New proof-theoretic facts about $\mathsf{KP}\omega$

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Abstract

The Σ_1-ordinal of $\mathsf{KP}\omega$ (Kripke-Platek set theory with infinity) is, by definition,

$$\min\{\alpha : L_\alpha \models \psi, \text{ for all } \Sigma_1\text{-sentences } \psi \text{ such that } \mathsf{KP}\omega \vdash \psi\}.$$

It is well-known that this is the Bachmann-Howard ordinal. We introduce a finite-order term language T_Ω with two ground types: N for the natural numbers and Ω for the countable construtive tree ordinals.

Let W the smallest set which contains 0 and is such that, whenever f is a function that maps ω into W, then $(1, f) \in W$. Each element a of W has a (set-theoretical) ordinal height $|a|$. Each closed term of T_Ω of type Ω denotes an element of W. Each closed term of type $\Omega \to \Omega$ denotes a function from W to W.

(a) The supremum of the ordinal heights of the (denotations of the) closed terms of T_Ω is the Σ_1-ordinal of $\mathsf{KP}\omega$. This is proved using a (bounded) functional interpretation.

(b) If $\mathsf{KP}\omega \vdash \forall x \exists y\, \phi(x, y)$, where ϕ is a bounded formula, then there is a closed term t of type $\Omega \to \Omega$ such that $\forall a \in W \forall x \in L_{|a|} \exists y \in L_{|t(a)|}\, \phi(x, y)$.

The above two results also hold for a second-order version $\mathsf{KP}\omega^2$ of $\mathsf{KP}\omega$ together with the schema of Δ_1-comprehension and of strict-Π^1_1 reflection. Moreover, this second-order theory is Σ_1-conservative over $\mathsf{KP}\omega$. It is an open question whether this conservativity result extends to Π_2-sentences.

References

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