DUALITY THEORY

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In [2], the authors make the seemingly paradoxical observation that "…an equation is only interesting or useful to the extent that the two sides are different!". Indeed, a moment's thought convinces us that an equation like $e^{i\omega} = \cos(\omega) + i\sin(\omega)$ is far more interesting than the rather dull statement that $3 = 3$, and a comparable remark applies if we go up in dimension: equivalent categories are thought to be essentially equal, but an equivalence is of more interest if the involved categories look different. Numerous examples of equivalences of "different" categories relate a category X and the dual of a category A; such an equivalence is called a *dual equivalence* or simply a *duality*. As examples we mention here the classical Stone-dualities (see [14, 15]) for Boolean algebras respectively distributive lattices, Esakia's duality theorem for Heyting algebras (see [4]), and the duality for Boolean algebras with operator of [9, 10, 11].

In these lectures we give a broad overview of several techniques and results concerning the study and construction of dual equivalences. We start by succinctly recalling the main ingredients from category theory, and then discuss the structure of dual equivalences and, more generally, of dual adjunctions between concrete categories. Here we will see that, under mild conditions, every such adjunction is induced by a so-called *dualising object*. Starting from the other end, we give sufficient conditions on a dualising object to yield a dual adjunction and illustrate this procedure with various examples. Dual equivalences constructed this way are often called *natural dualities*, for more information we refer to [8, 13, 3].

Other techniques leading to dual equivalences can be characterised by the slogan "move from models to syntax". For instance, the theory of monads is one of the main category theoretic formulations of universal algebra, and the dual equivalence between monads (syntax) and monadic categories (semantics) is at the heart of the proofs of the classical Gelfand and Pontryagin dualities presented in [12]. We present here a similar argumentation which was employed in [7] to derive in a uniform way several duality theorems involving categories of relations and categories of algebras with "hemimorphisms", generalising this way the approach of [6] to duality theory for Boolean algebras with operators.

Not surprisingly, it is often easier to construct a dual equivalence only between finite objects. In order to extend this equivalence to all objects, another duality between syntax and semantics comes handy, namely the Gabriel-Ulmer duality for locally presentable categories on one side and limit sketches on the other (see [5, 1]). If time permits, we sketch this approach and show in particular how it leads to a "two for the prize of one" principle: obtain a new duality from a given one simply by structure interchange.

References

[1] J. Adámek and J. Rosický, *Locally presentable and accessible categories*, vol. 189 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1994.

[2] J. Baez and J. Dolan, *From finite sets to Feynman diagrams*, in Mathematics Unlimited – 2001 and Beyond, B. Engquist and W. Schmid, eds., Springer Verlag, Mar. 2001, pp. 29–50, arXiv:0004133 [math.QA].

[3] D. M. Clark and B. A. Davey, *Natural dualities for the working algebraist*, vol. 57 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1998.

[4] L. Esakia, *Topological Kripke models*, Doklady Akademii Nauk SSSR, 214 (1974), pp. 298–301.
[5] P. Gabriel and F. Ulmer, *Lokal präsentierbare Kategorien*, Lecture Notes in Mathematics, Vol. 221, Springer-Verlag, Berlin, 1971.
[6] P. R. Halmos, *Algebraic logic I. Monadic Boolean algebras*, Compositio Mathematica, 12 (1956), pp. 217–249.
[7] D. Hofmann and P. Nora, *Dualities for modal algebras from the point of view of triples*, Algebra Universalis, 73 (2015), pp. 297–320, arXiv:1302.5609 [math.LO].
[8] P. T. Johnstone, *Stone spaces*, vol. 3 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1986. Reprint of the 1982 edition.
[9] B. Jónsson and A. Tarski, *Boolean algebras with operators. I*, American Journal of Mathematics, 73 (1951), pp. 891–939.
[10] ———, *Boolean algebras with operators. II*, American Journal of Mathematics, 74 (1952), pp. 127–162.
[11] C. Kupke, A. Kurz, and Y. Venema, *Stone coalgebras*, Theoretical Computer Science, 327 (2004), pp. 109–134.
[12] J. W. Negrepontis, *Duality in analysis from the point of view of triples*, Journal of Algebra, 19 (1971), pp. 228–253.
[13] H.-E. Porst and W. Tholen, *Concrete dualities*, in Category theory at work, H. Herrlich and H.-E. Porst, eds., vol. 18 of Research and Exposition in Mathematics, Heldermann Verlag, Berlin, 1991, pp. 111–136. With Cartoons by Marcel Erné.
[14] M. H. Stone, *The theory of representations for Boolean algebras*, Transactions of the American Mathematical Society, 40 (1936), pp. 37–111.
[15] ———, *Topological representations of distributive lattices and Brouwerian logics*, Časopis pro pěstování matematiky a fysiky, 67 (1938), pp. 1–25, eprint: http://dml.cz/handle/10338.dmlcz/124080.

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