)}\sin\left(\frac{2N+1}{2}y\right)dy$$

and

$$\int_{0}^{\pi}\frac{f(x+0)-f(x+y)}{y}\,\frac{y}{2\pi\sin(y/2)}\sin\left(\frac{2N+1}{2}y\right)dy$$

tends to zero as $N\to\infty$, because of the Riemann-Lebesgue lemma. **q.e.d.**

3.5 Examples of Fourier series

Here we show some computations of simple Fourier series. For simplicity, we only consider functions with period 2π (or, better, the 2π-periodic extensions of functions defined in the interval $-\pi \le x < \pi$). If you want other periods, say 2ℓ, just change variable, from x to $\ell x/\pi$.

Example (linear). Find the Fourier series of $f(x) = x$, defined for $-\pi \le x \le \pi$. Since f is odd, all a_n's are zero. Moreover,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)dx = \frac{2}{\pi}\int_{0}^{\pi} x\sin(nx)dx \\ &= \frac{2}{\pi}\left[\frac{\sin(nx) - nx\cos(nx)}{n^2}\right]_0^{\pi} = 2\frac{\cos(n\pi)}{n} = 2\frac{(-1)^{n+1}}{n} \end{aligned}$$

Hence,

$$\begin{aligned} x &\sim 2\sum_{n\ge 1}\frac{(-1)^{n+1}}{n}\sin(nx) \\ &\sim 2\left(\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \frac{1}{4}\sin(4x) + ...\right). \end{aligned}$$

Example (modulus). Find the Fourier series of $f(x) = |x|$, defined for $-\pi \le x < \pi$. Since f is even, all b_n's are zero. The mean value is

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}|x|dx = \frac{\pi}{2},$$

, and

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos(\pi nx)dx = \frac{2}{\pi}\int_{0}^{\pi} x\cos(nx)dx \\ &= \frac{2}{\pi}\left[\frac{\cos(nx) + nx\sin(nx)}{n^2}\right]_0^{\pi} = \frac{2}{\pi}\frac{\cos(n\pi) - 1}{n^2} = \begin{cases} -\frac{4}{\pi n^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even and positive} \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} |x| &\sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n \text{ odd}}\frac{\cos(nx)}{n^2} \\ &\sim \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(x) + \frac{1}{9}\cos(3x) + \frac{1}{25}\cos(5x) + \frac{1}{49}\cos(7x) + ...\right). \end{aligned}$$

Example (square). Find the Fourier series of $f(x) = x^2$, defined for $-\pi \le x \le \pi$. Since f is even, all b_n's are zero. The mean value is

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}x^2dx = \frac{\pi^2}{3},$$

and other a_n's are

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos(nx)dx = \frac{2}{\pi}\int_{0}^{\pi}x^2\cos(nx)dx \\ &= \frac{2}{\pi}\left[\frac{x^2\sin(nx)}{n}\right]_0^{\pi} - \frac{4}{n\pi}\int_0^{\pi}x\sin(nx)dx = 4\frac{\cos(n\pi)}{n^2} = 4\frac{(-1)^{n-1}}{n^2} \end{aligned}$$

Hence,

$$\begin{aligned} x^2 &\sim \frac{\pi^2}{3} - 4\sum_{n\ge 1}\frac{(-1)^{n-1}}{n^2}\cos(nx) \\ &\sim \frac{\pi^2}{3} - 4\left(\cos(x) - \frac{1}{4}\cos(2x) + \frac{1}{9}\cos(3x) - \frac{1}{16}\cos(4x) + ...\right). \end{aligned}$$

This Fourier series is famous, since it allows to compute the value of the *Riemann's zeta function*

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

at the point $z = 2$, which is (the volume of the unit tangent bundle of the modular orbifold $\mathbb{H}^2/PSL(2,\mathbf{Z})$)

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + ... = \frac{\pi^2}{6} \,.$$

Example (periodic step functions). Find the Fourier series of 2π-periodic extension of the Heaviside function, namely

$$\Theta(x) = \begin{cases} 1 & \text{if } 0 \le x < \pi \\ 0 & \text{if } -\pi \le x < 0 \end{cases}$$

The mean value is

$$\frac{a_0}{2} = \frac{1}{2\pi} \int_0^{\pi} dx = \frac{1}{2} \,,$$

and other a_n's are

$$a_n = \frac{1}{\pi} \int_0^{\pi} \cos(nx) dx = \frac{1}{n\pi} \left[\sin(n\pi)\right]_0^{\pi} = 0 \qquad \text{if } n = 1, 2, 3, ...$$

The b_n's coefficients are

$$b_n = \frac{1}{\pi} \int_0^{\pi} \sin(nx) dx = \frac{1 - \cos(n\pi)}{n\pi} = \begin{cases} \frac{2}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Hence,

$$\begin{aligned} \Theta(x) &\sim \frac{1}{2} + \frac{2}{\pi} \sum_{n \text{ odd}} \frac{\sin(nx)}{n} \\ &\sim \frac{1}{2} + \frac{2}{\pi} \left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + ... \right) . \end{aligned}$$

Also interesting in applications is the odd function

$$2\Theta(x) - 1 = \begin{cases} 1 & \text{if } 0 \le x < \pi \\ -1 & \text{if } -\pi \le x < 0 \end{cases} \,,$$

whose Fourier series is

$$\begin{aligned} 2\Theta(x) - 1 &\sim \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nx)}{n} \\ &\sim \frac{4}{\pi} \left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + ... \right) . \end{aligned}$$

Example (periodic delta functions). The distributional derivative of the Heaviside function is the Dirac delta function (at the origin). Actually, if you formally derive term by term the Fourier series of the periodic $\Theta(x)$, you get

$$\Theta'(x) \sim \frac{2}{\pi} \left(\cos(x) + \cos(3x) + \cos(5x) + \cos(7x) + ...\right) .$$

which may be interpreted as the formal Fourier series of the 2π-periodic extension of the Dirac function $\delta(x)$. In more generality, here we compute the formal Fourier series of the odd and even 2π-periodic extensions of $\delta(x - \alpha)$, for some $0 < \alpha < 2\pi$. A straightforward computation shows that

$$\delta(x - \alpha) - \delta(x + \alpha) \sim \frac{2}{\pi} \sum_{n=1}^{\infty} \sin(n\alpha)\sin(nx)$$

and

$$\delta(x-\alpha)+\delta(x+\alpha) \sim \frac{1}{\pi}+\frac{2}{\pi}\sum_{n=1}^{\infty}\cos(n\alpha)\cos(nx)\,.$$

Example (zig-zag). Find the Fourier series of the zig-zag, the odd 2π-periodic extension of

$$Z(x)=\begin{cases} x & \text{if } 0 \le x < \pi/2 \\ \pi - x & \text{if } \pi/2 \le x < \pi \end{cases}\,.$$

Since Z is odd, all a_n's are zero. The b_n's coefficients are

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi/2} x\sin(nx)dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi-x)\sin(nx)dx \\ &= \frac{4}{\pi n^2}\sin(n\pi/2) = \begin{cases} \frac{2}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} Z(x) &\sim \frac{4}{\pi}\sum_{n \text{ odd}}\frac{(-1)^{(n-1)}}{n}\sin(nx) \\ &\sim \frac{4}{\pi}\left(\sin(x)-\frac{1}{9}\sin(3x)+\frac{1}{25}\sin(5x)-\frac{1}{49}\sin(7x)+...\right)\,. \end{aligned}$$

3.6 Fourier series solutions of the wave equation

Example (playing cavaquinho). When you play your “cavaquinho”, you leave the strings with no appreciable initial velocity and more or less triangular initial shape. To understand the sound, we must solve the suitable wave equation

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=0$$

for a string of length ℓ, with initial displacement given by

$$u(x,0)\simeq\begin{cases} \frac{h}{\alpha}x & \text{se } 0\le x<\alpha \\ \frac{h}{\alpha(\ell-\alpha)}(\ell-x) & \text{se } \alpha\le x<\ell \end{cases}$$

where $0<\alpha<\ell$ is the point where you touch the string, and $h=\varepsilon\alpha$ is the maximal initial displacement, and zero initial velocity $\frac{\partial u}{\partial t}(x,0)=0$. Computing the Fourier coefficients of the odd 2ℓ-periodic extension of the initial displacement,

$$\begin{aligned} b_n &= \frac{2h}{\alpha\ell}\int_0^{\alpha} x\sin\left(\frac{\pi n}{\ell}x\right)dx + \frac{2h}{\ell\alpha(\ell-\alpha)}\int_{\alpha}^{\ell}(\ell-x)\sin\left(\frac{\pi n}{\ell}x\right)dx \\ &= \frac{2h}{\alpha\ell}\left[\frac{\ell^2}{\pi^2 n^2}\sin\left(\frac{\pi n}{\ell}x\right)-\frac{\ell\alpha}{\pi n}\cos\left(\frac{\pi n}{\ell}x\right)\right]_0^{\alpha}+ \\ &\qquad +\frac{2h}{\ell\alpha(\ell-\alpha)}\left[-\frac{\ell^2}{\pi n}\cos\left(\frac{\pi n}{\ell}x\right)-\frac{\ell^2}{\pi^2n^2}\sin\left(\frac{\pi n}{\ell}x\right)+\frac{\ell\alpha}{\pi n}\cos\left(\frac{\pi n}{\ell}x\right)\right]_{\alpha}^{\ell} \\ &= \text{... some cancellations and rewriting ...} \\ &= \frac{2h\ell^2}{\pi^2\alpha(\ell-\alpha)n^2}\,, \end{aligned}$$

we get the amplitude of the n-th excited harmonic

$$A_n=\frac{2h\ell^2}{\pi^2\alpha(\ell-\alpha)}\frac{\sin(n\pi\alpha/\ell)}{n^2}\,.$$

The corresponding energy is

$$E_n=\frac{mh^2\ell^2c^2}{\pi^2\alpha^2(\ell-\alpha)^2}\frac{\sin^2(n\pi\alpha/\ell)}{n^2}\,,$$

where $m = \rho\ell$ is the mass of the string. You see that the intensities of the different harmonics decrease as $1/n^2$, so that the sound is essentially given by the fundamental frequency $\nu_1 = c/2\ell$ and few others.

You can also observe that choosing the point α near a rational multiple of the length ℓ, it is possible to kill, or at least to dump, all multiples of some given harmonic. Of course, our model is but a first and poor approximation, but you may experience some difference in timbre playing a guitar at different heights of its strings.

Example (playing piano). When you play a piano, strings are excited by a hammer driven by the key that you're pressing. In a first approximation, we can imagine that the hammer gives to the string an instantaneous impulse p concentrated at some point $0 < \beta < \ell$ of the string. So, we may try to solve the suitable wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

for a string of length ℓ, with zero initial displacement $u(x, 0) = 0$ and initial velocity given by

$$\frac{\partial u}{\partial t}(x, 0) = \frac{p}{\rho\ell}\delta(x - \beta) \, .$$

Computing the Fourier coefficients of the odd 2ℓ-periodic extension of the Dirac delta function at β, we get the amplitude of the n-th excited harmonic

$$A_n = \frac{2p}{\pi m} \frac{\sin(n\pi\beta/\ell)}{n}$$

The corresponding energy is

$$E_n = \frac{p^2}{m} \sin^2(n\pi\beta/\ell) \, .$$

You see that all the different harmonics have nearly equal intensities, as long as β is not a special point. This is a first explanation for the piano's timbre being more "important" than the one of your cavaquinho. The problem with this first approximation is that it gives the unrealistic value ∞ for the total energy $E = \sum_{n\geq 1} E_n$ of the vibrating string! A more realistic model could be an initial velocity v equally distributed along a portion of length 2δ where the real hammer hits the string. So, we must solve the wave equation with initial velocity

$$\frac{\partial u}{\partial t}(x, 0) \simeq \begin{cases} v & \text{if } |x - \beta| \leq \delta \\ 0 & \text{if } |x - \beta| > \delta \end{cases}$$

The non-zero Fourier coefficients of the odd 2ℓ-periodic extension of the initial velocity are

$$\begin{aligned} b_n &= \frac{2v}{\ell} \int_{\beta-\delta}^{\beta+\delta} \sin\left(\frac{\pi n}{\ell} x\right) dx \\ &= \frac{2v}{\pi n} \left(\cos\left(\frac{\pi n}{\ell}(\beta - \delta)\right) - \cos\left(\frac{\pi n}{\ell}(\beta + \delta)\right)\right) = \frac{4v}{\pi n} \sin(n\pi\beta/\ell) \sin(n\pi\delta/\ell) \, . \end{aligned}$$

The energy of the n-th excited harmonic is then

$$E_n = \frac{4mv^2}{\pi^2} \frac{\sin^2(n\pi\beta/\ell)\sin(n\pi\delta/\ell)}{n^2} \, .$$

The total energy is now finite. Moreover, if δ is much smaller that the string's length ℓ, the approximation $\sin(n\pi\delta/\ell) \simeq n\pi\delta/\ell$ shows that the first harmonics (those for which $n\delta$ is still much smaller that ℓ) still have energy

$$E_n \simeq \frac{(mv2\delta/\ell)^2}{m} \sin^2(n\pi\beta/\ell)$$

(observe that we recover the previous solution for an impulse $p = mv2\delta/\ell$, as it should be!) nearly constant, in accordance with our previous model.

3.7 Fourier series solutions of the heat equation

3.8 Harmonic extensions

Fórmula integral de Poisson no disco unitário. Considere o problema de determinar uma extensão harmónica $f(re^{i\theta})$, ou seja, uma solução da equação de Laplace

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0\,,$$

no disco $D = \{(x, y) \in \mathbf{R}^2 \text{ t.q. } x^2 + y^2 \le 1\} \simeq \{z = re^{i\theta} \in \mathbf{C} \text{ t.q. } |z| = r \le 1\}$ de uma função $g(e^{i\theta})$ definida na circunferência $S^1 = \{(x, y) \in \mathbf{R}^2 \text{ t.q. } x^2 + y^2 = 1\} \simeq \{z = e^{i\theta} \in \mathbf{C} \text{ t.q. } |z| = 1\}$.

- Verifique que $r^{|n|}e^{in\theta}$ é uma extensão harmónica de $e^{in\theta}$.
- Deduza que

  $$f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta} \qquad\qquad re^{i\theta} \in D$$

  é uma extensão harmónica da série de Fourier (suposta convergente)

  $$g(e^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \qquad\qquad e^{i\theta} \in S^1$$

- Use a definição dos coeficientes de Fourier complexos para mostrar que uma extensão harmónica de $g(e^{i\theta})$ é dada pela *fórmula integral de Poisson*

  $$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{i\varphi}) P_r(\theta - \varphi) d\varphi$$

  onde o *núcleo de Poisson* é definido por

  $$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

- Mostre que o núcleo de Poisson admite as seguintes expressões,

  $$\begin{aligned} P_r(\theta) &= 1 + \sum_{n=1}^{\infty} z^n + \sum_{n=1}^{\infty} \overline{z}^n \\ &= 1 + \frac{z}{1-z} + \frac{\overline{z}}{1-\overline{z}} \\ &= \frac{1-|z|^2}{|1-z|^2} \end{aligned}$$

  onde $z = re^{i\theta}$, e portanto

  $$P_r(\theta) = \frac{1-r^2}{1+r^2-2r\cos(\theta)}\,.$$

4 Fourier transform

The Fourier transform is one of the basic tools in analysis. Let's illustrate it with Fourier's original idea.

Heat flow on a infinite rod and Fourier's idea. Here we look for bounded solutions of the heat problem

$$\frac{\partial u}{\partial t} - \beta \frac{\partial^2 u}{\partial x^2} = 0$$

Assuming $u(x,t) = f(x)g(t)$, we get, as usual

$$f'' = \lambda f \qquad \text{and} \qquad g' = -\beta\lambda g$$

for some constant λ. Bounded (complex valued) solutions of the first equation are proportional to

$$f_{\pm\xi}(x) = e^{\pm i\xi x}$$

where the parameter ξ now can take any value in $\mathbf{R}$, and $\lambda = -\xi^2$. The second equation then gives

$$g_{\pm\xi}(t) = e^{-\xi^2 t} g_{\pm\xi}(0)$$

so that we are left with the one-parameter family of separable solutions

$$u_\xi(x,t) = e^{-\xi^2 t}\left(g(\xi)e^{i\xi x} + g(-\xi)e^{-i\xi x}\right)$$

for some constants $g(\xi) = g_\xi(0)$. We are lead to try a solution as an integral $\int_0^\infty u_\xi(x,t)d\xi$ over all these possible separable solutions, hence as

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi) e^{-\xi^2 t} e^{i\xi x} d\xi$$

where we inserted a factor $1/\sqrt{2\pi}$ which will simplify some future formulae. Apart from convergence issues, the above is a solution of the heat equation with initial condition

$$u(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi) e^{i\xi x} d\xi$$

as you may prove deriving twice.

Now, following Fourier, we ask: "when is it possible to write such an integral formula

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi) e^{i\xi x} d\xi$$

for a generic function of $f(x)$ defined on the real line? If so, what is the function $g(\xi)$?"

As with Fourier series, the second question is much easier. If we integrate the product $f(x)\overline{e^{i\xi x}}$ over the real line we get, formally (i.e. assuming everything converge and commute!),

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\overline{e^{i\xi x}} dx &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\lim_{\ell\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\ell}^{\ell} g(\eta) e^{i\eta x} d\eta \right) e^{-i\xi x} dx \\
&= \lim_{\ell\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\ell}^{\ell} e^{i(\eta-\xi)x} dx \right) g(\eta) d\eta \\
&= \lim_{\ell\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin((\xi-\eta)\ell)}{\xi-\eta} g(\eta) d\eta \\
&= \lim_{\ell\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(t)}{t} g(\xi - t/\ell) dt \\
&= g(\xi) \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(t)}{t} dt \\
&= g(\xi)
\end{aligned}$$

So, the answer is that $g(\xi)$ must be the integral

$$g(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$$

4.1 Fourier transform of integrable functions

The space $L^1(\mathbf{R})$ das *funções integráveis* na recta real é o espaço das funções $f(x)$ tais que, equipped with the norm

$$\|f\|_{L^1} = \int_{-\infty}^{\infty} |f(x)| dx < \infty$$

A *transformada de Fourier* da função integrável $f(x)$ é a função $\mathcal{F}\{f(x)\}(\xi) = \widehat{f}(\xi)$ definida pelo integral impróprio

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx\,.$$

The existence of the above integral follows from the estimate

$$\left| \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx \right| \leq \int_{-\infty}^{\infty} |f(x)| dx = \|f\|_{L^1}$$

Riemann-Lebesgue lemma. *The Fourier transform of a function $f \in L^1(\mathbf{R})$ is continuous, bounded and $\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$.*

Proof.

Elementary properties. The Fourier transform is linear, i.e.

$$\mathcal{F}\{\lambda f(x) + \mu g(x)\}(\xi) = \lambda \widehat{f}(\xi) + \mu \widehat{g}(\xi)$$

and behaves as follows under conjugation and dilatations:

$$\mathcal{F}\{\overline{f}(x)\}(\xi) = \overline{\widehat{f}(-\xi)} \qquad\qquad \mathcal{F}\{f(\lambda x)\}(\xi) = \frac{1}{\lambda} \widehat{f}(\xi/\lambda)$$

Translations $x \mapsto x - a$ are sent into multiplications by the character $e^{-ia\xi}$,

$$\mathcal{F}\{f(x-a)\}(\xi) = e^{-ia\xi} \widehat{f}(\xi) \qquad\qquad \mathcal{F}\{e^{ibx} f(x)\}(\xi) = \widehat{f}(\xi - b)$$

Exercise. Calcule a transformada de Fourier de $1_{[-\ell,\ell]}$, a função característica do intervalo $[-\ell, \ell]$. Show that

...

Calcule, usando a técnica dos resíduos, a transformada de Fourier de

$$f(x) = \frac{1}{x^2 + a^2} \qquad \text{e} \qquad f(x) = e^{-a|x|}\,.$$

Example (Gaussian and heat kernel). Show that the Fourier transform of the Gaussian

$$g(x) = e^{-x^2/2} \qquad \text{is} \qquad \widehat{g}(\xi) = e^{-\xi^2/2}\,,$$

In other words, the Gaussian is a fixed point of the Fourier transform, a first evidence of its importance. Deduce that the Fourier transform of the heat kernel

$$P_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \qquad \text{is} \qquad \widehat{P}(\xi) = \frac{1}{\sqrt{2\pi}} e^{-t\xi^2/2}\,,$$

Point-wise converge of the Fourier transform.

4.2 Fourier transform on the Schwartz space

The Schwartz space. O *espaço de Schwartz* $\mathcal{S} = \mathcal{S}(\mathbf{R})$ é o espaço das funções $f(x)$ infinitamente diferenciáveis na recta real que decrescem $|f(x)| \to 0$ quando $x \to \pm\infty$, com todas as suas derivadas, mais rápido que o inverso de qualquer polinómio. Ou seja, o espaço das funções $f(x)$ tais que $\forall\ \alpha, \beta \in \mathbf{N}_0$

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbf{R}} \left| x^\alpha \frac{\partial^\beta f}{\partial x^\beta}(x) \right| < \infty\,.$$

Examples are $p(x)e^{-x^2}$, where $p(x)$ is a polynomial.

The Schwartz space is a locally convex topological vector space, with the natural topology defined by the system of seminorms $\| \cdot \|_{\alpha,\beta}$.

Fourier transform on the Schwartz space. Mostre que, se $f \in \mathcal{S}$,

$$\widehat{\left(\frac{df}{dx}\right)}(\xi) = i\xi \widehat{f}(\xi) \qquad \text{e} \qquad \widehat{(-ixf)}(\xi) = \frac{d}{d\xi}\widehat{f}(\xi)$$

Define the *inverse Fourier transform* $\mathcal{F}^{-1} : \mathcal{S} \to \mathcal{S}$, sending $g \mapsto \check{g}$, as

$$\check{g}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi) e^{i\xi x} d\xi$$

Fourier inversion theorem. *The Fourier transform is a linear bijection $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ of the Schwartz space, which inverse is the inverse Fourier transform $g \mapsto \check{g}$. In particular, the Fourier inversion formula holds,*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y) e^{-i\xi y} dy \right) e^{i\xi x} d\xi \,.$$

Riesz's proof. First observe that, for $f, g \in \mathcal{S}$,

$$\begin{aligned}
\int g(\xi)\widehat{f}(\xi)e^{i\xi x}d\xi &= \frac{1}{\sqrt{2\pi}} \int g(\xi) \left(\int_{-\infty}^{\infty} f(y)e^{-i\xi y}dy \right) e^{i\xi x}d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int f(y) \left(\int_{-\infty}^{\infty} g(\xi)e^{-i\xi(y-x)}d\xi \right) dy \\
&= \int f(y)\widehat{g}(y-x)dy \\
&= \int f(x+y)\widehat{g}(y)dy
\end{aligned}$$

Now, take $g(\xi) = e^{-\xi^2/2}$, so that $g(\sqrt{t}\xi)$ is the Fourier inverse transform of the heat kernel $\frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}$. Indeed, $\widehat{g}(y) = e^{-y^2/2}$ and $\int \widehat{g}(y)dy = \sqrt{2\pi}$. We get, for $t > 0$,

$$\begin{aligned}
\int g(t\xi)\widehat{f}(\xi)e^{i\xi x}d\xi &= \int f(x+y)t^{-1}\widehat{g}(y/t)dy \\
&= \int f(x+ty)\widehat{g}(y)dy
\end{aligned}$$

and, letting $t \to 0$, this gives

$$g(0) \int \widehat{f}(\xi)e^{i\xi x}d\xi = f(x)\sqrt{2\pi}$$

Since $g(0) = 1$, this is the Fourier inversion formula. **q.e.d.**

Produto de convolução. The *convolution product* de $f(x)$ e $g(x)$ é a função $(f * g)(x)$ definida por

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy$$

It is easy to see that $f * g \in \mathcal{S}$ whenever $f, g \in \mathcal{S}$. Moreover, Fubini theorem and change of variable show that the following basic identity holds:

$$\widehat{(f * g)}(\xi) = \sqrt{2\pi}\hat{f}(\xi)\hat{g}(\xi)$$

This says that the Fourier transform sends convolution products into ordinary products in the frequency space.

Example (convolutions in probability) If X and Y are two independent discrete random variables taking integers values with discrete densities $p_X(n)$ and $p_Y(n)$ respectively, then their sum $X+Y$ has density

$$p_{X+Y}(n) = \sum_k p_X(k)p_Y(n-k)\,.$$

The obvious generalization of this formula for absolutely continuous random variables says that the density of the sum $X+Y$ is the convolution product of the two densities, namely

$$f_{X+Y}(t) = \int_{-\infty}^{\infty} f_X(s)f_Y(t-s)ds\,.$$

Exercise (Brownian motion) Considere o *núcleo de Poisson*

$$P_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

This is the probability density of a Brownian motion W_t, namely

$$\mathbf{P}\left(W_t \in A \,|\, W_0 = 0\right) = \int_A P_t(x)dx\,.$$

Show that $P_t * P_s = P_{t+s}$ and interprete this fact.

Parseval identities. Se $f \in \mathcal{S}$, então vale a *identidade de Parseval*

$$\boxed{\|f\|_2 = \|\widehat{f}\|_2 \qquad \text{ou seja,} \qquad \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2 d\xi}$$

O *teorema de Plancherel* afirma que a transformada de Fourier extende a uma transformação unitária $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ do espaço de Hilbert $\mathcal{H} = L^2(\mathbf{R})$ das funções de quadrado integrável[3], cuja inversa é a própria adjunta, $\mathcal{F}^{-1} = \mathcal{F}^*$.

- Calcule a norma L^2 da gaussiana $e^{-x^2/2}$.

Equação e funções de Hermite. As *funções de Hermite* $\phi_n(x) = e^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2}\right)$, soluções da *equação de Hermite*

$$\frac{d^2}{dx^2}\phi_n(x) - x^2\phi_n(x) = \lambda_n \phi_n(x)$$

com $\lambda_n = -(2n+1)$, formam uma base ortogonal do espaço de Hilbert $L^2(\mathbf{R})$.

- Mostre que a transformada de Fourier transforma a equação diferencial

$$\frac{d^2}{dx^2} f(x) - x^2 f(x) = \lambda f(x) \qquad \text{em} \qquad \frac{d^2}{d\xi^2}\widehat{f}(\xi) - \xi^2\widehat{f}(\xi) = \lambda\widehat{f}(\xi)\,.$$

- Verifique que as funções de Hermite $\phi_n(x)$ são funções próprias, de valores próprios $(-i)^n$, da transformada de Fourier, ou seja

$$\mathcal{F}\{\phi_n\} = (-i)^n \phi_n\,.$$

The Poisson summation formula. Dada uma função $f(x) \in \mathcal{S}(\mathbf{R})$, seja

$$f_{2\pi\mathbf{Z}}(x) = \sum_{n\in\mathbf{Z}} f(x + 2\pi n)$$

Então $f_{2\pi\mathbf{Z}}(x)$ é uma função de classe $\mathcal{C}^\infty$, periódica de período 2π, e como tal admite uma expansão em série de Fourier

$$f_{2\pi\mathbf{Z}}(x) = \sum_{n\in\mathbf{Z}} e^{inx}\widehat{f_{\mathbf{Z}}}(n)$$

com coeficientes

$$\widehat{f_{2\pi\mathbf{Z}}}(n) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{-inx} f_{\mathbf{Z}}(x)dx$$

- Mostre que

$$\begin{aligned}
\widehat{f_{2\pi\mathbf{Z}}}(n) &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{-inx}\left(\sum_{m\in\mathbf{Z}} f(x+2\pi m)\right)dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-inx} f(x)dx \\
&= \widehat{f}(n)\,.
\end{aligned}$$

[3] O espaço $\mathcal{H} = L^2(\mathbf{R})$ das *funções de quadrado integrável* é o espaço das funções $f(x)$ tais que

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$$

- Deduza a *fórmula do somatório de Poisson*

$$\sum_{n\in\mathbf{Z}} f(2\pi n) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbf{Z}} \widehat{f}(n)$$

Poisson formula as a "trace formula". The Laplacian $\Delta = -\partial^2/\partial x^2$ on the circle $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ has eigenvalues $\lambda_n = n^2$, with $n = 0, 1, 2, 3, ...$ Indeed, its eigenfunctions are

$$\Delta e^{inx} = n^2 e^{inx}$$

4.3 Distributions and their Fourier transform

Distribuições. O espaço das *distribuições (temperadas)* $\mathcal{S}'$ é o espaço dual do espaço de Schwartz $\mathcal{S}$, o espaço dos funcionais lineares contínuos $T : \mathcal{S} \to \mathbf{C}$. A derivada da distribuição T é a distribuição T' definida pela identidade

$$T'(f) = -T(f') \qquad \forall f \in \mathcal{S}\,.$$

- Dada $g(x) \in \mathcal{S}$, seja T_g a distribuição induzida, definida por $T_g(f) = \int_{-\infty}^{\infty} f(x)g(x)dx$. Mostre que $(T_g)' = T_{g'}$.

- A *delta de Dirac* é a distribuição δ definida por

$$\delta(f) = \int_{-\infty}^{\infty} \delta(x)f(x)dx = f(0)$$

  (o integral é apenas uma notação, $\delta(x)$ não é uma função!). A delta de Dirac em $x_0 \in \mathbf{R}$ é a distribuição δ_{x_0} definida por $\delta_{x_0}(f) = \int_{-\infty}^{\infty} \delta(x-x_0)f(x)dx = f(x_0)$. Calcule as derivadas de δ_{x_0}.

- A *distribuição de Heaviside* é a distribuição $H = T_{u_0}$ definida por

$$H(f) = \int_{-\infty}^{\infty} u_0(x)f(x)dx = \int_0^{\infty} f(x)dx$$

  onde a *função de salto unitário* (ou *função de Heaviside*) é definida por

$$u_{x_0}(x) = \begin{cases} 0 & \text{se } x < x_0 \\ 1 & \text{se } x \geq x_0 \end{cases}\,.$$

  Calcule as derivadas de H.

- Considere o *núcleo de Poisson*

$$P_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

  Verifique que $P_t(x) \to \delta(x)$ quando $t \downarrow 0$, ou seja, que, se $f(x)$ é contínua e limitada,

$$\lim_{t\downarrow 0} \int_{-\infty}^{\infty} P_t(y)f(y)dy = \delta(f) = f(0)$$

Transformada de Fourier das distribuições. Se $T \in \mathcal{S}'$ é uma distribuição, a sua transformada de Fourier é a distribuição $\widehat{T}$ definida pela identidade

$$\widehat{T}(f) = T(\widehat{f}) \qquad \forall f \in \mathcal{S}\,.$$

- Verifique que se $g(x)$ é uma função integrável e T_g é a distribuição induzida, definida por $T_g(f) = \int_{-\infty}^{\infty} f(x)g(x)dx$, então $\widehat{T_g} = T_{\widehat{g}}$.

- Calcule a tranformada de Fourier da distribuição T_1, induzida pela função constante $g(x) = 1$.

- Calcule a transformada de Fourier da delta de Dirac δ.

- Calcule a transformada de Fourier da distribuição de Heaviside H.

4.4 Fourier transform on L^2

The Hilbert space of square integrable functions. O produto interno e a norma L^2 são definidos por

$$\boxed{(f,g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx \qquad \|f\|_2 = \sqrt{(f,f)} = \sqrt{\int_{-\infty}^{\infty} |f(x)|^2 dx}}$$

Plancherel's theorem *The Fourier transform extends to a unitary linear map $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ of the Hilbert space $\mathcal{H} = L^2(\mathbf{R})$, whose inverse is its adjoint $\mathcal{F}^{-1} = \mathcal{F}^*$.*

Proof. A function $f \in \mathcal{H}$ defines a distribution $T_f \in \mathcal{S}'$. Take its Fourier transform $\widehat{T_f}$. By the Schwarz inequality

$$\left|\widehat{T_f}(\varphi)\right| = |T_f(\widehat{\varphi})| = \left|\int f(x)\widehat{\varphi}(x)dx\right| \leq \|f\|_2 \cdot \|\varphi\|_2$$

The Riesz's representation theorem says that there exists a unique $\widehat{f} \in \mathcal{H}$ such that

$$\widehat{T_f}(\varphi) = (\widehat{f}, \varphi)$$

for any $\varphi \in \mathcal{H}$.

4.5 Fourier transform solutions of heat and wave equations

Difusão na recta e núcleo de Poisson. Considere a equação de calor

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

na recta, com condição inicial $u(x,0) = u_0(x)$. Se $u(x,t)$ é suficientemente regular, a sua transformada de Fourier (apenas na variável x),

$$\widehat{u}(\xi,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} u(x,t)dx$$

satisfaz a equação diferencial

$$\frac{\partial \widehat{u}}{\partial t} = -\xi^2 \widehat{u} \qquad \text{com condição inicial} \qquad \widehat{u}(\xi,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} u_0(x)dx = \widehat{u_0}(\xi)$$

A solução é

$$\widehat{u}(\xi,t) = e^{-\xi^2 t}\widehat{u_0}(\xi)$$

- Verifique que $\frac{1}{\sqrt{2\pi}} e^{-\xi^2 t}$ é a transformada de Fourier do *núcleo de Poisson* (ou *heat kernel*)

$$P_t(x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$$

- Deduza que a solução da equação de calor na recta com condição inicial $u_0(x)$ é

$$\begin{aligned} u(x,t) &= (P_t * u_0)(x) \\ &= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} u_0(y)dy \end{aligned}$$

Movimento Browniano. No modelo do movimento Browniano proposto por Einstein em 1905[4], a densidade de probabilidade $P(x,t)$ de encontrar a partícula Browniana na posição x no tempo t sabendo que ela estava na posição 0 no tempo 0 é a solução não-negativa da equação da difusão

$$\frac{\partial P}{\partial t} - \beta \frac{\partial^2 P}{\partial x^2} = 0$$

tal que $\lim_{t\to 0} P(x,t) = 0$ para todo o $x \neq 0$, e $\int_{-\infty}^{\infty} P(x,t)dx = 1$ para todo o tempo $t > 0$. O "coeficiente de difusão" é $\beta = \frac{RT}{N\alpha}$, onde R é a constante de gás perfeito, T a temperatura absoluta, N o número de Avogadro, e $\alpha = 6\pi\eta\rho$ um coeficiente de fricção (que depende da viscosidade dinâmica η do líquido e do raio ρ da partícula Browniana).

[4] A. Einstein, Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen, *Ann. Phys.* **17**, 549, 1905. Traduzido em A. Einstein, *Investigations on the Theory of Brownian Movement*, Dover, New York, 1956.

- Verifique que a Gaussiana
$$P_t(x) = \frac{1}{2\sqrt{\pi\beta t}} e^{-x^2/(4\beta t)} .$$
resolve o problema do movimento Browniano.

- Verifique que
$$P_{t+s}(x) = \int_{-\infty}^{\infty} P_t(y) P_s(x-y) dy$$
e interprete este facto.

- Calcule o caminho quadrático médio da partícula Browniana no tempo t, definido por
$$\langle x(t)^2 \rangle = \int_{-\infty}^{\infty} x^2 P_t(x) dx .$$

Exercise: equação de calor com dissipação. Use a transformada de Fourier para achar a solução formal do problema
$$\frac{\partial u}{\partial t} - \beta \frac{\partial^2 u}{\partial x^2} = -\alpha u$$
com condição inicial $u(x,0) = f(x)$ contínua e limitada.

Exercise: equação de ondas com dissipação. Use a transformada de Fourier para achar a solução formal do problema
$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = -\alpha u$$
com condições iniciais $u(x,0) = f(x)$ e $\frac{\partial u}{\partial t}(x,0) = g(x)$ contínuas e limitadas.

4.6 Kernels for the Laplace equation

Poisson formula in the unit disk.

Fórmula integral de Poisson no semi-plano. Considere o problema de determinar uma extensão harmónica $f(x,y)$, ou seja
$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$
no semi-plano superior $\mathbf{H} = \{(x,y) \in \mathbf{R}^2 \text{ t.q. } y > 0\}$ de uma função $g(x)$ definida na fronteira $\partial \mathbf{H} = \{(x,y) \in \mathbf{R}^2 \text{ t.q. } y = 0\}$. Se $f(x,y)$ é suficientemente regular, a sua transformada de Fourier (apenas na variável x),
$$\widehat{f}(\xi, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f(x,y) dx$$
satisfaz a equação diferencial
$$-\xi^2 \widehat{f} + \frac{\partial^2 \widehat{f}}{\partial y^2} = 0 \qquad \text{com condição inicial} \qquad \widehat{f}(\xi, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} g(x) dx = \widehat{g}(\xi)$$
A solução limitada em $y \geq 0$ é
$$\widehat{f}(\xi, y) = e^{-|\xi| y}\, \widehat{g}(\xi)$$

- Verifique que $\frac{1}{\sqrt{2\pi}} e^{-|\xi| y}$ é a transformada de Fourier do *núcleo de Poisson* em $\mathbf{H}$
$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2}$$

- Deduza que a extensão harmónica de $g(x)$ em $\mathbf{H}$ é
$$\begin{aligned} f(x,y) &= (P_y * g)(x) \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-s)^2 + y^2} g(s) ds \end{aligned}$$

Conformal transformations.

4.7 Quantum mechanics

Oscilador harmónico quântico. Considere a equação de Schrödinger

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{2}m\omega^2x^2\Psi$$

para a função de onda $\Psi(x,t) \in \mathbf{C}$ de uma partícula de massa m num potencial $V(x) = \frac{1}{2}m\omega^2x^2$, onde $\hbar$ é a constante de Planck reduzida. Determine as soluções separáveis, ou seja do género

$$\Psi_n(x,t) = e^{-i\frac{E_n t}{\hbar}} f_n(x)\,,$$

com $f_n(x) \in L^2(\mathbf{R})$, e mostre que os "níveis de energia" são

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right) \qquad \text{com} \qquad n = 0,1,2,3,...$$

4.8 Characteristic functions and the central limit theorem

Função característica e teorema limite central. A *função característica* de uma variável aleatória real X, com lei $\mathbf{P}$, é a média/esperaça da variável $e^{i\xi X}$, com $\xi \in \mathbf{R}$, ou seja,

$$\phi_X(\xi) = \mathbf{E}\, e^{i\xi X} = \int e^{i\xi x}\mathbf{P}(dx)\,.$$

Em particular, se X é absolutamente contínua com lei $\mathbf{P}(X \in A) = \int_A p(x)dx$, então a função característica $\phi_X(\xi)$ é a transformada de Fourier da densidade de X, pois

$$\phi_X(\xi) = \int_{-\infty}^{\infty} e^{i\xi x}p(x)dx = \sqrt{2\pi}\cdot\widehat{p}(\xi)\,.$$

O *teorema de Lévy* afirma que uma sequência de variáveis aleatórias (X_n) converge "em lei"[5] para a variável X quando $n\to\infty$, i.e. $X_n \to^{\mathcal{L}} X$, sse $\phi_{X_n}(\xi) \to \phi_X(\xi)$ para cada $\xi \in \mathbf{R}$.

- Verifique que as derivadas da função característica na origem são

  $$\phi_X(0) = 1 \qquad \phi_X'(0) = i\mathbf{E}X \qquad \phi_X''(0) = -\mathbf{E}X^2 \qquad ...$$

  (desde que os momentos da variável X sejam finitos).

- Mostre que, se a variável aleatória X tem média $\mathbf{E}X = m$ e variância $\mathbf{V}X = \mathbf{E}(X-m)^2 = \sigma^2$, e se $Y = \frac{X-m}{\sigma}$, então[6]

  $$\phi_Y(\xi) = 1 - \frac{1}{2}\xi^2 + o(\xi^2)$$

  numa vizinhança da origem.

- Sejam X_1, X_2, X_3, ... variáveis aleatórias independentes e identicamente distribuídas, com média m e variância σ^2, seja $S_n = X_1 + X_2 + ... + X_n$, e seja

  $$S_n^* = \frac{S_n - nm}{\sigma\sqrt{n}} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{X_i - m}{\sigma}$$

  Verifique que

  $$\phi_{S_n^*}(\xi) = \left(\phi_Y(\xi/\sqrt{n})\right)^n$$

  onde $Y = \frac{X-m}{\sigma}$, e portanto

  $$\phi_{S_n^*}(\xi) \to e^{-\xi^2/2} \qquad \text{quando } n\to\infty$$

[5] *Convergência em lei:* $X_n \to^{\mathcal{L}} X$ se, para cada função $f(x)$ contínua e limitada, $\mathbf{E}f(X_n) \to \mathbf{E}f(X)$, ou seja, se

$$\lim_{n\to\infty}\mathbf{P}(X_n \le x) = \mathbf{P}(X \le x)\,.$$

para cada ponto de continuidade x da função de repartição de X.

[6] Notação de Landau: $f(\xi) = o(\xi^k)$ quando $\xi \to 0$ significa que $\lim_{\xi\to 0} f(\xi)/\xi^k = 0$.

- Deduza o *teorema limite central*, que afirma que a sequência de variáveis (S_n^*) converge em lei para uma variável normal $N(0,1)$, ou seja,

$$\mathbf{P}\left(S_n^* \leq x\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt\,.$$

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