# Global Asymptotic Stability for Systems of Delayed Differential Equations with Applications to Neural Networks

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## Some important references


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**[3]** Y. Chen, Global asymptotic stability of delayed Cohen-Grossberg neural networks, *IEEE Trans. Circuits Syst.* vol.53, no.2, (2006), 351-357

**[4]** M. Wang, L. Wang, Global asymptotic robust stability of static neural network models with S-type distributed delays, *Math. Comput. Modelling*, **44** (2006), 218-222

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## Notation

- $n \in \mathbb{N}$, $x \in \mathbb{R}^n$,

$$|x|_\infty = \max\{|x_i| : i \in \{1,\ldots,n\}\};$$

- $\tau \in \mathbb{R}^+$, $C_n := C([-\tau, 0]; \mathbb{R}^n)$

$$\|\varphi\| = \sup_{\theta\in[-\tau,0]} |\varphi(\theta)|_\infty;$$

- FDE in $C_n$

$$\dot{x}(t) = f(t, x_t)$$

$x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$

- $N(1,n) := \{1,\ldots,n\}$;

- $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is a **non-singular M-matrix** if $a_{ij} \leq 0$, $i \neq j$ and Re $\sigma(A) > 0$.

## Neural Network Models

Hopfield with delays

$$\dot{x}_i(t) = -b_i x_i(t) + \sum_{j=1}^{n} c_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t-\tau_{ij})) + J_i, \quad (1)$$

$i \in N(1, n)$

Cohen-Grossberg with delays

$$\dot{x}_i(t) = -k_i(x_i(t)) \left[ b_i(x_i(t)) - \sum_{j=1}^{n} \sum_{p=1}^{P} a_{ij}^{(p)} f_j(x_j(t-\tau_{ij}^{(p)})) + J_i \right], \quad (2)$$

$i \in N(1, n)$

Bidirectional associative memory with delays

$$\begin{cases} \dot{x}_i(t) = -x_i(t) + c_{ii} g_i(x_i(t-d_{ii})) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t-\tau_{ij})) + I_i \\ \\ \dot{y}_i(t) = -y_i(t) + l_{ii} f_i(y_i(t-m_{ii})) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\sigma_{ij})) + J_i \end{cases} \quad (3)$$

$i \in N(1, n)$.

Static model with S-type distributed delays

$$\dot{x}_i(t) = -b_i x_i(t) + f_i \left( \sum_{j=1}^{n} \omega_{ij} \int_{-\tau}^{0} x_j(t+\theta) d\eta_{ij}(\theta) + J_i \right) \quad (4)$$

$i \in N(1, n)$

## General situation

$$\dot{x}_i(t) = -k_i(x_i(t)) \left[ b_i(x_i(t)) + f_i(x_t) \right], \quad i = 1, \dots, n$$

$$\dot{x}_i(t) = -k_i(x_i(t))\left[b_i(x_i(t)) + f_i(x_t)\right], \quad (5)$$

$k_i : \mathbb{R} \to (0, +\infty)$ continuous,

**(A1)** $\exists \beta_i > 0, \forall u, v \in \mathbb{R}, u \neq v$:

$$(b_i(u) - b_i(v))/(u - v) \geq \beta_i;$$

[In particular, for $b_i(u) = \beta_i u$.]

**(A2)** $f_i : C_n \to \mathbb{R}$ are Lipschitz functions with Lipschitz constants $l_i$.

$$\dot{x}_i(t) = -k_i(x_i(t))\left[b_i(x_i(t)) + \sum_{j=1}^{n} f_{ij}(x_{j,t})\right] \quad (6)$$

$$\dot{x}_i(t) = -k_i(x_i(t))\left[b_i(x_i(t)) + f_i\left(\sum_{j=1}^{n} \omega_{ij} \int_{-\tau}^{0} x_i(t+\theta)d\eta_{ij}(\theta)\right)\right] \quad (7)$$

## Main Objective

To obtain sufficient conditions for the existence and global asymptotic stability of the equilibrium point of neural network models written in the form (5).

## 1. Main Result

We consider the functional differential system in general form

$$\dot{x}_i(t) = f_i(x_t), \quad t \geq 0, \tag{8}$$

$i \in N(1,n) = \{1, \ldots, n\}$,
$C_n := C([-\tau, 0]; \mathbb{R}^n)$, $x_t(\theta) = x(t+\theta)$
$f_i : C_n \to \mathbb{R}$ are continuous.

**Hypotheses:**
**(H1)** $f_i$ bounded on bounded sets of $C_n$;
**(H2)** $\forall \varphi \in C_n, \forall i \in N(1,n)$,

$$\|\varphi\| = |\varphi(0)|_\infty = |\varphi_i(0)| > 0 \Rightarrow \varphi_i(0) f_i(\varphi) < 0.$$

**(H2)**$\Rightarrow x \equiv 0$ is the unique equilibrium

**Theorem 1**
Assume **(H1)** and **(H2)**
Then $x \equiv 0$ is globally asymptotically stable.

**Proof (idea)**
Let $x(t)$ be a solution of (8)

•**(H1)+(H2)** $\Rightarrow x(t)$ defined and bounded on $[-\tau, +\infty)$

$$-v_i = \liminf_{t\to+\infty} x_i(t), \quad u_i = \limsup_{t\to+\infty} x_i(t)$$

$$v = \max_i\{v_i\}, \quad u = \max_i\{u_i\},$$

$u, v \in \mathbb{R}$, $-v \le u$.
We have to show $\max(u, v) = 0$.
We suppose $|v| \le u$ ($|u| \le v$ is similar).
Let $i \in N(1, n)$ such that $u_i = u$.

$\epsilon > 0, \exists T > 0 : \|x_t\| < u + \epsilon,\ t \ge T$

•We can show that exists $(t_k)_{k\in\mathbb{N}}$ such that

$$t_k \nearrow +\infty, \quad x_i(t_k) \to u, \quad \text{and} \quad f_i(x_{t_k}) \to 0$$

•**(H1)+(H2)** $\Rightarrow \dot{x}(t)$ is bounded $\Rightarrow \{x_{t_k}\}$ is bounded and equicontinuous $\Rightarrow \exists \varphi \in C_n$

$$x_{t_k} \to \varphi \text{ on } C_n$$

with $\|\varphi\| \le u$, $\varphi_i(0) = u$ and $f_i(\varphi) = 0$

**(H2)** $\Rightarrow u = 0.\square$

## 2. General Neural Network

$$\dot{x}_i(t) = -k_i(x_i(t))\,[b_i(x_i(t)) + f_i(x_t)]\,, \quad (9)$$

**Theorem 2**
Assume **(A1)**, **(A2)**, and $k_i : \mathbb{R} \to (0, +\infty)$ continuous.
If $\beta_i > l_i, \forall i$, then (9) has an equilibrium point $x^* \in \mathbb{R}^n$, which is globally asymptotically stable.

**Proof (idea)**
•Existence of equilibrium point

$$\begin{array}{rccl} H: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & x & \mapsto & (b_1(x_1) + f_1(x), \ldots, b_n(x_n) + f_n(x)) \end{array}$$

is a homeomorphism.
Then there exists $x^* \in \mathbb{R}^n$, $H(x^*) = 0$, i.e. $x^*$ is an equilibrium.

•By translation, we may suppose $x^* = 0$, $b_i(0) + f_i(0) = 0, \forall i$.

• $\beta_i > l_i, \forall i \Rightarrow$**(H1)** and **(H2)**

From Theorem 1, we have the result.$\Box$

## 3. Cohen Grossberg Model

$$\dot{x}_i(t) = -k_i(x_i(t)) \left[ b_i(x_i(t)) + \sum_{j=1}^{n} f_{ij}(x_{j,t}) \right] \qquad (10)$$

- $k_i : \mathbb{R} \to (0, +\infty)$ continuous
- Assume **(A1)**
- $f_{ij} : C_1 \to \mathbb{R}$ Lipchitz with Lipchitz constant $l_{ij}$

$$B = diag(\beta_1, \ldots, \beta_n), \quad A = [l_{ij}], \quad N = B - A$$

**Theorem 3**
If $N$ is a non-singular M-matrix, then there is an equilibrium of (10), which is globally asymptotically stable.

**Proof (idea)**
- $N$ non-singular M-matrix $\Rightarrow$

Exists $d = (d_1, \ldots, d_n) > 0$ such that $Nd > 0$,

$$\beta_i d_i - \sum_{j=1}^{n} l_{ij} d_j > 0, \quad \forall i \in N(1, n). \qquad (11)$$

- The change of variables

$$y_i(t) = d_i^{-1} x_i(t)$$

transform (10) into

$$\dot{y}_i(t) = -\bar{k}_i(y_i(t))[\bar{b}_i(y_i(t)) + \bar{f}_i(y_t)],$$

$$\bar{f}_i(\varphi) = d_i^{-1} \sum_{j=1}^{n} f_{ij}(d_j \varphi_j), \quad \varphi \in C_n$$

$$\bar{b}_i(u) = d_i^{-1} b_i(d_i u), \quad \bar{k}_i = k_i(d_i u), \quad u \in \mathbb{R}$$

- $\bar{f}_i$ satisfies **(A2)**, $i \in N(1, n)$:

$$|\bar{f}_i(\varphi) - \bar{f}_i(\psi)| \leq \left( d_i^{-1} \sum_{j=1}^{n} l_{ij} d_j \right) \|\varphi - \psi\|$$

and $\bar{b}_i$ satisfies **(A1)** with, by (11),

$$\bar{\beta}_i = \beta_i > l_i := d_i^{-1} \sum_{j=1}^{n} l_{ij} d_j,$$

and the result follows from Theorem 2.□

## Example 1.

$$\dot{x}_i(t) = -k_i(x_i(t)) \left[ b_i(x_i(t)) - \sum_{j=1}^{n} \sum_{p=1}^{P} a_{ij}^{(p)} f_j(x_j(t - \tau_{ij}^{(p)})) + J_i \right] \quad (12)$$

- $J_i, a_{ij}^{(p)}, \tau \in \mathbb{R}$, $0 \le \tau_{ij}^{(p)} \le \tau$, $i, j \in N(1, n)$, $p \in N(1, P)$
- $k_i : \mathbb{R} \to (0, +\infty)$ continuous
- Assume **(A1)**
- $f_i : \mathbb{R} \to \mathbb{R}$ Lipschitz with Lipschitz constant $l_i$

$$N := diag(\beta_1, \ldots, \beta_n) - [l_{ij}], \text{ with } l_{ij} = \sum_{p=1}^{P} |a_{ij}^{(p)}| l_j$$

## Corollary

If $N$ is a non-singular M-matrix, then there is an equilibrium point of (12) which is globally asymptotically stable.

## Remark

In [Y. Chen, 2005], the same result was proved with the additional hypotheses:

**(i)** $\exists \underline{k}_i, \overline{k}_i > 0 : \underline{k}_i \le k_i(u) \le \overline{k}_i, \quad \forall u, \forall i;$

**(ii)** $\underline{N} := B\underline{K} - [l_{ij}]\overline{K}$ non-singular M-matrix, for $\underline{K} = diag(\underline{k}_1, \ldots, \underline{k}_n)$, $\overline{K} = diag(\overline{k}_1, \ldots, \overline{k}_n)$.

**(ii)** $\Rightarrow N$ non-singular M-matrix

## 4. Neural network model with time-varing delay

$$\dot{x}_i(t) = -k_i(x_i(t)) \left[ b_i(x_i(t)) + \sum_{j=1}^{n} \sum_{p=1}^{P} h_{ij}^{(p)}(x_j(t - \tau_{ij}^{(p)}(t))) \right] \quad (13)$$

- $\tau_{ij}^{(p)} : [0, +\infty) \to [0, +\infty)$ bounded and continuous;
- $k_i : \mathbb{R} \to (0, +\infty)$ continuous;
- Assume **(A1)**
- $h_{ij}^{(p)} : \mathbb{R} \to \mathbb{R}$ Lipschitz with Lipschitz constant $l_{ij}^{p}$

$$N := B - [l_{ij}], \text{ with } l_{ij} = \sum_{p=1}^{P} l_{ij}^{(p)}$$

**Theorem 4**

If $N$ is a non-singular M-matrix, then there is an equilibrium of (13) which is globally asymptotically stable.

## Example 2.

Bidirectional associative memory neural network

$$\begin{cases} \dot{x}_i(t) = -x_i(t) + g_i(x_i(t-d_i(t))) + \sum_{j=1}^{m} f_{ij}(y_j(t-\tau_{ij}(t))) \\ \\ \dot{y}_j(t) = -y_j(t) + f_j(y_j(t-m_j(t))) + \sum_{i=1}^{n} g_{ji}(x_i(t-\sigma_{ji}(t))) \end{cases} \tag{14}$$

$i \in N(1,n), j \in N(1,m)$
$\tau_{ij}, \sigma_{ji} : [0,+\infty) \to [0,+\infty)$ continuous
$g_i, f_j, f_{ij}, g_{ji} : \mathbb{R} \to \mathbb{R}$ Lipschitz with Lipschit constant $G_i$, $F_j$, $F_{ij}$ and $G_{ij}$, respectively

$$N := \begin{bmatrix} I_n - G_d & -F \\ \\ -G & I_m - F_d \end{bmatrix}_{(n+m)\times(n+m)}$$

$$G_d = diag(G_1, \ldots, G_n), \quad F_d = diag(F_1, \ldots, F_m),$$

$$G = [G_{ji}]_{m\times n}, \quad F = [F_{ij}]_{n\times m}$$

**Corollary**

If $N$ is a non-singular M-matrix, then there is an equilibrium of (14) which is globally asymptotically stable.

**Remark**
The model (3), studied in [L.Wang and X.Zou, 2005], is a subclass of (14).

## 5. Static neural network model with S-type distributed delays

$$\dot{x}_i(t) = -k_i(x_i(t)) \left[ b_i(x_i(t)) + f_i \left( \sum_{j=1}^{n} \omega_{ij} \int_{-\tau}^{0} x_j(t+\theta) d\eta_{ij}(\theta) + J_i \right) \right] \quad (15)$$

• $\tau > 0, J_i, \omega_{ij} \in \mathbb{R}$

• $k_i : \mathbb{R} \to (0, +\infty)$ continuous

• Assume **(A1)**

• $f_i : \mathbb{R} \to \mathbb{R}$ Lipschitz with Lipschitz constant $l_i$

• $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ are normalized bounded variation functions

$$M = diag(\beta_1, \ldots, \beta_n) - [l_i |\omega_{ij}|]$$

## Theorem 5

If $M$ is a non-singular M-matrix, then there is an equilibrium point of (15) which is globally asymptotically stable.

## Example 3.

$$\dot{x}_i(t) = -b_i x_i(t) + f_i\left(\sum_{j=1}^{n} \omega_{ij} \int_{-\tau}^{0} x_j(t+\theta)\, d\eta_{ij}(\theta) + J_i\right) \qquad (16)$$

- $b_i > 0$, $J_i, \omega_{ij} \in \mathbb{R}$
- $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ are normalized bounded variation functions
- $f_i : \mathbb{R} \to \mathbb{R}$ Lipschitz with Lipschitz constant $l_i$

$$N = B - [l_i|\omega_{ij}|]$$

$$B = diag(b_1, \ldots, b_n)$$

## Corollary

If $N$ is a non-singular M-matrix, then there is an equilibrium point of (16) which is globally asymptotically stable.

## Remark

In [M. Wang and L. Wang, 2006], the same result was proved with $\eta_{ij}$ **nondecreasing bounded variation functions** on $[-\tau, 0]$.