(Introduction to) Duality theory

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January 25–27, 2017

One Example

Theorem (L. Pontrjagin; 1934)

$\mathbf{Ab} \sim \mathbf{CompHausAb}^{\mathrm{op}}$.

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That fact is a theorem of topological groups.

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That fact is a theorem of topological groups. That character groups yield an adjoint connection is a theorem of category theory.

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A seemingly paradoxical observation

"…an equation is only interesting or useful to the extent that the two sides are different!"

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One more example

"ordered sets = Heyting algebras": $\mathbf{Pos}_{\mathrm{fin}} \sim \mathbf{Heyt}_{\mathrm{fin}}^{\mathrm{op}}$.

. . . And Now for Something Completely Different!

Intuitionistic propositional logic

Heyting

"Instead of asking the question *When is a sentence* Φ *true*, we ask *What is a proof of* Φ*?*"

- ...

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- A proof of $\varphi \wedge \psi$ is a pair (p, q) consisting of a proof p of φ and a proof q of ψ.

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- A proof of $\varphi \wedge \psi$ is a pair (p, q) consisting of a proof p of φ and a proof q of ψ.
- A proof of $\varphi \vee \psi$ is a pair (i, p) where either $i = 0$ and p is a proof of φ or $i = 1$ and q is a proof of ψ.
- ...

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More formally: Natural deduction

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}\,\mathsf{I}_\wedge \qquad \frac{\varphi \wedge \psi}{\varphi}\,\mathsf{E1}_\wedge \qquad \frac{\varphi \wedge \psi}{\psi}\,\mathsf{E2}_\wedge$$

$$\frac{\varphi}{\varphi \vee \psi}\,\mathsf{I1}_\vee \qquad \frac{\psi}{\varphi \vee \psi}\,\mathsf{I2}_\vee \qquad \frac{\varphi \vee \psi \quad \boxed{\begin{array}{c}\varphi\\ \vdots\\ \theta\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \theta\end{array}}}{\theta}\,\mathsf{E}_\vee$$

$$\ldots$$

But not: $\dfrac{}{\varphi \vee \neg\varphi}$.

About $\varphi \vee \psi$?

Question

$$\nvdash \varphi \text{ and } \nvdash \psi \implies \nvdash (\varphi \vee \psi)?$$

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- First recall: $\models \theta$ means $[\![\theta]\!] = \top$, for all interpretations $[\![-]\!]$ in (finite) Heyting algebras H.

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- Hence our job is: If $[\![\varphi]\!]_{H_1} < \top$ and $[\![\psi]\!]_{H_2} < \top$, construct an interpretation in an Heyting algebra H so that $\varphi \vee \psi$ fails. . .

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$$\nvdash \varphi \text{ and } \nvdash \psi \implies \nvdash (\varphi \vee \psi)?$$

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$$\not\models \varphi \text{ and } \not\models \psi \implies \not\models (\varphi \vee \psi)$$

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- . . . does not seem to be easier!!?

☐

Kripke semantics

Definition

A Kripke model is a tuple of the form $\mathcal{C} = (C, \leq, \Vdash)$ where $(C, \leq)$ is a partially ordered set and $\Vdash$ is a binary relation between elements of C and propositional variables so that:

$$\text{if } c \leq c' \text{ and } c \Vdash p \text{ then } c' \Vdash p.$$

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$\mathcal{C} \Vdash \varphi$ whenever $c \Vdash \varphi$ for all $c \in C$ and $\Vdash \varphi$ whenever $\mathcal{C} \Vdash \varphi$ for all $\mathcal{C}$.

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Theorem

$$\models \varphi \iff \Vdash \varphi.$$

Returning to $\varphi \vee \psi$

Theorem

$$\nVdash \varphi \text{ and } \nVdash \psi \implies \nVdash (\varphi \vee \psi).$$

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$$\not\Vdash \varphi \text{ and } \not\Vdash \psi \implies \not\Vdash (\varphi \vee \psi).$$

Proof.

If φ fails in $\mathcal{C}_1$ and ψ fails in $\mathcal{C}_2$, then $\varphi \vee \psi$ fails in $\mathcal{C} = (C, \leq, \Vdash)$ where "$C = C_1 + C_2 + 1$." $\square$

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Why "Kripke=Heyting"?

- Kripke semantics in $\mathcal{C}$ = Heyting semantics in $\{$upsets of $C\}$:

$$c \Vdash \varphi \iff c \in [\![\varphi]\!].$$

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- In fact: $\mathbf{Pos}_{\mathrm{fin}}^{\mathrm{op}} \sim \mathbf{Heyt}_{\mathrm{fin}} \quad (\sim \mathbf{DL}_{\mathrm{fin}})$.

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & \mathrm{Up}(X) \\
Y \arrow[r, mapsto] & \mathrm{Up}(Y) \arrow[u, "\mathrm{Up}(f)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
H \arrow[r, mapsto] \arrow[d, "g"'] & \mathrm{spec}(H) \\
K \arrow[r, mapsto] & \mathrm{spec}(K) \arrow[u, "\mathrm{spec}(g)"']
\end{tikzcd}$$

What about the infinite case?

Stone's slogan:

"A cardinal principle of modern mathematical research may be stated as a maxim: *One must always topologize*."

Marshall Harvey Stone (1938a). "The representation of Boolean algebras". In: *Bulletin of the American Mathematical Society* **44**.(12), pp. 807–816.

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Examples

- **Spec** $\sim$ **DL**$^{\mathrm{op}}$ (certain compact spaces vs. distributive lattices).

Marshall Harvey Stone (1938b). "Topological representations of distributive lattices and Brouwerian logics". In: *Časopis pro pěstování matematiky a fysiky* **67**.(1), pp. 1–25.

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Examples

- $\mathbf{Spec} \sim \mathbf{DL}^{\mathrm{op}}$ (certain compact spaces vs. distributive lattices).
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Marshall Harvey Stone (1936). "The theory of representations for Boolean algebras". In: *Transactions of the American Mathematical Society* **40**.(1), pp. 37–111.

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Hilary A. Priestley (1970). "Representation of distributive lattices by means of ordered stone spaces". In: *Bulletin of the London Mathematical Society* **2**.(2), pp. 186–190.

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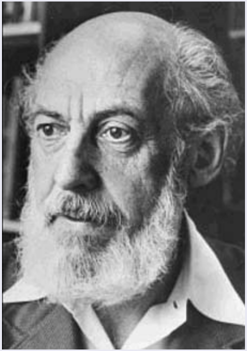
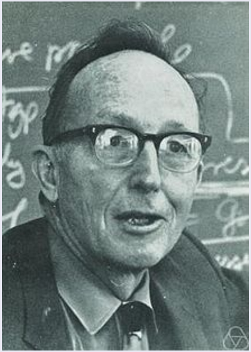
- **CompHaus** $\sim$ $C^*\text{-}\mathbf{Alg}^{\mathrm{op}}$ (compact T2 spaces vs. certain Banach algebras).

Izrail Gelfand (1941). “Normierte Ringe”. In: *Recueil Mathématique. Nouvelle Série* **9**.(1), pp. 3–24.

Category theory

Sammy Eilenberg (1913 – 1998) and Saunders MacLane (1909 – 2005)





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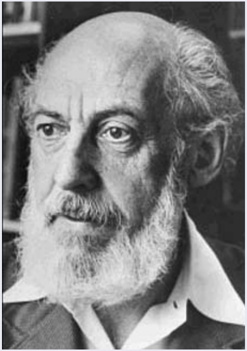

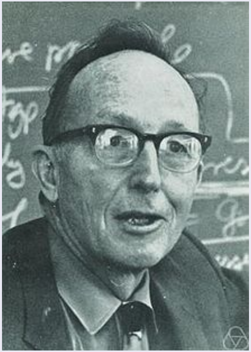

- Started in the 1940's in their work about algebraic topology.
- Is by now present in (almost) all areas of mathematics and also extensively used in physics and in computer science.

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Definition

A category $\mathbf{X}$ consists of

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A category **X** consists of

- a collection of objects $X, Y, \dots,$

Think of

- vector spaces, topological spaces, Banach spaces,. . .

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- arrows can be composed (associativity)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "g\cdot f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

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- the composite of linear maps is linear, . . .

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- for every object there is an identity arrow $1_X\colon X \to Y$.

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- The identity map is linear . . . .

Examples

Every field of mathematics defines (at least) one category

Top, **Ab**, $\mathbf{Vec}_{\mathrm{fd}}$,

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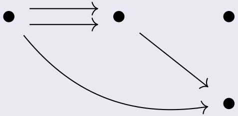
An abstract category . . .



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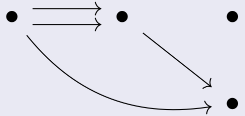

Definition

For every category $\mathbf{X}$, there is the dual category $\mathbf{X}^{\mathrm{op}}$ with the same objects but all arrows point in the opposite direction.

Examples

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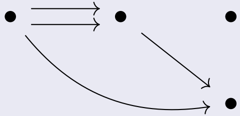

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$$\begin{tikzcd}
\bullet \arrow[r, shift left] \arrow[r, shift right] \arrow[rrd, bend right] & \bullet \arrow[rd] & \bullet \\
& & \bullet
\end{tikzcd}$$

. . . and its dual

$$\begin{tikzcd}
\bullet & \bullet \arrow[l, shift left] \arrow[l, shift right] & \bullet \\
& & \bullet \arrow[lu] \arrow[llu, bend left]
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Some typical categorical notions

Isomorphism

An arrow $f\colon X \to Y$ in a category $\mathbf{X}$ is called an isomorphism whenever there is some arrow $g\colon Y \to X$ with

$$g \cdot f = 1_X \qquad \text{and} \qquad f \cdot g = 1_Y .$$

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Product in $\mathbf{X}$

$$\begin{tikzcd}
& X \\
Z \arrow[ur, bend left] \arrow[r, dotted, "\text{pairing}"] \arrow[dr, bend right] & X \times Y \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\
& Y
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Sum in $\mathbf{X}$

$$\begin{tikzcd}
X \arrow[d, "i_1"'] \arrow[dr, bend left] & \\
X + Y \arrow[r, dotted, "\text{if then else}"] & Z \\
Y \arrow[u, "i_2"] \arrow[ur, bend right] &
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$$\begin{tikzcd}
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Z \arrow[ur, bend left] \arrow[r, dotted, "\text{pairing}"] \arrow[dr, bend right] & X \times Y \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\
 & Y
\end{tikzcd}$$

Sum in $\mathbf{X}$ = product in $\mathbf{X}^{\mathrm{op}}$

$$\begin{tikzcd}
X \arrow[d, "i_1"'] \arrow[dr, bend left] & \\
X + Y \arrow[r, dotted, "\text{if then else}"] & Z \\
Y \arrow[u, "i_2"] \arrow[ur, bend right] &
\end{tikzcd}$$

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$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"'] & GF(X) \arrow[d, "GF(f)"] \\
X' \arrow[r, "\eta_{X'}"'] & GF(X')
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "\varepsilon_Y"] \arrow[d, "g"'] & FG(Y) \arrow[d, "FG(g)"] \\
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Adjunction

As above but the arrows $\eta_X\colon X \to GFX$ and $\varepsilon_Y\colon FGY \to Y$ need not be isomorphisms; moreover:

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1"'] & FGF \arrow[d, "\varepsilon_F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[dr, "1"'] & GFG \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd}$$

Representable functors

Example

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Remark

Representable functors preserve limits (i.e. $\mathbf{X}(-,X)$ sends coproducts to products).

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- For $|-|\colon \mathbf{DL} \to \mathbf{Set}$: $|-| \simeq \mathbf{Vec}(3,-)$.

Next Goal

Analyse the structure of dual adjunctions and how to construct them.

The mother of all dual equivalences...

Theorem

The category **BA** *of Boolean algebras and homomorphisms is dually equivalent to the category* **BooSp** *of Boolean spaces (= zero-dimensional compact Hausdorff spaces) and continuous maps:*

$$\mathbf{BooSp} \sim \mathbf{BA}^{\mathrm{op}}.$$

References

Marshall Harvey Stone (1936). "The theory of representations for Boolean algebras". In: *Transactions of the American Mathematical Society* **40**.(1), pp. 37–111.

Marshall Harvey Stone (1937). "Applications of the theory of Boolean rings to general topology". In: *Transactions of the American Mathematical Society* **41**.(3), pp. 375–481.

M.H. Stone

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The structure of dual adjunctions

References

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We consider categories with respresentable forgetful functors

$$|-| \simeq \mathbf{X}(X_0, -)\colon \mathbf{X} \longrightarrow \mathbf{Set} \quad \textit{and} \quad |-| \simeq \mathbf{A}(A_0, -)\colon \mathbf{A} \longrightarrow \mathbf{Set}.$$

and an adjunction $\begin{tikzcd} \mathbf{X} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{A}^{\mathrm{op}} \arrow[l, bend left, "G"] \end{tikzcd}$.

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2. *The units are "essentially" given by*

$$\eta_X\colon |X| \longrightarrow |GFX| \simeq \mathbf{A}(FX, \widetilde{A}),\ x \longmapsto \mathrm{ev}_x$$

with ev_x *denoting the evaluation map (and similar for* ε*).*

How to construct dual adjunctions?

Assumption

We assume that $\widetilde{X}$ and $\widetilde{A}$ are objects in $\mathbf{X}$ and $\mathbf{A}$ respectively, with the same underlying set $|\widetilde{X}| = |\widetilde{A}|$.

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Goal

We wish to lift the hom-functors

$$\mathrm{hom}(-, \widetilde{X}) : \mathbf{X}^{\mathrm{op}} \to \mathbf{Set} \qquad \text{and} \qquad \mathrm{hom}(-, \widetilde{A}) : \mathbf{A}^{\mathrm{op}} \to \mathbf{Set}$$

to functors

$$F : \mathbf{X}^{\mathrm{op}} \to \mathbf{A} \qquad \text{and} \qquad G : \mathbf{A}^{\mathrm{op}} \to \mathbf{X}$$

in such a way that the maps defined by $x \mapsto \mathrm{ev}_x$ underlay an **X**-morphism respectively and **A**-morphism.

How to construct dual adjunctions? (cont.)

Initial structures

A family $\mathcal{C} = (f_i \colon A \to A_i)_{i \in I}$ in $\mathbf{A}$ is called initial with respect to $|-|\colon \mathbf{A} \to \mathbf{Set}$ if for every family $\mathcal{D} = (g_i \colon B \to A_i)_{i \in I}$ and every map $h : |B| \to |A|$ such that $|\mathcal{D}| = |\mathcal{C}| \cdot h$, there exists a unique $\mathbf{A}$-morphism $\overline{h}\colon D \to C$ with $\mathcal{D} = \mathcal{C} \cdot \overline{h}$ and $h = |\overline{h}|$.

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Main conditions

(Init X) For each object X in $\mathbf{X}$, the cone

$$(\mathrm{ev}_x : \hom(X, \widetilde{X}) \to |\widetilde{A}|, \psi \mapsto \psi(x))_{x \in |X|}$$

admits an initial lift $(\mathrm{ev}_x : F(X) \to \widetilde{A})_{x \in |X|}$.

(Init A) . . .

How to construct dual adjunctions? (cont.)

Theorem

If conditions (InitX) and (InitA) are fulfilled, then these initial lifts define the object parts of an adjunction $\begin{tikzcd}\mathbf{X} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{A}^{\mathrm{op}} \arrow[l, bend left, "G"]\end{tikzcd}$.

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Proof.

Consider $f : X \to Y$ in $\mathbf{X}$. Then

$$\begin{tikzcd} \mathbf{X}(Y, \widetilde{X}) \arrow[r, "\mathbf{X}(f{,}\widetilde{X})"] \arrow[dr, "\mathrm{ev}_{fx}"'] & \mathbf{X}(X, \widetilde{X}) \arrow[d, "\mathrm{ev}_{x}"] \\ & |\widetilde{A}| \end{tikzcd}$$

commutes, hence $\mathbf{X}(f, \widetilde{X})$ is an $\mathbf{A}$-morphism $Ff : FY \to FX$.

□

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Proof.

For every $\psi : X \to \widetilde{X}$, the diagram

$$\begin{tikzcd}|X| \arrow[r, "\eta_X"] \arrow[dr, "\psi"'] & |G(FX)| \arrow[d, "\mathrm{ev}_\psi"] \\ & |\widetilde{X}|\end{tikzcd}$$

commutes. Hence η_X is an **X**-morphism.

$\square$

How to guarantee (InitX) and (InitA)?

The trivial case

If $|-|\colon \mathbf{X} \to \mathbf{Set}$ admits all initial lifts (is topological), then (InitX).

Examples: **Top**, **Ord**, **Met**, . . .

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Proposition

Let **A** *be the category of algebras for a signature* Ω *of operation symbols and assume that* **X** *is complete and* $|-|\colon \mathbf{X} \to \mathbf{Set}$ *preserves limits. Furthermore, assume that, for every operation symbol* $\omega \in \Omega$, *the corresponding operation* $|\widetilde{A}|^{I} \to |\widetilde{A}|$ *underlies an* **X***-morphism* $\widetilde{X}^{I} \to \widetilde{X}$. *Then both (InitX) and (InitA) are fulfilled.*

Joachim Lambek and Basil A. Rattray (1979). "A general Stone–Gelfand duality". In: *Transactions of the American Mathematical Society* **248**.(1), pp. 1–35.

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Proof.

For (InitA): Define the operations on $\mathbf{X}(X, \widetilde{X})$ "pointwise".

For instance, $\omega(h_1, h_2)$ is the composite $X \xrightarrow{\langle h_1, h_2\rangle} \widetilde{X} \times \widetilde{X} \xrightarrow{\omega} \widetilde{X}$. □

From Adjunctions to Equivalences

Theorem

Every adjunction $\mathbf{X} \begin{tikzcd}[ampersand replacement=\&] {} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] \& {} \arrow[l, bend left, "G"] \end{tikzcd} \mathbf{A}^{\mathrm{op}}$ *can be restricted to the full subcategories* $\mathsf{Fix}(\eta)$ *and* $\mathsf{Fix}(\varepsilon)$ *of* $\mathbf{X}$ *respectively* $\mathbf{A}$*, defined by the classes of objects*

$$\{X \mid \eta_X \text{ is an isomorphism}\} \quad \text{and} \quad \{A \mid \varepsilon_A \text{ is an isomorphism}\},$$

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Remark

These fixed subcategories might be empty.

Improving the adjunction

Assumption

Assume that the dual adjunction is constructed using (InitX) and (InitA).

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Remark

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Recall that

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Remark

Our adjunction restricts to $\mathrm{InitCog}(\widetilde{X})$ and $\mathrm{InitCog}(\widetilde{A})$.

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Here we consider:

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$$\begin{aligned} \mathrm{InitCog}(\widetilde{X}) &= \{\text{zero-dimensional compact Hausdorff spaces}\} \\ &= \mathbf{BooSp}. \end{aligned}$$

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  where the units are pointwise embeddings.

Stone duality (continuation)

Theorem (Stone-Weierstraß type)

Let X be a Boolean space and $m\colon B \to FX$ be an embedding in **BA** *so that $(m(x)\colon X \to \widetilde{X})_{x \in B}$ is (initial and) point-separating. Then m is an isomorphism.*

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$\mathbf{Spec} \sim \mathbf{DL}^{\mathrm{op}}$.

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> *"The topological spaces which arise as duals of Boolean algebras may be characterized as those which are compact and totally disconnected (i.e. the Stone spaces); the corresponding purely topological characterization of the duals of distributive lattices obtained by Stone is less satisfactory. In the present paper we show that a much simpler characterization in terms of ordered topological spaces is possible."*

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Spectral spaces also appear in:

Melvin Hochster (1969). “Prime ideal structure in commutative rings”. In: *Transactions of the American Mathematical Society* **142**, pp. 43–60.

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$$\mathbf{CompHaus}^{\mathrm{op}} \xrightarrow{\mathsf{hom}(-,[0,1])} \mathbf{Set},$$

it is faithful since $[0,1]$ is a cogenerator in **CompHaus** (Tietze-Urysohn).

This is what gave the general idea of the notion of mathematical structure. Let us say immediately that this notion has since been superseded by that of category and functor, which includes it under a more general and convenient form.

Jean A. Dieudonné (1970). "The work of Nicholas Bourbaki". In: The American Mathematical Monthly ***77****.(2), pp. 134–145.*

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The algebraic theory of **CompHaus**$^{\mathrm{op}}$

- **CompHaus**$^{\mathrm{op}}$ $\xrightarrow{\mathrm{hom}(-,[0,1])}$ **Set** is algebraic (monadic).

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Theorem

The category **A** *is monadic over* **Set** *iff* **A** *is an exact category and has a regularly projective regular generator with arbitrary copowers.*[a]

---

[a] John MacDonald and Manuela Sobral (2004). "Aspects of Monads". In: *Categorical Foundations: Special Topics in Order, Topology, Algebra, and Sheaf Theory*. Ed. by Maria Cristina Pedicchio and Walter Tholen. Cambridge: Cambridge University Press, pp. 213–268.

Recall: A category is exact if it has finite limits, coequalizers of kernel pairs, pullback stable regular epimorphisms and all equivalence relations are effective.

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- $\mathbf{CompHaus}^{\mathrm{op}} \xrightarrow{\hom(-,[0,1])} \mathbf{Set}$ is algebraic (monadic).
- $[0,1]$ is $\aleph_1$-copresentable in **CompHaus**.

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  Recall: An object X of a category $\mathbf{X}$ is called λ-presentable provided that its hom-functor $\hom(X,-) : \mathbf{X} \to \mathbf{Set}$ preserves λ-directed (equivalently, λ-filtered) colimits.

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  Recall: An object X of a category $\mathbf{X}$ is called λ-presentable provided that its hom-functor $\hom(X,-)\colon \mathbf{X} \to \mathbf{Set}$ preserves λ-directed (equivalently, λ-filtered) colimits.

Theorem

A compact Hausdorff space is

- *finitely copresentable iff it is finite;*
- $\aleph_1$*-copresentable iff it is metrisable.*

The algebraic theory of $\mathbf{CompHaus}^{\mathrm{op}}$

- $\mathbf{CompHaus}^{\mathrm{op}} \xrightarrow{\mathsf{hom}(-,[0,1])} \mathbf{Set}$ is algebraic (monadic).
- $[0,1]$ is $\aleph_1$-copresentable in **CompHaus**.
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Isbell uses four finitary and one infinitary operation, the interpretation of the latter in $[0,1]$ is "sum":

$$\sigma\colon [0,1]^{\mathbb{N}} \longrightarrow [0,1],\ (x_n) \longmapsto \sum_{n=1}^{\infty} \frac{x_n}{2^n}.$$

(Gives limits: $\lim\limits_{n\to\infty} \varphi_n = \varphi_1 + (\varphi_2 - \varphi_1) + \ldots$.)

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Remark

$\mathbf{CompHaus}^{\mathrm{op}}$ embeds fully into a *finitary* variety, the infinitary operation is only needed to describe the objects.

Gelfand duality

Izrail Gelfand (1941). “Normierte Ringe”. In: *Recueil Mathématique. Nouvelle Série* **9**.(1), pp. 3–24.

Here we consider:

- $\mathbf{X} = \textbf{CompHaus}$ and $\mathbf{A} = C^*\text{-}\textbf{Alg}$ (normed $\mathbb{C}$-algebras $\ldots$).

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- $\mathbb{D}$ is a cogenerator in **CompHaus** (Tietze-Urysohn) and $\mathbb{C}$ is a cogenerator in $C^*\text{-}\mathbf{Alg}$ (Gelfand): for every $x \in A$,

$$\|x\| = \sup\{|\varphi(x)| \mid \varphi \in C^*\text{-}\mathbf{Alg}(A, \mathbb{C})\}.$$

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$$\|x\| = \sup\{|\varphi(x)| \mid \varphi \in C^*\text{-}\mathbf{Alg}(A, \mathbb{C})\}.$$

- In fact, $(\mathbb{D}, \mathbb{C})$ induces an equivalence $\mathbf{CompHaus} \sim C^*\text{-}\mathbf{Alg}^{\mathrm{op}}$.

Next Goal

We are now going to present conditions which guarantee that our adjunction

$$\begin{tikzcd}\mathbf{X} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{A}^{\mathrm{op}} \arrow[l, bend left, "G"]\end{tikzcd}$$

is already an equivalence provided that its restriction to the full subcategories $\mathbf{X}_{\mathrm{fin}}$ and $\mathbf{A}_{\mathrm{fin}}$ of finite objects of $\mathbf{X}$ and $\mathbf{A}$ respectively is.

Some References:

Peter T. Johnstone (1986). *Stone spaces*. Vol. 3. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press. xxii + 370. Reprint of the 1982 edition.

David M. Clark and Brian A. Davey (1998). *Natural dualities for the working algebraist*. Vol. 57. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press. xii + 356.

Dirk Hofmann (2002). "A generalization of the duality compactness theorem". In: *Journal of Pure and Applied Algebra* **171**.(2-3), pp. 205–217.

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2. F sends cofiltered limits of finite objects to colimits.
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Remark

Being part of an adjunction, G preserves limites (sends colimits in $\mathbf{A}$ to limits in $\mathbf{X}$).

Describing the strategy (cont.)

The “todo-list”

1. Each object A in $\mathbf{A}$ is a filtered (directed) colimit of finite objects.
2. F sends cofiltered limits of finite objects to colimits.
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Remark

We have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "\varepsilon_A"] & FG(A) \\
A_i \arrow[u, "c_i"] \arrow[r, "\varepsilon_{A_i}"'] & FG(A_i) \arrow[u, "FG(c_i)"']
\end{tikzcd}$$

where the left hand side and the right hand side are colimit cones. Then, if all ε_{A_i} are isomorphisms, also ε_A is an isomorphism (and similar for η).

Regarding (1)

Recall:

In many categories **A** with "algebraic flavour" (i.e. sets equipped with finitary operations or relations; more technical: a model category of a limit sketch in **Set**[a]), every object is a directed colimit of *finitely presentable objects*.

---

[a] Peter Gabriel and Friedrich Ulmer (1971). *Lokal präsentierbare Kategorien*. Lecture Notes in Mathematics, Vol. 221. Berlin: Springer-Verlag, pp. v + 200.

Jiří Adámek and Jiří Rosický (1994). *Locally presentable and accessible categories*. Vol. 189. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press. xiv + 316.

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Proposition

Under “mild” conditions, every object of $\mathrm{InitCog}(\widetilde{A})$ *is a directed colimit of finite objects.*
(Since the reflector $R\colon \mathbf{A} \to \mathrm{InitCog}(\widetilde{A})$ *sends finitely presentable to finite objects.)*

Regarding (2)

Proposition

Assume $\mathbf{A} = \mathrm{InitCog}(\widetilde{A})$*. Let* $D : I \to \mathbf{X}$ *be a diagram in* $\mathbf{X}$ *with a concrete limit* $(p_i\colon L \to D(i))_{i \in I}$ *such that, for each* $i \in I$*,* $\eta_{D(i)}$ *is an isomorphism.*
Then $(F(p_i)\colon F(L) \to FD(i))_{i \in I}$ *is a colimit of* $FD : I^{\mathrm{op}} \to \mathbf{A}$ *if* $\mathbf{X}(-, \widetilde{X})$ *sends* $(p_i\colon L \to D(i))_{i \in I}$ *to a colimit of* $\mathbf{X}(D(-), \widetilde{X})\colon I^{\mathrm{op}} \to \mathbf{Set}$*.*

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In a nutshell. . .

The proposition above reduces (2) to the corresponding condition on $\mathbf{X}(-, \widetilde{X}) \colon \mathbf{X} \to \mathbf{Set}$ which is often more easily verified.

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Theorem

Assume that $\mathbf{X}$ *is a category of Boolean spaces equipped with finitary operations or relations (more technical: a model category of a limit sketch in* **BooSp***).*

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Theorem

Assume that $\mathbf{X}$ *is a category of Boolean spaces equipped with finitary operations or relations (more technical: a model category of a limit sketch in* **BooSp***).*
If "the theory of $\mathbf{X}$ *is finitely generated", then* $\mathbf{X}(-, \widetilde{X})\colon \mathbf{X} \to \mathbf{Set}$ *sends cofiltered limits to filtered colimits.*[a]

---

[a] Recall that $\widetilde{X}$ is finite

Regarding (3)

Theorem

Let $D\colon I \to \mathbf{CompHaus}$ *be a cofiltered diagram and consider* $(p_i\colon X \to D(i))_{i\in I}$*. The following assertions are equivalent.*

- (i) $(p_i\colon X \to D(i))_{i\in I}$ *is a limit of* D*.*
- (ii) *The following two conditions are fulfilled.*
  1. $(p_i\colon X \to D(i))_{i\in I}$ *is point separating.*
  2. *For each* $i \in I$*,* $\operatorname{im} p_i = \bigcap_{j \xrightarrow{k} i} \operatorname{im} D(k)$*.*

Nicolas Bourbaki (1942). *Éléments de mathématique. 3. Pt. 1: Les structures fondamentales de l'analyse. Livre 3: Topologie générale.* Paris: Hermann & Cie.

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- **BooSp** is closed in **CompHaus** under limits, therefore this characterization holds for cofiltered limits in **BooSp** as well.

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Remark

- **BooSp** is closed in **CompHaus** under limits, therefore this characterization holds for cofiltered limits in **BooSp** as well.
- If we have a "nice" $|-|\colon \mathbf{X} \to \mathbf{BooSp}$, then $(p_i\colon L \to D(i))_{i\in I}$ is a limit in $\mathbf{X}$ iff it is a limit in **BooSp** and initial with respect to $|-|$.

Regarding (3) (cont.)

Recall the "Bourbaki conditions":

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- $(p_i \colon X \to D(i))_{i \in I} =$ *all morphisms into finite objects.*

Then $(p_i \colon X \to D(i))_{i \in I}$ *is*

- *point-separating and initial since it “contains”* $(X \to \widetilde{X})$, *and*

Regarding (3) (cont.)

Recall the "Bourbaki conditions":

1. $(p_i \colon X \to D(i))_{i \in I}$ is point separating.
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hence $(p_i \colon X \to D(i))_{i \in I}$ *is a limit.*

Putting everything together

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Theorem

Then the adjunction

$$\begin{tikzcd}
\mathrm{InitCog}(\widetilde{X}) \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{InitCog}(\widetilde{A})^{\mathrm{op}} \arrow[l, bend left, "G"]
\end{tikzcd}$$

is an equivalence provided that its restriction to the finite objects is an equivalence.

The "two for the price of one" principle

Recall

Priest $\sim$ $\mathbf{DL}^{\mathrm{op}}$ (a Priestley space is a special ordered Boolean space).

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Back to all objects

$\widetilde{A}$ is an initial cogenerator in **Pos** (easy) and $\widetilde{X}$ is an initial cogenerator in **BooSpDL** (not so easy); hence **BooSpDL** $\sim$ **Pos**.

Next Goal

We consider now “categories of relations” (instead of functions). This way we include Heyting algebras and modal algebras.

Motivating Example: $\mathrm{CoAlg}(V) \sim \mathbf{BAO}^{\mathrm{op}}$

References:

Bjarni Jónsson and Alfred Tarski (1951). "Boolean algebras with operators. I". In: *American Journal of Mathematics* **73**.(4), pp. 891–939.

Bjarni Jónsson and Alfred Tarski (1952). "Boolean algebras with operators. II". In: *American Journal of Mathematics* **74**.(1), pp. 127–162.

Giovanni Sambin and Virginia Vaccaro (1988). "Topology and duality in modal logic". In: *Annals of Pure and Applied Logic* **37**.(3), pp. 249–296.

Clemens Kupke, Alexander Kurz, and Yde Venema (2004). "Stone coalgebras". In: *Theoretical Computer Science* **327**.(1-2), pp. 109–134.

Motivating Example: $\mathrm{CoAlg}(V) \sim \mathbf{BAO}^{\mathrm{op}}$

Definitions

- The Vietoris functor[a] $V\colon \mathbf{BooSp} \to \mathbf{BooSp}$ sends X to the space VX of all closed subsets of X, and $Vf\colon VX \to VY$, $A \mapsto f[A]$.

---

[a] Leopold Vietoris (1922). "Bereiche zweiter Ordnung". In: *Monatshefte für Mathematik und Physik* **32**.(1), pp. 258–280.

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Proof of $\mathrm{CoAlg}(V) \sim \mathbf{BAO}^{\mathrm{op}}$.

The functor $F\colon \mathrm{CoAlg}(V) \to \mathbf{BAO}$ sends (X, α) to $\mathbf{BooSp}(X, 2)$ . . . □

A better way . . .

“A posteriori, the first and fundamental result in duality theory is Jónsson-Tarski representation theorem for modal algebras [19], which was substantially improved by Halmos [20], who implicitly introduced categories.” [a]

---

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A better way . . .

Theorem (Halmos duality, 1956)

Consider the category **BooSpRel** *of Boolean spaces and continuous relations*[a] *and the category* $\mathbf{BA}_{\perp,\vee}$ *of Boolean algebras and hemimorphisms*[b]. *Then*

$$\mathbf{BooSpRel} \sim \mathbf{BA}^{\mathrm{op}}_{\perp,\vee}.$$

---

[a] will be explained in a minute (or two or three . . .)

[b] maps preserving finite suprema

Paul R. Halmos (1956). "Algebraic logic I. Monadic Boolean algebras". In: *Compositio Mathematica* **12**, pp. 217–249.

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Corollary

$\mathbf{BooSp} \sim \mathbf{BA}^{\mathrm{op}}$.

Since a relation is a function iff the corresponding hemimorphism is a homomorphism.

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A short trip to algebra

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Universal algebras

Fix a collection of operation symbols

Think of

- $+, -, 0 \ldots$

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Algebra: $(X, \alpha\colon TX \to X)$

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- $(3 + 4) + 5$

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$TX = \{ \text{ terms on } X \}/\sim$
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More abstract

A monad on a category $\mathbf{X}$ is a triple (T, m, e) such that. . .

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Examples

- Power set monad $\mathbb{P} = (P, m, e)$ on **Set**
- Vietoris monad $\mathbb{V} = (V, m, e)$ on **BooSp** (or **Priest** or ...)

The Kleisli construction

Adjunctions induce monads

Every adjunction

$$F\colon \mathbf{X} \to \mathbf{A}, \qquad G\colon \mathbf{A} \to \mathbf{X}, \qquad \eta_X\colon X \to GFX, \qquad \varepsilon_A\colon FGA \to A$$

induces a monad (T, e, m) on $\mathbf{X}$ with $T = GF$, $e = \eta$ and $m = G\varepsilon F$.

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Monads come from adjunctions: the Kleisli construction

For a monad $\mathbb{T} = (T, m, e)$ on $\mathbf{X}$, we have the Kleisli category $\mathbf{X}_{\mathbb{T}}$:

Heinrich Kleisli (1965). "Every standard construction is induced by a pair of adjoint functors". In: *Proceedings of the American Mathematical Society* **16**.(3), pp. 544–546.

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We get an adjunction $$\begin{tikzcd} \mathbf{X}_{\mathbb{T}} \arrow[r, bend left, "G"] \arrow[r, phantom, "\top"] & \mathbf{X} \arrow[l, bend left, "F"] \end{tikzcd}$$ which induces $\mathbb{T} = (T, m, e)$.

Adjunctions vs. monads

Examples

$\mathbf{Set}_{\mathbb{P}} \sim \mathbf{Rel}$ and $\mathbf{BooSp}_{\mathbb{V}} \sim \mathbf{BooSpRel}$

(and hence a coalgebra $\alpha\colon X \to VX$ is an endomorphism in **BooSpRel**).

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An adjunction from adjunctions

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Here a left morphism of adjunctions over $\mathbf{X}$ is a functor $J : \mathbf{A} \to \mathbf{A}'$ with $F' = JF$.

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This adjunction restricts to an "equivalence" between monads and Kleisli adjunctions (i.e. where F is surjective on objects).

Halmos revisited

Using Kleisli adjunctions

$$\begin{tikzcd}
\mathbf{BooSp}_{\mathbb{V}} \arrow[rr, "C=\hom(-,1)"] \arrow[dr, bend left] & & \mathbf{BA}^{\mathrm{op}}_{\perp,\vee} \arrow[dl, bend right] \\
& \mathbf{BooSp} \arrow[ul, bend left] \arrow[ur, bend right, "C=\hom(-,2)"'] &
\end{tikzcd}$$

induces a monad morphism with components

$$VX \longrightarrow \hom(CX,2),\ A \longmapsto \{B \in CX \mid B \cap A \neq \varnothing\};$$

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which is indeed an isomorphism. Hence

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Similarly:

$\mathbf{Priest}_{\mathbb{V}} \sim \mathbf{DL}_{\perp,\vee}$, $\mathbf{Top}_{\mathbb{V}} \sim \mathbf{Frm}_{\bigvee}$, ...

Duality for (co)Heyting algebras

Theorem

A distributive lattice L is a co-Heyting algebra iff L is a split subobject of a Boolean algebra in $\mathbf{DL}_{\perp,\vee}$.[a]

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[a] J. C. C. McKinsey and Alfred Tarski (1946). "On closed elements in closure algebras". In: *Annals of Mathematics. Second Series* **47**.(1), pp. 122–162.

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Recall:

- $e\colon X \to X$ idempotent if $e \cdot e = e$
- if $r \cdot s = 1$, then $e = s \cdot r$ is idempotent.
- A category is idempotent split complete whenever every idempotent is of this form.
- Most "everyday categories" are idempotent split complete.
- **Rel** is not.
- A bit surprisingly, $\mathbf{Priest}_{\mathbb{V}}$ is (because $\mathbf{DL}_{\perp,\vee}$ is).

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Definition

A Priestley X is called an Esakia space whenever, for every open subset A of X, its down-closure $\downarrow A$ is open in X.[a]

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[a] Leo Esakia (1974). "Topological Kripke models". In: *Doklady Akademii Nauk SSSR* **214**, pp. 298–301.

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Theorem

From $\mathbf{BooSp}_{\mathbb{V}} \sim \mathbf{BA}^{\mathrm{op}}_{\perp,\vee}$ *we get* $\mathbf{EsaRel} \sim {\mathbf{coHeyt}_{\perp,\vee}}^{\mathrm{op}}$ $(\sim {\mathbf{Heyt}_{\top,\wedge}}^{\mathrm{op}})$*.*

Next Goal

- Start with $\begin{tikzcd} \mathbf{X} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{A}^{\mathrm{op}} \arrow[l, bend left, "G"] \end{tikzcd}$ (with units embeddings).

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Liftings to Kleisli categories

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Let be $\mathbb{T} = (T, m, e)$ *a monad on* $\mathbf{X}$ *and* $F \dashv G$ *an adjunction*

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(i) Functors $F \colon \mathbf{X}_{\mathbb{T}} \to \mathbf{A}^{\mathrm{op}}$ *so that*

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(iii) $\mathbb{T}$-algebra structures $\sigma\colon T\widetilde{X} \to \widetilde{X}$ such that the map

$$\hom(X, \widetilde{X}) \longrightarrow \hom(TX, \widetilde{X}),\ \psi \longmapsto \sigma \cdot T\psi$$

is an $\mathbf{A}$*-morphism $\kappa_X\colon FX \to FTX$, for every object X in* $\mathbf{X}$.

Duality for $\mathbf{PosComp}^{\mathrm{op}}$

We consider the Vietoris monad $\mathbb{V}$ on $\mathbf{X} = \mathbf{PosComp}$, with $\widetilde{X} = [0,1]^{\mathrm{op}}$ and $\mathbb{V}$-algebra structure

$$V([0,1]^{\mathrm{op}}) \longrightarrow [0,1]^{\mathrm{op}},\ A \longmapsto \sup_{x\in A} x.$$

Then, for a category $\mathbf{A}$ and an adjunction

$$\begin{tikzcd}
\mathbf{PosComp} \arrow[r, bend left, "C"] \arrow[r, phantom, "\perp"] & \mathbf{A}^{\mathrm{op}} \arrow[l, bend left, "G"]
\end{tikzcd}$$

induced by $([0,1]^{\mathrm{op}}, [0,1])$ and compatible with the $\mathbb{V}$-algebra structure on $[0,1]^{\mathrm{op}}$, the corresponding monad morphism j has as components the maps

$$j_X : VX \longrightarrow G(CX),\ A \longmapsto (\Phi_A : CX \to [0,1],\ \psi \mapsto \sup_{x\in A} \psi(x)).$$

We wish to find an appropriate category $\mathbf{A}$ so that j is an isomorphism.

Duality for **PosComp**$^{\mathrm{op}}$

Our approach

First recall: Compare metrics $a\colon X \times X \to [0,\infty]$ with order relations $X \times X \to \mathbf{2}$:

$$\begin{array}{ccc} 0 \geqslant a(x,x) & \text{and} & a(x,y)+a(y,z) \geqslant a(x,z). \\ \top \implies (x \leq x) & \text{and} & (x \leq y)\,\&\,(y \leq z) \implies (x \leq z). \end{array}$$

F. William Lawvere (1973). “Metric spaces, generalized logic, and closed categories”. In: *Rendiconti del Seminario Matemàtico e Fisico di Milano* **43**.(1), pp. 135–166. Republished in: Reprints in Theory and Applications of Categories, No. 1 (2002), 1–37.

Felix Hausdorff (1914). *Grundzüge der Mengenlehre*. Leipzig: Veit & Comp. viii + 476.

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where (??) is the category of

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Hausdorff (1914), *Grundzüge der Mengenlehre*

Thinking of an order relation on a set M as a function

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Hausdorff observes that

> *"Nun steht einer Verallgemeinerung dieser Vorstellung nichts im Wege, und wir können uns denken, daß eine beliebige Funktion der Paare einer Menge definiert, d.h. jedem Paar (a, b) von Elementen einer Menge M ein bestimmtes Element $n = f(a, b)$ einer zweiten Menge N zugeordnet sei. In noch weiterer Verallgemeinerung können wir eine Funktion der Elementtripel, Elementfolgen, Elementkomplexe, Teilmengen u. dgl. von M in Betracht ziehen."*

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> *Now there stands nothing in the way of a generalisation of this idea, and we can think of an arbitrary function of pairs of points which associates to each pair (a, b) of elements of a set M a specific element $n = f(a, b)$ of a second set N. Generalising further, we can consider a function of triples, sequences, complexes, subsets, etc.*

Categories are everywhere . . .

> *The kinds of structures which actually arise in the practice of geometry and analysis are far from being 'arbitrary' . . . , as concentrated in the thesis that fundamental structures are themselves categories.*
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Quantales and enriched categories

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Samuel Eilenberg and G. Max Kelly (1966). "Closed categories". In: *Proceedings of the Conference on Categorical Algebra: La Jolla 1965*. Ed. by Samuel Eilenberg, DK Harrison, H Röhrl, and Saunders MacLane. Springer Verlag, pp. 421–562.

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Enriched categories as actions

Some facts about $\mathcal{V}$-categories

- "$x \leq y$ whenever $k \leq a(x, y)$" defines a functor $\mathcal{V}$-**Cat** $\to$ **Ord**.

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- For X copowered and separated, we have $\otimes\colon X \times \mathcal{V} \to X$ with

$$x \otimes k = x, \quad (x \otimes u) \otimes v = x \otimes (u \otimes v), \quad x \otimes \bigvee_{i \in I} u_i = \bigvee_{i \in I} (x \otimes u_i).$$

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The bottom line

copowered $\mathcal{V}$-categories $=$ ordered sets with an action of $\mathcal{V}$.

Duality for $\mathbf{PosComp}^{\mathrm{op}}$ (part II)

$$\begin{tikzcd}
\mathbf{PosComp}_{\mathbb{V}} \arrow[rr, "C"] & & \mathrm{LaxMon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}, \\
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The induced monad morphism j is precisely given by the family of maps

$$j_X \colon VX \longrightarrow [CX,[0,1]],\; A \longmapsto \Phi_A, \qquad \Phi_A(\psi)=\sup_{x\in A}\psi(x).$$

Dirk Hofmann and Pedro Nora (2016). *Enriched Stone-type dualities*. Tech. rep. arXiv: `1605.00081 [math.CT]`.

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Remark

Does not work for $\otimes = \wedge$ (but can be fixed by adding unary operations).

A Stone–Weierstraß theorem for $[0,1]_\odot$-categories

Let **A** be the category with objects all $[0,1]_\odot$-powered objects in

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Let $m : A \to CX$ be a mono in **A** *so that the cone $(m(a) : X \to [0,1]^{\mathrm{op}})_{a \in A}$ is point-separating and initial w.r.t.* $\mathbf{PosComp} \to \mathbf{Set}$*. Then m is an isomorphism in* **A** *if and only if A is Cauchy-complete (as $[0,1]_{\odot}$-category).*

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We apply this to $X = \hom(A,[0,1]), \quad A \longrightarrow C(X),\ a \longmapsto \mathrm{ev}_a.$

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Definition

We say that an object A of **A** has enough characters whenever the cone $(\varphi : A \to [0,1])_{\varphi}$ of all morphisms into $[0,1]$ separates the points of A.

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For $\otimes = \odot$, $\mathbf{PosComp}^{\mathrm{op}}_{\mathbb{V}} \simeq \mathbf{LaxA}_{[0,1],\mathrm{cc}}$ *and* $\mathbf{PosComp}^{\mathrm{op}} \simeq \mathbf{A}_{[0,1],\mathrm{cc}}$.

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We went from

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That is, *all of* **2** *is occupied by the* $[0,1]$.[a]

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[a] To paraphrase Asterix: *All of Gaul is occupied by the Romans.*

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Duality for metric compact Hausdorff spaces (main idea)

A brief description of the setting

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“metric compact Hausdorff space”

“metric topological space” with convergence $UX \times X \to [0,1]$[a]

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[a] Robert Lowen (1997). *Approach Spaces: The Missing Link in the Topology-Uniformity-Metric Triad*. Oxford Mathematical Monographs. Oxford: Oxford University Press. x + 253.

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[a] Dirk Hofmann (2014). "The enriched Vietoris monad on representable spaces". In: *Journal of Pure and Applied Algebra* **218**.(12), pp. 2274–2318.

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