Lecture notes on “Análise Matemática 3”
Laplace transform

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Abstract

This is not a book! These are personal notes written while preparing lectures on “Análise Matemática 3” for students of FIS in the a.y. 2007/08 and then 2009/10. They are based on previous notes on “Complementos de Análise Matemática” for students of ENGSI, FIS, FQ(E), QP and QT. They are rather informal and may even contain mistakes. I tried to be as synthetic as I could, without missing the observations that I consider important.

I probably will not lecture all I wrote, and did not write all I plan to lecture. So, I included empty or sketched paragraphs, about material that I think should/could be lectured within the same course.

References contain some introductory manuals, some classics, and other books where I have learnt things in the past century. Besides, good material and further references can easily be found on the web, for example in Wikipedia.

Pictures were made with “Grapher” on my MacBook.

Contents

1 Laplace transform and applications to linear ODEs

The Laplace transform is a powerful and deep tool which exploits the superposition principle to find solutions of a linear differential equation with constant coefficients. It gives a particularly enlightening description of the response of a linear system to a given input. It is just the top of an iceberg, a long story which includes the Fourier transform, the Mellin transform, the Radon transform, ...

Solutions of a homogeneous differential equations with constant coefficients $Lx = 0$ are, at least when the characteristic polynomial has no repeated roots, linear combinations of exponentials $e^{z_i t}$, for some complex frequencies z_i's. Moreover, when the r.h.s. of a non-homogeneous linear differential equation with constant coefficients $Lx = f$ is a superposition of exponentials, a particular solution is again a superposition $\sum_i c_i e^{z_i t}$ of exponentials, for some coefficients c_i determined by the equation (using the method of undetermined coefficients). All this suggests to look for solutions of a linear ODE with constant coefficients inside the space of linear combinations of exponentials. The space of finite linear combinations of exponentials being too small, we may try infinite linear combinations, namely integrals. So, we make the conjecture

$$x(t) = \frac{1}{2\pi i} \int_C X(z) e^{zt} dz \,.$$

Above, the function $X(z)$ plays the role of the coefficients, the factor $1/2\pi i$ is a normalization, and the integral is over some line C in the complex plane. Assume that the r.h.s. also admits an expression like

$$f(t) = \frac{1}{2\pi i} \int_C F(z) e^{zt} dz \,.$$

The above conjecture gives, thanks to linearity and provided we may pass the derivatives inside the integral,

$$Lx = r \qquad \Rightarrow \qquad \frac{1}{2\pi i} \int_C P(z) X(z) e^{zt} dz = \frac{1}{2\pi i} \int_C F(z) e^{zt} dz \,,$$

where $P(z)$ is the characteristic polynomial associated to the differential operator L (and this observation reveals the true meaning of the characteristic polynomial!). The above identity suggests that, if we take $X(z) = F(z)/P(z)$, then the above formula gives a (at least formal) solution $x(t)$ of the linear differential equation. We may resume saying that the Laplace transform translates a problem formulated in the time domain, in our case solving a differential equation, into a problem formulated in the frequency domain, which in our case is an algebraic problem (hence much simpler!). Observe also that, in the frequency domain, the response $X(z)$ of the system takes the simple form of a product of the input $F(z)$ times a proportionality factor $H(z) = 1/P(z)$, called transfer function of the system. Back to the time domain, this will produce a nice integral formula for the response $x(t)$ which displays its causal dependence from the values $f(\tau)$, with $\tau \leq t$, of input force.

What is left is to give a meaning to the above integrals. The function $X(z)$ will be called Laplace transform of the function $x(t)$ (after Pierre Laplace, who first used this device in probability, where it goes under the name of generating function of moments), and consequently the function $x(t)$ will be called inverse Laplace transform of the function $X(z)$.

The Laplace transform is an instance of a duality, time versus frequency (that is, energy), which is a central theme both in Newtonian and in quantum physics.

1.1 Laplace transform

First, we must define a class of functions that are allowed to have a Laplace transform. To simplify things, we'll only consider

Piecewise continuous functions. A real valued function $f(t)$, defined in a bounded interval $[a, b]$, is said *piecewise continuous* if there exist $a = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = b$ such that

- f is continuous in any open subinterval $]t_i, t_{i+1}[$,
- and admits finite lateral limits $f\left(t_i^\pm\right) = \lim_{t \to t_i^\pm} f(t)$ at all points t_i's.

A real function f, defined in an interval I of the real line, is said *piecewise continuous* if it is piecewise continuous once restricted to any bounded subinterval $[a, b] \subset I$.

What will be important, for us, is that piecewise continuous function are Riemann integrable in any bounded subinterval of their domains.

Laplace transform (for causal systems). Let $f : [0, \infty) \to \mathbf{R}$ be a piecewice continuous function. The *Laplace transform* of f is the function $F(z)$ defined as the improper integral

$$F(z) = \int_0^\infty e^{-zt} f(t) dt \,,$$

for z in some region of the complex plane $\mathbf{C}$, called *Region of Convergence* (or simply *R.o.C.*), where the integral is absolutely convergent.

The traditional notation for the Laplace transform of f is

$$F(z) = \mathcal{L}\{f\}(z) \qquad \text{or} \qquad F(z) = \mathcal{L}\{f(t)\}(z) \,,$$

if it s important to remind the independent variable of f. You should think at the Laplace transform as an operator $\mathcal{L}$ sending certain functions (for us, piecewice continuous) of time t into functions of the "complex frequency" z.

It is also possible to define a *two-sided Laplace transform* as the integral

$$\int_{-\infty}^\infty e^{-zt} f(t) dt \,.$$

It is suited to treat non-causal problems, as differential equations where the independent variable does not have the interpretation of time but of some other observable ...

There are no general rules to decide what the R.o.C. of the Laplace transform is. Actually, it will also be useful to define the Laplace transform of objects that are not even functions. Meanwhile, we get a simple answer for the case of

Laplace transform of functions with exponential growth. A function $f : [0, \infty) \to \mathbf{R}$ has *exponential growth* if there exist non-negative constants M and m such that

$$|f(t)| \leq M e^{mt}$$

for any sufficiently large t (say bigger than some t_0). The infimum of those m's for which the inequality above is satisfied is called *exponential order* of the function f. Let $\mathcal{E}$ denotes the space of piecewise continuous functions of exponential growth.

Examples of function with exponential growth are polynomials, trigonometric functions, exponentials, and products between them. That is, most of the functions you are used to see in applications. A function which is not of exponential growth is e^{e^t}.

Let $f : [0, \infty) \to \mathbf{R}$ be a piecewice continuous function with exponential growth and exponential order m. The Laplace transform of f,

$$F(z) = \int_0^\infty e^{-zt} f(t) dt \,,$$

is absolutely convergent for $\Re(z) > m$, because of the bound

$$|e^{-zt} f(t)| \leq M e^{-(\Re(z) - m - \varepsilon)t} \,,$$

for any positive $\varepsilon > 0$. Hence, its R.o.C. is the half-plane $\{z \in \mathbf{C} \, \text{s.t.} \, \Re(s) > m\}$. Moreover, you may notice that $F(z) \to 0$ for $\Re(z) \to \infty$, uniformly in $\Im(z)$. Actually, it follows from general principles that $F(z)$ is a holomorphic function in the region of convergence.

For the application we have in mind, it will be sufficient to consider only real values of the frequency z, that we'll call, following the tradition, s.

Derivatives of the Laplace transform. The Laplace transform of a piecewise continuous function with exponential growth is actually infinitely differentiable in its R.o.C.. Indeed, its derivatives are

$$\begin{aligned}
F'(s) &= \int_0^\infty (-t) e^{-st} f(t) dt \,, \\
F''(s) &= \int_0^\infty t^2 e^{-st} f(t) dt \,, \\
&\vdots \qquad \vdots \\
\frac{d^n F}{ds^n}(s) &= \int_0^\infty (-1)^n t^n e^{-st} f(t) dt \,,
\end{aligned}$$

and they are absolutely convergent in $f(t)$'s R.o.C. because the $t^n f(t)$'s have the same exponential order that $f(t)$. This formula is the key of Laplace use of the transformation nowadays bearing his name, originating in probability and later incorporated in the modern formalism of statistical mechanics.

Elementary properties of the Laplace transform. The Laplace transform is linear, namely

$$\mathcal{L}\{\lambda f\} = \lambda \mathcal{L}\{f\} \qquad \text{and} \qquad \mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\} ,$$

for any piecewise continuous functions f and g with exponential growth, and any constant λ, in a region where both the Laplace transforms are defined.

The Laplace transform is also well behaved under homotheties in the time-space, since

$$\mathcal{L}\{f(\lambda t)\}(s) = \frac{1}{\lambda}\mathcal{L}\{f\}(s/\lambda) \qquad \text{for } s > \lambda m ,$$

for any piecewise continuous function f with exponential order m and any positive λ, and under translations in the frequency-space, since

$$\mathcal{L}\{f\}(s-k) = \mathcal{L}\left\{e^{kt} f(t)\right\}(s) \qquad \text{for } s > k+m ,$$

for any piecewise continuous function f with exponential order m and any $k \in \mathbf{R}$.

Examples (Laplace transform of elementary functions). Here we collect the Laplace transform of simple functions useful for applications.

Computations and induction show that

$$\mathcal{L}\{1\}(s) = \frac{1}{s} \qquad \mathcal{L}\{t\}(s) = \frac{1}{s^2} \qquad \ldots \qquad \mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}} \qquad \text{for } s > 0 .$$

Using linearity, the above formulas give the Laplace transform of any polynomial. Moreover,

$$\mathcal{L}\left\{e^{kt}\right\}(s) = \frac{1}{s-k} \qquad \text{for } s > k$$

and, taking $k = i\omega$ and then real and imaginary parts,

$$\mathcal{L}\{\cos(\omega t)\}(s) = \frac{s}{s^2+\omega^2} \qquad \text{and} \qquad \mathcal{L}\{\sin(\omega t)\}(s) = \frac{\omega}{s^2+\omega^2} \qquad \text{for } s > 0 .$$

Also interesting in applications is the Laplace transform of (translates of) the *Heaviside function* (also known as *unitary jump function*) $\Theta(t)$, also called $u(t)$ by engineers, defined as

$$\Theta(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases} .$$

Observe that the function $u_\tau(t) = \Theta(t-\tau)$ has a jump (from 0 to 1) at the point τ. Computation shows that

$$\mathcal{L}\{\Theta(t-\tau)\}(s) = \frac{e^{-\tau s}}{s} \qquad \text{for } s > 0$$

Gamma function. The Laplace transform of nonnegative powers $f(t) = t^q$, with $q \geq 0$, is

$$\mathcal{L}\{t^q\}(s) = \frac{\Gamma(q+1)}{s^{q+1}} \qquad \text{for } s > 0 ,$$

where the *Gamma function* is defined, in $\Re(z) > 0$, by the integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

The Gamma function satisfies the functional identity $\Gamma(z+1) = z\Gamma(z)$. Since $\Gamma(1) = 1$, the Gamma function actually extends the "factorial", namely $\Gamma(n+1) = n!$ if $n = 0, 1, 2, 3, \ldots$.

Exercise. Show that

$$\mathcal{L}\{\Theta(t-\tau)f(t)\} = e^{-\tau s}\mathcal{L}\{f(t+\tau)\} .$$

Laplace transform of periodic functions. If $f : [0,\infty[\to \mathbf{R}$ is a periodic piecewice continuous function, with period T, then its Laplace transform is

$$\mathcal{L}\{f(t)\}(s) = \frac{F_T(s)}{1-e^{-sT}} \qquad \text{where} \qquad F_T(s) = \int_0^T e^{-st} f(t)dt\,.$$

Indeed,

$$\begin{aligned}
\int_0^\infty e^{-st} f(t)dt &= \sum_{k\geq 0} \int_{kT}^{(k+1)T} e^{-st} f(t)dt \\
&= \sum_{k\geq 0} \int_0^T e^{-s(t+kT)} f(t+kT)dt \\
&= \sum_{k\geq 0} e^{-skT} \int_0^T e^{-st} f(t)dt \\
&= F_T(s) \sum_{k\geq 0} e^{-skT} \\
&= \frac{F_T(s)}{1-e^{-sT}}\,.
\end{aligned}$$

Exercise (Laplace transform of periodic step functions. Find the Laplace transform of the following periodic functions (below, $[t]$ denotes the integer part of t, i.e. the biggest integer $n \in \mathbf{Z}$ such that $n \leq t$):

$$f(t) = t - [t] \qquad f(t) = \begin{cases} 0 & \text{if } [t] \text{ is even} \\ 1 & \text{if } [t] \text{ is odd} \end{cases} \qquad f(t) = \begin{cases} t-[t] & \text{if } [t] \text{ is even} \\ 1+[t]-t & \text{if } [t] \text{ is odd} \end{cases}$$

Laplace transform and convolutions. Let f and $g : [0,\infty) \to \mathbf{R}$ be two piecewice continuous functions. The *convolution (product)* between f and g is the function $f * g : [0,\infty) \to \mathbf{R}$ defined as

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau\,.$$

Observe that the convolution is commutative, namely $f*g = g*f$, and linear in both the arguments.

If f and g are piecewice continuous and have exponential growth, then

$$\mathcal{L}\{f*g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}\,,$$

in a region where both Laplace transforms are defined. Indeed, using the substitution $\eta = t - \tau$ and Fubini theorem, we see that

$$\begin{aligned}
\mathcal{L}\{f*g\}(s) &= \int_0^\infty e^{-st} \left(\int_0^t f(\tau)g(t-\tau)d\tau \right) dt \\
&= \int_0^\infty \int_0^\infty e^{-s(\tau+\eta)} f(\tau)g(\eta)d\tau d\eta \\
&= \left(\int_0^\infty e^{-s\tau} f(\tau)d\tau \right) \left(\int_0^\infty e^{-s\eta} g(\eta)\eta \right)\,.
\end{aligned}$$

In other words, the Laplace operator $\mathcal{L}$ sends a convolution (in the time-space) into a (ordinary) point-wise product between functions (in the frequency-space). This fact is useful when trying to compute inverse Laplace transforms. More important will be the possibility to write solutions of a linear differential equation as a convolution of the "external force" times a certain function which only depends on the differential operator.

Exercise. Show that, if f is a piecewise continuous function with exponential order m, then

$$\mathcal{L}\left\{ \int_0^t f(x)dx \right\}(s) = \frac{1}{s}\mathcal{L}\{f(t)\}(s) \quad \text{with } s > m\,.$$

Deduce that

$$\mathcal{L}\left\{f(t)\right\}(s) - \mathcal{L}\left\{f(0)\right\}(s) = \frac{1}{s}\mathcal{L}\left\{f'(t)\right\}(s) \quad \text{with } s > m\,.$$

Laplace transform of time derivatives. The relation between Laplace transform and derivatives (both in time and frequency spaces) is what makes the Laplace transform useful in applications.

Let $F(s) = \int_0^\infty e^{-st} f(t)dt$ be the Laplace transform of $f(t)$, and assume that both f and its derivative f' are piecewise continuous and have exponential growth. Then integration by parts (or use of convolution) shows that the Laplace transform of f' is

$$\mathcal{L}\left\{f'(t)\right\}(s) = sF(s) - f(0)\,.$$

If f and all its derivatives up to $d^n f/dt^n$ are piecewice continuous functions with exponential growth, then induction shows that

$$\mathcal{L}\left\{\frac{d^n f}{dt^n}(t)\right\}(s) = s^n F(s)(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ... - sf^{(n-2)}(0) - f^{(n-1)}(0)\,.$$

For applications, you may not want to remind the above long formula, but only the fact that if the values of f and all its derivatives up to $f^{(n-1)}$ are zero at the point $t = 0$, then

$$\mathcal{L}\left\{\frac{d^n f}{dt^n}(t)\right\}(s) = s^n F(s)(s)\,.$$

It says that time derivatives $\frac{d^n}{dt^n}$ translate into powers s^n in the frequency domain.

Exercise. Find the Laplace transform of the following functions $f(t)$:

$$e^{-\alpha t}\cos(\omega t) \qquad e^{-\alpha t}\sin(\omega t) \qquad t\cos(\omega t) \qquad t\sin(\omega t)$$

$$u_a(t)e^{-\alpha t} \qquad u_a(t)\sin(\omega t) \qquad (1 - u_a(t))\,t\,.$$

$$A\sin(\omega t + \varphi) \qquad Ae^{-\alpha t}\sin(\omega t + \varphi) \qquad Ae^{-\alpha t}\sinh(\beta t + \varphi)$$

$$A\sin(\omega t + \varphi) + \frac{F_0}{\omega^2 - \gamma^2}\cos(\gamma t) \qquad A\sin(\omega t + \varphi) + \frac{F_0}{2\omega}t\sin(\omega t)$$

$$Ae^{-\alpha t}\sin\left(\sqrt{\omega^2 - \alpha^2}t + \varphi\right) + \frac{F_0}{\sqrt{(\omega^2 - \gamma^2)^2 + 4\alpha^2\gamma^2}}\sin(\gamma t + \phi)$$

Integral of Laplace transform. If the integral

$$\int_p^\infty F(p)dp$$

of the Laplace transform of $f(t)$ is convergent, then it is the Laplace transform of

$$\frac{f(t)}{t}$$

1.2 Inverse Laplace transform

Let $F(z)$ be a function holomorphic in some domain $R \subset \mathbf{C}$. A piecewise continuous function $f : [0, \infty) \to \mathbf{R}$ is called *inverse Laplace transform* of F if $F = \mathcal{L}\{f\}$, i.e. if

$$F(z) = \int_0^\infty e^{-zt} f(t)dt\,.$$

When $f(t)$ has exponential growth, hence the R.O.C. of its Laplace transform R contains a semi-plane $\Re(z) > m$, we can recover $f(t)$ using the

Mellin (or Bromwich) theorem. *Let $F(s)$ be the Laplace transform of a function $f(t)$ with exponential order α. Then, at points of continuity of f,*

$$f(t) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} e^{st} F(s) ds$$

where $\beta > \alpha$.

Proof. For $\xi > \alpha$, the function $\varphi(t) = e^{-\beta t} f(t)$ is piecewise continuous and decreases exponentially to zero for $t \to 0$. There follows that, at points of continuity, can be given as a Fourier integral

$$\varphi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi t} \left(\int_{-\infty}^{\infty} e^{-i\xi s} \varphi(s) ds \right) d\xi \, .$$

Substituting the definition of φ we get

$$\begin{aligned} e^{-\beta t} f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi t} \left(\int_{-\infty}^{\infty} e^{-(\beta - i\xi)s} f(s) ds \right) d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi t} \left(\int_{0}^{\infty} e^{-(\beta - i\xi)s} f(s) ds \right) d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi t} F(\beta - i\xi) d\xi \\ &= \frac{e^{-\beta t}}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} e^{st} F(s) ds \, . \end{aligned}$$

q.e.d.

In [SvesnikovTichonov] (page 227) you may find a sufficient conditions for a function $F(s)$ to be the Laplace transform of a function of exponential order.

Of course, different piecewise continuous functions may have the same Laplace transform, since you can change the value of a continuous function in a countable number of points without changing its (Riemann's) integral. Despite of this fact, it is customary to use the notation

$$f = \mathcal{L}^{-1} \{F\}$$

to denote any inverse Laplace transform of F. Uniqueness is recovered only when we restrict to continuous functions, namely we state the following

Uniqueness theorem. *If two continuous functions f and $g : [0, \infty[\to \mathbf{R}$ with exponential growth have the same Laplace transform, then $f = g$.*

Exercise. Show that

$$\mathcal{L}^{-1} \{F(s-a)\} (t) = e^{at} \mathcal{L}^{-1} \{F(s)\} (t)$$

and

$$\mathcal{L}^{-1} \left\{ \frac{1}{a} F(s/a) \right\} (t) = \mathcal{L}^{-1} \{F(s)\} (at) \qquad \text{for } a > 0 \, ,$$

Hint. More useful, if you want to compute inverse Laplace transforms of rational functions times exponentials (which are the typical Laplace transforms occurring while solving linear differential equations with constant coefficients), is the following recipe: factorize the rational function, find the inverse Laplace transform of each factor of degree two or one, do the appropriate translations in the frequency domain due to the exponentials, and finally take the convolution product of the inverse Laplace transforms of the factors.

2 Laplace transform solutions of linear ODEs

The basic idea to solve a linear differential equation with constant coefficients using the Laplace transform is that it translates the differential equation $Lx = f$ into an algebraic equation for the Laplace transform $X(s) = \int_0^\infty e^{-st}x(t)dt$ of the solution. This happens because the Laplace transform of the derivatives $x^{(k)}(t)$ are polynomials in the frequency s, with leading term $s^k X(s)$. We first illustrate the naive method with a simple example.

Example. We want to solve $\dot{x} + \beta x = e^{\alpha t}$ with initial condition $x(0) = x_0$.

First, we assume that the solution $x(t)$ has exponential growth, and denote by $X(s)$ its Laplace transform. Taking the Laplace transform of both sides of the differential equation we get

$$sX(s) - x(0) + \beta X(s) = \frac{1}{s-\alpha}$$

so that

$$X(s) = \frac{x_0}{s+\beta} + \frac{1}{(s+\beta)(s-\alpha)}\,.$$

Taking the inverse Laplace transform (and using its properties) we get

$$\begin{aligned} x(t) &= x_0 e^{-\beta t} + \int_0^t e^{-\beta(t-\tau)} e^{\alpha\tau} d\tau \\ &= x_0 e^{-\beta t} + \frac{1}{\alpha+\beta}\left(e^{\alpha t} - e^{\beta t}\right)\,. \end{aligned}$$

It is clear that the above argument does work for any linear differential ODE with arbitrary order. The reason why we like the Laplace transform method is actually in the integral formula that appeared above, giving a part of the solution as a convolution product of the r.h.s. term and of a particular solution of the homogeneous equation.

2.1 Transfer function and impulse response function

Transfer function and impulse response function. Let $L = m\frac{d^2}{dt^2} + \alpha\frac{d}{dt} + \beta$ be a differential operator with constant coefficients (the order two is not necessary for what follows, but it is important to see mechanical interpretations of the objects that we are going to define!). A differential equation $Lx = f$, namely

$$m\ddot{x} + \alpha\dot{x} + \beta x = f(t)\,,$$

describes a linear system (mechanical, electronic, or other devices ...) subject to an external force $f(t)$. By the superposition principle, its solution with some given initial data x_0 and v_0 may be written as

$$x_{\mathrm{free}}(t) + x(t)\,,$$

where $x_{\mathrm{free}}(t)$ is the solution of the homogeneous equation $Lx_{\mathrm{free}} = 0$ with the given initial conditions $x_{\mathrm{free}}(0) = x_0$ and $x'_{\mathrm{free}}(0) = v_0$, and $x(t)$ is the particular solution of the non-homogeneous equation with zero initial conditions. We already know that finding $x_{\mathrm{free}}(t)$, the free system solution, reduces to the algebraic problem of factorizing the characteristic polynomial of the differential operator. You may think that $x(t)$ describe the system's response to the input $f(t)$, schematically

$$f(t) \Rightarrow \boxed{\text{device (differential operator } L\text{)}} \Rightarrow x(t)$$

If $X(s)$ denotes the Laplace transform of $x(t)$, from the equation $Lx = f$ we get

$$P(s)X(s) = F(s)\,,$$

where $F(s)$ denotes the Laplace transform of the external force $f(t)$ and $P(s) = s^2 + 2\alpha s + \beta$ is the characteristic polynomial of the differential operator L. The function

$$H(s) = \frac{1}{s^2 + 2\alpha s + \beta}\,,$$

i.e. the inverse of the characteristic polynomial, is called *transfer function* ("função de transferência") of the linear system, as it is the ratio $H(s) = X(s)/F(s)$ between the Laplace transforms of the response $x(t)$ (given zero initial conditions) and of the input force $f(t)$. So, the analytic problem posed in the time domain has this simple answer in the frequency domain:

$$F(s) \Rightarrow \boxed{\text{device (transfer function } H = 1/P)} \Rightarrow X(s) = H(s)F(s)$$

The continuous inverse Laplace transform of the transfer function, namely the continuous function $h(t)$ such that

$$H(s) = \int_0^\infty e^{-st} h(t) dt \, ,$$

is called *impulse response function* ("resposta impulsiva") of the system.

Since $X(s) = H(s)F(s)$, we may finally write the particular solution with zero initial data as the convolution $x(t) = (h * f)(t)$, that is in the suggestive form

$$x(t) = \int_0^t h(t-\tau) f(\tau) d\tau \, .$$

This formula displays cleanly the casual dependance of the response x from the input f, since the values of $f(\tau)$ only play a role for times $t \geq \tau$.

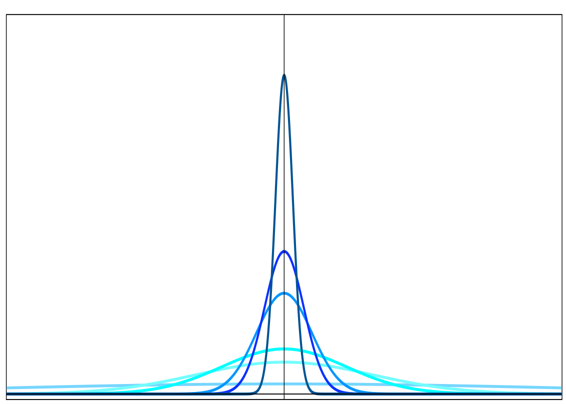
$$f(t) \Rightarrow \boxed{\text{device (impulse response function } h(t))} \Rightarrow x(t) = (h * f)(t)$$

Observe that the impulse response function may also be characterized as the solution of the homogeneous equation $Lh = 0$ with zero initial "position" $h(0) = 0$ and unit initial "moment" $mh'(0) = 1$. This fact could be used to prove the above formula for $x(t)$ without the help of the Laplace transform!

Dirac delta function and interpretation of the impulse response function. To reveal the meaning of the impulse response function (and its etymology), consider a sequence of forcing terms f_n which are supported on smaller and smaller intervals around a fixed time $t_0 > 0$, say $[t_0 - 1/n, t_0 + 1/n]$, but having constant time integral $\int_{t_0-1/n}^{t_0+1/n} f_n(t)dt = 1$, for $n = 1, 2, 3, ...$ The time integral of the force is equal to the impulse transferred to the system. Hence, if $1/n$ is much smaller than our time-resolution, we see f_n as an instantaneous impulse given to our system at time t_0 (a hammer striking an object). But taking the limit for $n \to \infty$ in the response to such forces, we get

$$\begin{aligned} x_n(t) &= \int_0^t h(t-\tau) f_n(\tau) d\tau \\ &= \int_{t_0-1/n}^{t_0+1/n} h(t-\tau) f_n(\tau) d\tau \\ &\to h(t-t_0) \, , \end{aligned}$$

as long as h is continuous at $t - t_0$ and $t > t_0$. Hence, $h(t)$ is the response of the system to an instantaneous unit impulse delivered at time 0.



Sequence of gaussians $\frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$ with decreasing time t =1, 0.1, 0.05, 0.01, 0.005, 0.001.

It is useful, following P.A.M. Dirac, to introduce a notation for the formal limit "$\lim_{n\to\infty} r_n(\tau)$". The *Dirac delta function*, $\delta(t)$, is defined by the formal identity

$$\int_{-\infty}^{\infty} \varphi(t)\delta(t)dt = \varphi(0) \, ,$$

which holds for any continuous function φ. Of course, there does not exist such a function (but this is not more surprising than the discovery that the dt's that you see inside integrals like $\int \varphi(t)dt$ do not mean too much if left alone!). What δ really is, is a linear functional defined on the space of continuous functions (which mathematicians call distribution of order zero, or signed measure), and the above integral is just a suggestive notation for $\langle\delta|\varphi\rangle$, the value of the Dirac delta functional $\langle\delta|$ at the continuous function $|\varphi\rangle$. The integral notation is useful, and can be justified (as it was by Laurent Schwartz some years after Dirac's suggestion and physicists' daily use) at the expense of some hard functional analysis.

Meanwhile, once we accept it, we see that we may manipulate integrals involving the Dirac delta function as if $\delta(t)$ were an ordinary function. For example, from the formula

$$\int_0^\infty e^{-st}\delta(t-t_0)dt = e^{-t_0 s}$$

we see that the exponential $e^{-t_0 s}$ may be interpreted as the Laplace transform of $\delta(t-t_0)$. If we take $t_0 = 0$, we see that the Dirac delta function δ has Laplace transform equal to 1, i.e. δ may be interpreted as the inverse Laplace transform of the constant function $F(s) = 1$. Moreover, since the Laplace transform of the Heaviside jump function $\Theta(t-t_0)$ is $e^{-t_0 s}/s$, we also see that the Dirac delta function $\delta(t-t_0)$ may be interpreted as the derivative of $\Theta(t-t_0)$, a rather meaningful fact if you think at their physical interpretations.

Now we go back to our differential equation $Lx = f$. Since the Laplace transform of $h(t)$ is $1/P(s)$, the impulse response function $h(t)$ may be interpreted as the solution of the formal differential equation

$$Lh = \delta(t)\,.$$

As such, it is also called *fundamental solution* of the differential operator L.

The integral formula for the system's response is sometimes best understood in a discrete time setting. Suppose that the system is subject to a series of impulses $f(k)\delta(t-k)$ delivered at integer times $k = 0, 1, 2, 3, ...$ (of course, changing the time scale, these can be integer multiples of any interval of time!). Then the convolution product giving the response takes the form

$$\begin{aligned} x(n) &= \int_0^n h(n-\tau)\left(\sum_{k\geq 0} f(k)\delta(\tau-k)\right)d\tau \\ &= \sum_0^n h(n-k)f(k) \\ &= h(n)f(0) + h(n-1)f(1) + h(n-2)f(2) + ... + h(0)f(n)\,. \end{aligned}$$

You may think that, once you gave an impulse $f(k)$ at time k, you receive a response $h(n-k)f(k)$ after time $n-k$, ...

Training. Solve the following Cauchy problems using the Laplace transform technique:

$$\dot{x} + x = -e^t \quad \text{with } x(0) = \sqrt{2}$$

$$\ddot{x} + 4x = 3t \quad \text{with } x(0) = 0 \text{ and } \dot{x}(0) = 2$$

$$\ddot{x} + 2\dot{x} = t - [t] \quad \text{with } x(0) = 0 \text{ and } \dot{x}(0) = 0$$

$$\ddot{x} + 2\dot{x} + 5x = \delta(t-t_0) \quad \text{with } x(0) = 0 \text{ and } \dot{x}(0) = 1$$

$$\ddot{x} + \pi^2 x = 3\left(1 - u_{t_0}(t)\right) \quad \text{with } x(0) = 1 \text{ and } \dot{x}(0) = 0$$

Exercise. Consider a RL circuit, whose current $I(t)$ is determined by the ODE

$$L\dot{I} + RI = V\,.$$

- Find the current if the generator, initially supplying a constant tension $V(t) = V_0$, is suddenly switched off at time $t_0 > 0$, given an initial stationary current satisfying Ohm's law $I(0) = V_0/R$.
- Find the current if the generator is suddenly switched on at time $t_0 > 0$, i.e. if $V(t) = V_0\left(1 - u(t-t_0)\right)$ or $V(t) = V_0 u(t-t_0)\sin(\omega t)$, given an initial zero current $I(0) = 0$.

Exercise. Consider a RLC circuit, whose current $I(t)$ is determined by the ODE

$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = \dot{V}\,.$$

- Find the transfer function of the circuit, and an integral formula for the current $I(t)$, given a trivial initial current $I(0) = 0$ and $\dot{I}(0) = 0$.
- Find the current if the generator, initially supplying a constant tension $V(t) = V_0$, is suddenly switched off at time $t_0 > 0$, given an initial stationary current $I(0) = V_0/R$ and $\dot{I}(0) = 0$. .
- Find the current if the generator is suddenly switched on at time $t_0 > 0$, i.e. if $V(t) = V_0\,(1 - u(t - t_0))$ or $V(t) = V_0 u(t - t_0)\sin(\omega t)$, given an initial zero current $I(0) = 0$ and $\dot{I}(0) = 0$.

Exercise. Consider the equations of *forced oscillations* and *dumped forced oscillations*

$$\ddot{q} = -\omega^2 q + f(t) \qquad \text{e} \qquad \ddot{q} = -2\alpha\dot{q} - \omega^2 q + f(t)\,.$$

Find the solutions of the Cauchy problem with generic initial conditions $q(0) = q_0$ and $\dot{q}(0) = v_0$, when the force is

$$f(t) = f_0\delta(t - t_0) \qquad f(t) = f_0 u_{t_0}(t) \qquad f(t) = f_0\,(1 - u_{t_0}(t)) \qquad f(t) = f_0\cos(\gamma t)\,.$$

Example (signal processing).

2.2 Laplace transform in probability and statistical mechanics

Moment generating function. Let ξ be a real valued random variable in some probability space, and let $\mathbf{P}$ be its law. The *(moment) generating function* of ξ is the mean value of the variable $e^{\beta\xi}$, namely the function

$$\beta \mapsto \mathbf{E}e^{\beta\xi} = \int e^{\beta x} d\mathbf{P}\,(x)\;.$$

Observe that if ξ is absolutely continuous with density $f(x)$, i.e. if $d\mathbf{P}\,(x) = f(x)dx$, then this is the two-sided Laplace transform of the density computed at the point $-\beta$. Assume that $\mathbf{E}e^{\beta\xi}$ is defined for β in some neighborhood of the origin, as is the case when ξ is bounded, for example. Then its derivatives at the origin gives the moments of the random variable, since

$$\begin{aligned} \frac{d^n\mathbf{E}e^{\beta\xi}}{d\beta^n}(0) &= \int x^n d\mathbf{P}\,(x) \\ &= \mathbf{E}\xi^n\,. \end{aligned}$$

This was Laplace's original use of the idea. It has many applications, and for some time it was used to derive limit theorems.

Example. If $\xi \sim B(p, 1)$, then

$$\mathbf{E}e^{\beta\xi} = e^{\beta}\cdot p + q = 1 + p(e^{\beta} - 1)$$

Example. If $\xi \sim N(0, 1)$, then

$$\mathbf{E}e^{\beta\xi} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\beta x} e^{-x^2/2} dx = e^{\beta^2/2}$$

Large deviations. The *free energy* of a system described by the variable ξ is the function

$$F(\beta) = \log \mathbf{E} e^{\beta\xi}$$

If X is bounded, then $F(\beta)$ is holomorphic and strictly convex (unless X is constant w.p. 1). Indeed,

$$\begin{aligned} F'(\beta) &= e^{-F(\beta)} \mathbf{E}\left(\xi e^{\beta\xi}\right) \\ F''(\beta) &= e^{-F(\beta)} \left(e^{-F(\beta)} \mathbf{E}\left(\xi^2 e^{\beta\xi}\right) - \left(\mathbf{E}\left(\xi e^{\beta\xi}\right)\right)^2 \right) \end{aligned}$$

The Cauchy-Schwarz inequality, applied to $\xi e^{\beta\xi/2}$ and $e^{\beta\xi/2}$, gives $F''(\beta) > 0$, unless $\xi e^{\beta\xi/2}$ is proportional to $e^{\beta\xi/2}$ with probability one, which means that ξ is constant with probability one.

Computation shows that $F(0) = 0$ and $F'(0) = \mathbf{E}X$.

The *Chebyshev exponencial inequality*

$$\mathbf{P}(X \geq \varepsilon) = \mathbf{P}\left(e^{\beta X} \geq e^{\beta\varepsilon}\right) \leq e^{-\beta\varepsilon} \mathbf{E} e^{\beta X} \qquad \forall\, \beta > 0\,,$$

shows that

$$\mathbf{P}(X \geq \varepsilon) \leq e^{-H(\varepsilon)}$$

where the *entropy* $H(\lambda)$ is defined as the Legendre transform of the free energy, i.e.

$$H(\lambda) = \sup_{\beta>0} (\beta\lambda - F(\beta))$$

Now, let X_1, X_2, X_3, ... be i.i.d.r.v., with the law of X and mean $\mathbf{E}X = m$, and let $S_n = X_1 + X_2 + ... + X_n$ be their partial sums. Then

$$\begin{aligned} \mathbf{P}(S_n/n \geq \lambda) &= \mathbf{P}\left(e^{\beta S_n} \geq e^{n\beta\lambda}\right) \\ &\leq e^{-n(\beta\lambda - F(\beta)} \end{aligned}$$

for any $\beta > 0$, hence

$$\begin{aligned} \mathbf{P}(S_n/n \geq \lambda) &\leq \inf_{\beta>0} e^{-n(\beta\lambda - F(\beta))} \\ &\leq e^{-n\cdot\sup_{\beta>0}(\beta\lambda - F(\beta))} \end{aligned}$$

There follows that the probability of "large deviations" can be estimated as

$$\mathbf{P}(|S_n/n - m| \geq \varepsilon) \leq 2e^{-n\cdot\min\{H(m+\varepsilon), H(m-\varepsilon)\}}\,.$$

Example (large deviations for Bernoulli trials). Show that the free energy of $X \sim B(1,p)$ (i.e., $\mathbf{P}(X = 1) = p$ and $\mathbf{P}(X = 0) = q = 1 - p$) is

$$F(\beta) = \log\left(e^{\beta} p + q\right)$$

and that the entropy is

$$H(\lambda) = \lambda \log(\lambda/p) + (1-\lambda) \cdot \log((1-\lambda)/(1-p)) \geq 2(\lambda - p)^2\,,$$

for $p < \lambda < 1$. Deduce that, if $S_n \sim B(n,p)$ (i.e., $S_n = X_1 + X_2 + ... + X_n$, where X_1, X_2, X_3, ... are i.i.d.r.v. with law $X_i \sim B(1,p)$), then the probability of large deviations is bounded by

$$\mathbf{P}(|S_n/n - p| \geq \varepsilon) \leq 2e^{-2n\varepsilon^2}\,.$$

Partition function. Consider a system of particles which energy may take the values e_n's. The *(canonical) partition function* of the system is defined as

$$Z(\beta) = \sum_n e^{-\beta e_n}\,.$$

The physical interpretation is that $\beta = 1/k_{\mathrm{B}}T$, where $k_{\mathrm{B}} \simeq 1.3806505(24) \times 10^{-23}$ joule/kelvin is the Boltzmann constant and T is the temperature of the system, and $p_n = e^{-\beta e_n}/Z(\beta)$ is the probability to find the system in the state with energy e_n. The idea is that the partition function, or better its logarithm

$F(\beta) = \log Z(\beta)$ called *free energy*, contains all the thermodynamics of the system. For example, its first derivative gives the mean energy $\langle E \rangle$, since

$$\begin{aligned} -\frac{d \log Z}{d\beta}(\beta) &= \frac{\sum_n e_n e^{-\beta e_n}}{\sum_n e^{-\beta e_n}} \\ &= \langle E \rangle \ , \end{aligned}$$

and its second derivative gives the energy fluctuation, proportional to the heath capacity (at constant volume) $c_v = \partial \langle E \rangle / \partial T$, since

$$\begin{aligned} \frac{d^2 \log Z}{d\beta^2}(\beta) &= \frac{\sum_n (e_n - \langle E \rangle)^2 e^{-\beta e_n}}{\sum_n e^{-\beta e_n}} \\ &= \left\langle (E - \langle E \rangle)^2 \right\rangle \\ &= k_{\mathrm{B}} T^2 c_v \ . \end{aligned}$$

The (Boltzmann's) entropy S is proportional to the Legendre transform of the free energy, since

$$\begin{aligned} S &= -k_{\mathrm{B}} \sum_n \log(p_n) p_n \\ &= k_{\mathrm{B}} \left(\log Z(\beta) + \beta \langle E \rangle \right) \\ &= k_{\mathrm{B}} \sup_{\lambda} \left(\log Z(\beta) - \beta \lambda \right) \ , \end{aligned}$$

... and so on.

References

[Ahlfors] Lars V. Ahlfors, *Complex analysis*, McGraw-Hill 1979.

[Apostol] Tom M. Apostol, *Calculus*, John Wiley & Sons, New York 1969.

[Arnold$_{78}$] Vladimir I. Arnold, *Metodi geometrici della teoria delle equazioni differenziali ordinarie*, Editori Riuniti - MIR, Roma 1978.

[Arnold$_{79}$] Vladimir I. Arnold, *Metodi matematici della meccanica classica*, Edizioni MIR - Editori Riuniti, Roma 1978.

[Arnold$_{85}$] Vladimir I. Arnold, *Equações diferenciais ordinárias*, MIR 1985.

[Beals] Richard Beals, *Advanced mathematical analysis: periodic functions and distributions, complex analysis, Laplace transform and applications*, Springer-Verlag 1987.

[Berkeley] C. Kittel, W.D. Knight and M.A. Ruderman, *Berkeley Physics*, McGraw-Hill 1962.

[BoyceDiPrima] William E. Boyce and Richard C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, John Wiley 1992.

[BrownChurchill] James W. Brown and Ruel V. Churchill, *Fourier Series and Boundary Value Problems*, McGraw-Hill 1993.

[ButtàNegrini] P. Buttà e P. Negrini, *Note del corso di Sistemi Dinamici*, Università di Roma "La Sapienza", 2005.
http://www.mat.uniroma1.it/people/butta/didattica/index.html

[ChurchillBrown] Ruel V. Churchill and James W. Brown, *Complex Variables and Applications*, McGraw-Hill 1990.

[CoddingtonLevinson] E.A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill 1955.

[Dyke] P.P.G. Dyke, *An Introduction to Laplace Transforms and Fourier Series*, Springer 2002.

[Feller] W. Feller, *An introduction to probability theory and its applications, vol. 1 & 2,* John Wiley & Sons, New York 1968

[Feynman] R.P. Feynman, R.B. Leighton and M. Sands, *The Feynman lectures on physics*, Addison-Wesley, Reading, 1963.

[Figueiredo] Djairo Guedes de Figueiredo, *Análise de Fourier e equações diferenciais parciais*, Projeto Euclides, IMPA 1987.

[Folland] Gerald B. Folland, *Fourier analysis and its applications*, Brooks/Cole Publishing Company 1992.

[Gamelin] Theodore W. Gamelin, *Complex analysis*, Springer 2001.

[GuckenheimerHolmes] J. Guckenheimer and P. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, A.M.S. **42**, Springer, Berlin 1993.

[Haberman] Richard Haberman, *Mathematical Models*, SIAM's Classics in Applied Mathematics 1998.

[Hale] J.K. Hale, *Ordinary differential equations*, Wiley-Interscience 1969.

[HasselblattKatok] B. Hasselblatt and A. Katok, *A first course in dynamics: with a panorama of recent developments*, Cambridge University Press, Cambridge 2003.

[HirschSmale] M.W. Hirsch and S. Smale, *Differential equations, dynamical systems and linear algebra*, Academic Press (Pure and Applied Mathematics. A series of Monographs and Textbooks), San Diego 1974.

[KatokHasselblatt] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of mathematics and its applications **54**, Cambridge University Press, Cambridge 1995.

[KolmogorovFomin] A.N. Kolmogorov e S.V. Fomin, *Elementos de Teoria das Funções e de Análise Funcional*, MIR 1983.

[Iório] Valéria Iório, *EDP, um Curso de Graduação*, Coleção Matemática Universitária, IMPA 2005.

[LandauLifshitz] L.D. Landau and E.M.Lifshitz, *Physics*,

[Lang] Serge Lang, *Complex Analysis*, Forth Edition, Springer 1999.

[LaurentievChabat] M. Laurentiev et B. Chabat, *Méthodes de la théorie des fonctions d'une variable complexe*, Éditions de Moscou MIR, 1972.

[MorseFeshbach] Philip McCord Morse and Herman Feshbach, *Methods of Theoretical Physics*, McGraw-Hill 1953. Reprinted by Feshbach Publishing 2005.

[NagleSaffSnider] R.K. Nagle, E.B. Saff and A.D. Snider, *Fundamentals of differential equations*, Addison-Wesley 2000.

[Olver] Peter J. Olver, *Applied Mathematics Lecture Notes.*
http://www.math.umn.edu/~olver/appl.html

[O'Neil] Peter V. O'Neil, *Beginning Partial Differential Equations*, John Wiley & Sons 1999.

[Marsden] Jerrold E. Marsden, *Basic complex analysis*, W.H. Freeman 1999.

[Pinsky] Mark A. Pinsky, *Partial Differential Equations and Boundary-Value Problems with Applications*, McGraw-Hill 1991

[Robinson] J.C. Robinson, *An introduction to ordinary differential equations*, Cambridge University Press, Cambridge 2004.

[Ross] S.L. Ross, *Differential equations*, John Wiley & Sons, 1984.

[Rudin] Walter Rudin, *Real and complex analysis*, McGraw-Hill 1987.

[Schiff] Joel L. Schiff, *The Laplace transform: theory and applications*, Springer 1999.

[Simmons] G.F. Simmons, *Differential equations with applications and historical notes*, McGraw-Hill, 1991.

[Spiegel] Murray R. Spiegel, *Análise de Fourier*, McGraw-Hill 1976.

Murray R. Spiegel, *Transformadas de Laplace*, McGraw-Hill 1971.

[SvesnikovTichonov] A.G. Svesnikov e A.N. Tichonov, *The Theory of Functions of a Complex Variables*, MIR 1971.

[TichonovSamarskij] A.N. Tichonov, A.A. Samarskij *Equazioni della fisica matematica*, Edizioni MIR, Mosca 1981.