Boundedness and Global Exponential Stability for Delayed Differential Equations with Applications

Teresa Faria[a] and José J. Oliveira[b]

August 21, 2009

[a] Departamento de Matemática, CMAF, Universidade de Lisboa
[b] Departamento de Matemática, CMAT, Universidade do Minho

1. Boundedness of solutions

$$\dot{y}(t) = f(t, y_t)$$

2. Exponential stability

$$\dot{x}_i(t) = -\rho_i(x_i(t))(b_i(x_i(t)) + f_i(x_t)), \qquad i = 1, \dots, n$$

3. Neural Network Models

$$\dot{x}_i(t) = -\rho_i(x_i(t)) \left(b_i(x_i(t)) + \sum_{j=1}^{n} f_{ij}(x_{j,t}) \right), i = 1, \dots, n$$

Notation

- $n \in \mathbb{N}$, $x, y \in \mathbb{R}^n$, $d = (d_1, \dots, d_n) > 0$, i.e. $d_i > 0$,

$$< x, y >_d = \sum_{i=1}^{n} d_i x_i y_i, \;\; |x|_{2,d} = \left(\sum_{i=1}^{n} d_i x_i^2 \right)^{1/2},$$

$$|x|_{\infty,d} = \max_{1 \le i \le n} d_i |x_i|;$$

Notation

- $n \in \mathbb{N}$, $x, y \in \mathbb{R}^n$, $d = (d_1, \dots, d_n) > 0$, i.e. $d_i > 0$,

$$< x, y >_d = \sum_{i=1}^{n} d_i x_i y_i, \;\; |x|_{2,d} = \left(\sum_{i=1}^{n} d_i x_i^2 \right)^{1/2},$$

$$|x|_{\infty,d} = \max_{1 \le i \le n} d_i |x_i|;$$

- $\tau \in \mathbb{R}^+$, $C_n := C([-\tau, 0]; \mathbb{R}^n)$

$$\|\varphi\|_{2,d} = \sup_{\theta \in [-\tau,0]} |\varphi(\theta)|_{2,d}, \qquad \|\varphi\|_{\infty,d} = \sup_{\theta \in [-\tau,0]} |\varphi(\theta)|_{\infty,d};$$

Notation

- $n \in \mathbb{N}$, $x, y \in \mathbb{R}^n$, $d = (d_1, \dots, d_n) > 0$, i.e. $d_i > 0$,

$$< x, y >_d = \sum_{i=1}^{n} d_i x_i y_i, \ |x|_{2,d} = \left(\sum_{i=1}^{n} d_i x_i^2 \right)^{1/2},$$

$$|x|_{\infty,d} = \max_{1 \le i \le n} d_i |x_i|;$$

- $\tau \in \mathbb{R}^+$, $C_n := C([-\tau, 0]; \mathbb{R}^n)$

$$\|\varphi\|_{2,d} = \sup_{\theta \in [-\tau, 0]} |\varphi(\theta)|_{2,d}, \qquad \|\varphi\|_{\infty,d} = \sup_{\theta \in [-\tau, 0]} |\varphi(\theta)|_{\infty,d};$$

- $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is a **non-singular M-matrix** if $a_{ij} \le 0$, $i \ne j$ and Re $\sigma(A) > 0$.

1. Boundedness of solutions

- FDE in C_n

$$\dot{y}(t) = f(t, y_t), \qquad t \geq t_0, \tag{1}$$

$t_0 \in \mathbb{R}$,
$f = (f_1, \dots, f_n) : [t_0, +\infty) \times C_n \to \mathbb{R}^n$ continuous,
$y_t \in C_n$, $y_t(\theta) = y(t + \theta)$, $\theta \in [-\tau, 0]$.

1. Boundedness of solutions

- FDE in C_n

$$\dot{y}(t) = f(t, y_t), \qquad t \geq t_0, \tag{1}$$

  $t_0 \in \mathbb{R}$,
  $f = (f_1, \dots, f_n) : [t_0, +\infty) \times C_n \to \mathbb{R}^n$ continuous,
  $y_t \in C_n$, $y_t(\theta) = y(t + \theta)$, $\theta \in [-\tau, 0]$.
- $y(t) = y(t, t_0, \varphi)$ denote the solution of (1) such that $y_{t_0} = \varphi \in C_n$.

- **Proposition 1**
  Assume that, for some $d = (d_1, \dots, d_n) > 0$, $f = (f_1, \dots, f_n)$ satisfies
  $\mathbf{(H)}_\infty$ $\forall t \geq t_0, \forall \varphi \in C_n, \forall i \in \{1, \dots, n\}$,

$$\|\varphi\|_{\infty,d} = |\varphi(0)|_{\infty,d} = d_i|\varphi_i(0)| > 0 \Rightarrow \varphi_i(0)f_i(t,\varphi) < 0;$$

  Then, the solution $y(t) = y(t, t_0, \varphi)$ of (1) is defined and bounded on $[t_0, +\infty)$, and

$$|y(t)|_{\infty,d} \leq \|\varphi\|_{\infty,d}, \quad \text{for} \quad t \geq t_0.$$

Proof (idea)

- $y(t)$ solution on $[t_0 - \tau, a)$, $a > t_0$, and $\|y_{t_0}\|_{\infty,d} \leq K$;

Proof (idea)

- $y(t)$ solution on $[t_0 - \tau, a)$, $a > t_0$, and $\|y_{t_0}\|_{\infty,d} \leq K$;
- Suppose that $|y(t_1)|_{\infty,d} > K > 0$ for some $t_1 > t_0$;

Proof (idea)

- $y(t)$ solution on $[t_0 - \tau, a)$, $a > t_0$, and $\|y_{t_0}\|_{\infty,d} \leq K$;
- Suppose that $|y(t_1)|_{\infty,d} > K > 0$ for some $t_1 > t_0$;
- Define

$$T = \min \left\{ t \in [t_0, t_1] : |y(t)|_{\infty,d} = \max_{s \in [t_0, t_1]} |y(s)|_{\infty,d} \right\}.$$

Proof (idea)

- $y(t)$ solution on $[t_0 - \tau, a)$, $a > t_0$, and $\|y_{t_0}\|_{\infty,d} \leq K$;
- Suppose that $|y(t_1)|_{\infty,d} > K > 0$ for some $t_1 > t_0$;
- Define

$$T = \min \left\{ t \in [t_0, t_1] : |y(t)|_{\infty,d} = \max_{s \in [t_0, t_1]} |y(s)|_{\infty,d} \right\}.$$

- We have
  $|y_T(\theta)|_{\infty,d} = |y(T + \theta)|_{\infty,d} \leq |y(T)|_{\infty,d}, -\tau \leq \theta \leq 0.$
  Choosing $i \in \{1, \dots, n\}$ such that $\|y_T\|_{\infty,d} = d_i |y_i(T)|$,

$$\mathbf{(H)}_\infty \Rightarrow y_i(T) f_i(T, y_T) < 0.$$

If $y_i(T) > 0$ (analogous if $y_i(T) < 0$), then $\dot{y}_i(T) < 0$.

Proof (idea)

- $y(t)$ solution on $[t_0 - \tau, a)$, $a > t_0$, and $\|y_{t_0}\|_{\infty,d} \leq K$;
- Suppose that $|y(t_1)|_{\infty,d} > K > 0$ for some $t_1 > t_0$;
- Define

$$T = \min \left\{ t \in [t_0, t_1] : |y(t)|_{\infty,d} = \max_{s \in [t_0, t_1]} |y(s)|_{\infty,d} \right\}.$$

- We have
  $|y_T(\theta)|_{\infty,d} = |y(T+\theta)|_{\infty,d} \leq |y(T)|_{\infty,d}, -\tau \leq \theta \leq 0.$
  Choosing $i \in \{1, \dots, n\}$ such that $\|y_T\|_{\infty,d} = d_i |y_i(T)|$,

$$\mathbf{(H)}_\infty \Rightarrow y_i(T) f_i(T, y_T) < 0.$$

  If $y_i(T) > 0$ (analogous if $y_i(T) < 0$), then $\dot{y}_i(T) < 0$.
- $d_i y_i(t) \leq |y(t)|_{\infty,d} < |y(T)|_{\infty,d} = d_i y_i(T)$, $t \in [t_0 - \tau, T)$,

$$\Rightarrow \dot{y}_i(T) \geq 0.$$

Proof (idea)

- $y(t)$ solution on $[t_0 - \tau, a)$, $a > t_0$, and $\|y_{t_0}\|_{\infty,d} \le K$;
- Suppose that $|y(t_1)|_{\infty,d} > K > 0$ for some $t_1 > t_0$;
- Define

$$T = \min \left\{ t \in [t_0, t_1] : |y(t)|_{\infty,d} = \max_{s \in [t_0, t_1]} |y(s)|_{\infty,d} \right\}.$$

- We have
  $|y_T(\theta)|_{\infty,d} = |y(T+\theta)|_{\infty,d} \le |y(T)|_{\infty,d}, -\tau \le \theta \le 0.$
  Choosing $i \in \{1, \dots, n\}$ such that $\|y_T\|_{\infty,d} = d_i |y_i(T)|$,

$$\textbf{(H)}_\infty \Rightarrow y_i(T) f_i(T, y_T) < 0.$$

  If $y_i(T) > 0$ (analogous if $y_i(T) < 0$), then $\dot{y}_i(T) < 0$.
- $d_i y_i(t) \le |y(t)|_{\infty,d} < |y(T)|_{\infty,d} = d_i y_i(T)$, $t \in [t_0 - \tau, T)$,

$$\Rightarrow \dot{y}_i(T) \ge 0.$$

- $|y(t, t_0, \varphi)|_{\infty,d} \le \|\varphi\|_{\infty,d}, \quad \forall t \ge t_0.$

- **Proposition 2**
  Assume that, for some $d = (d_1, \dots, d_n) > 0$, f satisfies
  (H)$_2$ $\forall t \geq t_0, \forall \varphi \in C_n$,

$$\|\varphi\|_{2,d} = |\varphi(0)|_{2,d} > 0 \Rightarrow < \varphi(0), f(t, \varphi) >_d < 0.$$

  Then, the solution $y(t) = y(t, t_0, \varphi)$ of $\dot{y}(t) = f(t, y_t)$ is defined and bounded on $[t_0, +\infty)$, and

$$|y(t)|_{2,d} \leq \|\varphi\|_{2,d}, \quad \text{for} \quad t \geq t_0.$$

- **Proposition 2**
  Assume that, for some $d = (d_1, \dots, d_n) > 0$, f satisfies
  (H)$_2$ $\forall t \geq t_0, \forall \varphi \in C_n$,

$$\|\varphi\|_{2,d} = |\varphi(0)|_{2,d} > 0 \Rightarrow < \varphi(0), f(t, \varphi) >_d < 0.$$

  Then, the solution $y(t) = y(t, t_0, \varphi)$ of $\dot{y}(t) = f(t, y_t)$ is defined and bounded on $[t_0, +\infty)$, and

$$|y(t)|_{2,d} \leq \|\varphi\|_{2,d}, \quad \text{for} \quad t \geq t_0.$$

- Proof: Similar to the proof of Proposition 1.

2. Exponential stability

- In applications,

$$\dot{x}_i(t) = -\rho_i(x_i(t))[b_i(x_i(t)) + f_i(x_t)], \qquad t \geq 0, \; i = 1, \dots, n, \quad (2)$$

where $\rho_i : \mathbb{R} \to (0, +\infty)$, $b_i : \mathbb{R} \to \mathbb{R}$ and
$f = (f_1, \dots, f_n) : [0, +\infty) \times C_n \to \mathbb{R}^n$ are continuous.

Hypothesis:

2. Exponential stability

- In applications,

$$\dot{x}_i(t) = -\rho_i(x_i(t))[b_i(x_i(t)) + f_i(x_t)], \qquad t \geq 0, \ i = 1, \ldots, n, \quad (2)$$

where $\rho_i : \mathbb{R} \to (0, +\infty)$, $b_i : \mathbb{R} \to \mathbb{R}$ and
$f = (f_1, \ldots, f_n) : [0, +\infty) \times C_n \to \mathbb{R}^n$ are continuous.

Hypothesis:

- **(A1)** $\exists \beta_i > 0, \forall u, v \in \mathbb{R}, u \neq v$:

$$(b_i(u) - b_i(v))/(u - v) \geq \beta_i.$$

[In particular, for $b_i(u) = \beta_i u$.]

- **Definition**
  An equilibrium $x^* \in \mathbb{R}$ of (2) is said to be *globally exponentially stable* if there are $\varepsilon, M > 0$ such that

$$|y(t,0,\varphi) - x^*| \leq Me^{-\varepsilon t}\|\varphi - x^*\|, \qquad t \geq 0,\ \varphi \in C_n.$$

- **Definition**
  An equilibrium $x^* \in \mathbb{R}$ of (2) is said to be *globally exponentially stable* if there are $\varepsilon, M > 0$ such that

$$|y(t,0,\varphi) - x^*| \leq Me^{-\varepsilon t}\|\varphi - x^*\|, \qquad t \geq 0,\ \varphi \in C_n.$$

- **Theorem 1**
  Suppose that $0 < r \leq \rho_i(x)$, $\forall x \in \mathbb{R}$, $i = 1, \dots, n$
  and assume **(A1)** and
  $\textbf{(A2)}_\infty$ $\exists d = (d_1, \dots, d_n) > 0, \forall \varphi, \psi \in C_n$:

$$|f_i(\varphi) - f_i(\psi)| \leq l_i\|\varphi - \psi\|_{\infty,d},$$

  with $\beta_i > d_i l_i$, $i = 1, \dots, n$.
  Then there is a unique equilibrium x^* of (2) which is globally exponentially stable.

Proof (idea)

- Existence of equilibrium
  Consider

$$\begin{array}{rccl} H: & \mathbb{R}^n & \rightarrow & \mathbb{R}^n \\ & x & \mapsto & (b_1(x_1) + f_1(x), \ldots, b_n(x_n) + f_n(x)) \end{array}$$

H is injective and $|H(x)|_{\infty,d} \rightarrow +\infty$ as $|x|_{\infty,d} \rightarrow +\infty$ implying that H is a homeomorfism [1], then there exists $x^* \in \mathbb{R}^n$, $H(x^*) = 0$, i.e. x^* is an equilibrium of (2).

[1] M. Forti and A. Tesi, New conditions for global stability of neural networks with applications to linear and quadratic programming problems, *IEEE Trans. Circuits Syst. I* 42 (1995) 354-366.

Proof (idea)

- Existence of equilibrium
  Consider

$$\begin{array}{rccc} H: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & x & \mapsto & (b_1(x_1) + f_1(x), \dots, b_n(x_n) + f_n(x)) \end{array}$$

  H is injective and $|H(x)|_{\infty,d} \to +\infty$ as $|x|_{\infty,d} \to +\infty$ implying that H is a homeomorfism [1], then there exists $x^* \in \mathbb{R}^n$, $H(x^*) = 0$, i.e. x^* is an equilibrium of (2).
- By translation we may suppose $x^* = 0$, $b_i(0) + f_i(t, 0) = 0$.

[1] M. Forti and A. Tesi, New conditions for global stability of neural networks with applications to linear and quadratic programming problems, *IEEE Trans. Circuits Syst. I* 42 (1995) 354-366.

- Exponential stability
  Consider $x(t) = x(t, 0, \varphi)$ a solution of (2).

- Exponential stability
  Consider $x(t) = x(t, 0, \varphi)$ a solution of (2).
- The change of variables

$$z(t) = e^{\varepsilon t} x(t),$$

for $\varepsilon > 0$ small enough, transform (2) into

$$\dot{z}(t) = g(t, z_t), \tag{3}$$

$g = (g_1, \ldots, g_n)$ with

$$g_i(t, \varphi) = \\ \varepsilon \varphi_i(0) - \rho_i(t, e^{-\varepsilon(t+\cdot)}\varphi)e^{\varepsilon t}\left[b_i(e^{-\varepsilon t}\varphi_i(0)) + f_i(t, e^{-\varepsilon(t+\cdot)}\varphi)\right]$$

- Exponential stability
  Consider $x(t) = x(t, 0, \varphi)$ a solution of (2).
- The change of variables

$$z(t) = e^{\varepsilon t} x(t),$$

  for $\varepsilon > 0$ small enough, transform (2) into

$$\dot{z}(t) = g(t, z_t), \tag{3}$$

  $g = (g_1, \dots, g_n)$ with

$$g_i(t, \varphi) = \varepsilon\varphi_i(0) - \rho_i(t, e^{-\varepsilon(t+\cdot)}\varphi)e^{\varepsilon t}\left[b_i(e^{-\varepsilon t}\varphi_i(0)) + f_i(t, e^{-\varepsilon(t+\cdot)}\varphi)\right]$$

- **(A1)**+**(A2)**$_\infty \Rightarrow$**(H)**$_\infty$

- Exponential stability
  Consider $x(t) = x(t, 0, \varphi)$ a solution of (2).
- The change of variables

$$z(t) = e^{\varepsilon t} x(t),$$

for $\varepsilon > 0$ small enough, transform (2) into

$$\dot{z}(t) = g(t, z_t), \tag{3}$$

$g = (g_1, \dots, g_n)$ with

$$g_i(t, \varphi) = \varepsilon \varphi_i(0) - \rho_i(t, e^{-\varepsilon(t+\cdot)}\varphi) e^{\varepsilon t} \left[b_i(e^{-\varepsilon t}\varphi_i(0)) + f_i(t, e^{-\varepsilon(t+\cdot)}\varphi) \right]$$

- **(A1)**+**(A2)**$_\infty \Rightarrow$**(H)**$_\infty$
- From Proposition 1 $|z(t)|_{\infty,d} \le \|z_0\|_{\infty,d}$, $t \ge 0$, and

$$|x(t, 0, \varphi)|_{\infty,d} = |e^{-\varepsilon t} z(t, 0, e^{\varepsilon \cdot}\varphi)|_{\infty,d} \le e^{-\varepsilon t} \|\varphi\|_{\infty,d}$$

- **Theorem 2**
  Suppose that $0 < r \leq \rho_i(x) \leq R$, $\forall x \in \mathbb{R}$, $i = 1, \dots, n$
  and assume **(A1)** and
  (A2)$_2$ $\exists d = (d_1, \dots, d_n) > 0, \forall \varphi, \psi \in C_n$:

$$|f(\varphi) - f(\psi)|_{2,d} \leq l \|\varphi - \psi\|_{2,d},$$

  with $\beta_i > l\sqrt{R/r}$, $i = 1, \dots, n$.
  Then there a unique equilibrium x^* of (2) which is globally exponentially stable.

- **Theorem 2**
  Suppose that $0 < r \leq \rho_i(x) \leq R$, $\forall x \in \mathbb{R}$, $i = 1, \dots, n$
  and assume **(A1)** and
  (A2)$_2$ $\exists d = (d_1, \dots, d_n) > 0, \forall \varphi, \psi \in C_n$:

$$|f(\varphi) - f(\psi)|_{2,d} \leq l\|\varphi - \psi\|_{2,d},$$

  with $\beta_i > l\sqrt{R/r}$, $i = 1, \dots, n$.
  Then there a unique equilibrium x^* of (2) which is globally exponentially stable.
- Proof: Similar to the proof of Theorem 1.

- **Theorem 2**
  Suppose that $0 < r \leq \rho_i(x) \leq R$, $\forall x \in \mathbb{R}$, $i = 1, \dots, n$
  and assume **(A1)** and
  (A2)$_2$ $\exists d = (d_1, \dots, d_n) > 0, \forall \varphi, \psi \in C_n$:

$$|f(\varphi) - f(\psi)|_{2,d} \leq l \|\varphi - \psi\|_{2,d},$$

  with $\beta_i > l\sqrt{R/r}$, $i = 1, \dots, n$.
  Then there a unique equilibrium x^* of (2) which is globally exponentially stable.
- Proof: Similar to the proof of Theorem 1.
- **Remark**
  Assuming the existence of an equilibrium point, similar results can be obtained for the non-autonomous case

$$\dot{x}_i(t) = -\rho_i(t, x_t)[b_i(x_i(t)) + f_i(t, x_t)], \qquad t \geq 0, \; i = 1, \dots, n.$$

3. Neural Network Models

Cohen-Grossberg neural network with distributed delays

$$\dot{x}_i(t) = -\rho_i(x_i(t)) \left[b_i(x_i(t)) + \sum_{j=1}^{n} f_{ij}(x_{j,t}) \right], \qquad (4)$$

- $\rho_i : \mathbb{R} \to (0, +\infty)$ are continuous and $\rho_i(x) \geq r > 0$;

3. Neural Network Models

Cohen-Grossberg neural network with distributed delays

$$\dot{x}_i(t) = -\rho_i(x_i(t))\left[b_i(x_i(t)) + \sum_{j=1}^{n} f_{ij}(x_{j,t})\right], \qquad (4)$$

- $\rho_i : \mathbb{R} \to (0, +\infty)$ are continuous and $\rho_i(x) \geq r > 0$;
- $f_{ij} : C_1 \to \mathbb{R}$ are Lipschitzian with

$$|f_{ij}(\varphi) - f_{ij}(\psi)| \leq l_{ij}\|\varphi - \psi\|_\infty, \quad \varphi, \psi \in C_1 = C([-\tau, 0]; \mathbb{R});$$

3. Neural Network Models

Cohen-Grossberg neural network with distributed delays

$$\dot{x}_i(t) = -\rho_i(x_i(t))\left[b_i(x_i(t)) + \sum_{j=1}^{n} f_{ij}(x_{j,t})\right], \qquad (4)$$

- $\rho_i : \mathbb{R} \to (0, +\infty)$ are continuous and $\rho_i(x) \geq r > 0$;
- $f_{ij} : C_1 \to \mathbb{R}$ are Lipschitzian with

$$|f_{ij}(\varphi) - f_{ij}(\psi)| \leq l_{ij}\|\varphi - \psi\|_\infty, \qquad \varphi, \psi \in C_1 = C([-\tau, 0]; \mathbb{R});$$

- $b_i : \mathbb{R} \to \mathbb{R}$ are continuous satisfying **(A1)**;

3. Neural Network Models

Cohen-Grossberg neural network with distributed delays

$$\dot{x}_i(t) = -\rho_i(x_i(t)) \left[b_i(x_i(t)) + \sum_{j=1}^{n} f_{ij}(x_{j,t}) \right], \qquad (4)$$

- $\rho_i : \mathbb{R} \to (0, +\infty)$ are continuous and $\rho_i(x) \geq r > 0$;
- $f_{ij} : C_1 \to \mathbb{R}$ are Lipschitzian with

$$|f_{ij}(\varphi) - f_{ij}(\psi)| \leq l_{ij} \|\varphi - \psi\|_\infty, \qquad \varphi, \psi \in C_1 = C([-\tau, 0]; \mathbb{R});$$

- $b_i : \mathbb{R} \to \mathbb{R}$ are continuous satisfying **(A1)**;

3. Neural Network Models

Cohen-Grossberg neural network with distributed delays

$$\dot{x}_i(t) = -\rho_i(x_i(t))\left[b_i(x_i(t)) + \sum_{j=1}^{n} f_{ij}(x_{j,t})\right], \qquad (4)$$

- $\rho_i : \mathbb{R} \to (0, +\infty)$ are continuous and $\rho_i(x) \geq r > 0$;
- $f_{ij} : C_1 \to \mathbb{R}$ are Lipschitzian with

$$|f_{ij}(\varphi) - f_{ij}(\psi)| \leq l_{ij}\|\varphi - \psi\|_\infty, \qquad \varphi, \psi \in C_1 = C([-\tau, 0]; \mathbb{R});$$

- $b_i : \mathbb{R} \to \mathbb{R}$ are continuous satisfying **(A1)**;
- $N := diag(\beta_1, \dots, \beta_n) - [l_{ij}]$.

- **Proposition 3**
  If N is a non-singular M-matrix, then there is a unique equilibrium of (4), which is globally exponentially stable.

- **Proposition 3**
  If N is a non-singular M-matrix, then there is a unique equilibrium of (4), which is globally exponentially stable.
- Proof (idea)
  For $f_i(\varphi) := \sum_{j=1}^{n} f_{ij}(\varphi_j)$, $\varphi = (\varphi_1, \dots, \varphi_n) \in C_n$, (4) has the form of (2).

- **Proposition 3**
  If N is a non-singular M-matrix, then there is a unique equilibrium of (4), which is globally exponentially stable.
- Proof (idea)
  For $f_i(\varphi) := \sum_{j=1}^{n} f_{ij}(\varphi_j)$, $\varphi = (\varphi_1, \dots, \varphi_n) \in C_n$, (4) has the form of (2).
- N non-singular M-matrix $\Rightarrow$
  Exists $c = (c_1, \dots, c_n) > 0$ such that $Nc > 0$,

$$\beta_i > c_i^{-1} \sum_{j=1}^{n} l_{ij} c_j, \qquad i = 1, \dots, n; \tag{5}$$

- **Proposition 3**
  If N is a non-singular M-matrix, then there is a unique equilibrium of (4), which is globally exponentially stable.
- Proof (idea)
  For $f_i(\varphi) := \sum_{j=1}^{n} f_{ij}(\varphi_j)$, $\varphi = (\varphi_1, \dots, \varphi_n) \in C_n$, (4) has the form of (2).
- N non-singular M-matrix $\Rightarrow$
  Exists $c = (c_1, \dots, c_n) > 0$ such that $Nc > 0$,

$$\beta_i > c_i^{-1} \sum_{j=1}^{n} l_{ij} c_j, \qquad i = 1, \dots, n; \tag{5}$$

- $|f_i(\varphi) - f_i(\psi)| \leq \sum_{j=1}^{n} |f_{ij}(\varphi_i) - f_{ij}(\psi_i)| \leq \left(\sum_{j=1}^{n} l_{ij} c_j\right) \|\varphi - \psi\|_{\infty,d}$
  and f_i satisfies **(A2)**$_\infty$ with $d = (c_1^{-1}, \dots, c_n^{-1})$;

- **Proposition 3**
  If N is a non-singular M-matrix, then there is a unique equilibrium of (4), which is globally exponentially stable.
- Proof (idea)
  For $f_i(\varphi) := \sum_{j=1}^{n} f_{ij}(\varphi_j)$, $\varphi = (\varphi_1, \dots, \varphi_n) \in C_n$, (4) has the form of (2).
- N non-singular M-matrix $\Rightarrow$
  Exists $c = (c_1, \dots, c_n) > 0$ such that $Nc > 0$,

$$\beta_i > c_i^{-1} \sum_{j=1}^{n} l_{ij} c_j, \qquad i = 1, \dots, n; \tag{5}$$

- $|f_i(\varphi) - f_i(\psi)| \leq \sum_{j=1}^{n} |f_{ij}(\varphi_i) - f_{ij}(\psi_i)| \leq \left(\sum_{j=1}^{n} l_{ij} c_j\right) \|\varphi - \psi\|_{\infty, d}$
  and f_i satisfies $\mathbf{(A2)}_\infty$ with $d = (c_1^{-1}, \dots, c_n^{-1})$;
- The result follows from Theorem 1.

- Denote $f = (f_1, \dots, f_n) : C_n \to \mathbb{R}^n$

$$f_i(\varphi) = \sum_{j=1}^{n} f_{ij}(\varphi_j), \qquad \varphi = (\varphi_1, \dots, \varphi_n) \in C_n.$$

- Denote $f = (f_1, \dots, f_n) : C_n \to \mathbb{R}^n$

$$f_i(\varphi) = \sum_{j=1}^{n} f_{ij}(\varphi_j), \qquad \varphi = (\varphi_1, \dots, \varphi_n) \in C_n.$$

- **Proposition 4**
  Assume **(A1)** and suppose

$$0 < r \leq \rho_i(x) \leq R \qquad \forall x \in \mathbb{R}, i = 1, \dots, n,$$

$$|f(\varphi) - f(\psi)|_{2,d} \leq l \|\varphi - \psi\|_{2,d} \qquad \forall \varphi, \psi \in C_n,$$

  with $\beta_i > l\sqrt{R/r}$, $\forall i$.
  Then there is an equilibrium of (4), which is globally exponentially stable.

- Denote $f = (f_1, \dots, f_n) : C_n \to \mathbb{R}^n$

$$f_i(\varphi) = \sum_{j=1}^{n} f_{ij}(\varphi_j), \qquad \varphi = (\varphi_1, \dots, \varphi_n) \in C_n.$$

- **Proposition 4**
  Assume **(A1)** and suppose

$$0 < r \le \rho_i(x) \le R \qquad \forall x \in \mathbb{R}, i = 1, \dots, n,$$

$$|f(\varphi) - f(\psi)|_{2,d} \le l \|\varphi - \psi\|_{2,d} \qquad \forall \varphi, \psi \in C_n,$$

  with $\beta_i > l\sqrt{R/r}$, $\forall i$.
  Then there is an equilibrium of (4), which is globally exponentially stable.

- Proof: Similar to the proof of Proposition 3.

- Example 1: Consider the 2-neuron network

$$\begin{aligned} \dot{x}_1(t) &= -b_1 x_1(t) + a_{11} f_1(x_1(t-\tau)) + a_{12} f_2(x_2(t-\tau)) \\ \dot{x}_2(t) &= -b_2 x_2(t) + a_{21} f_1(x_1(t-\tau)) + a_{22} f_2(x_2(t-\tau)) \end{aligned} \tag{6}$$

where, for $i = 1, 2$, $\tau, b_i > 0$, $a_{ij} \in \mathbb{R}$, $a_{ii} \neq 0$, and $f_i : \mathbb{R} \to \mathbb{R}$ such that $|f_i(u) - f_i(v)| \leq |u - v|$, $u, v \in \mathbb{R}$.

- Example 1: Consider the 2-neuron network

$$\begin{aligned}\dot{x}_1(t) &= -b_1x_1(t) + a_{11}f_1(x_1(t-\tau)) + a_{12}f_2(x_2(t-\tau))\\ \dot{x}_2(t) &= -b_2x_2(t) + a_{21}f_1(x_1(t-\tau)) + a_{22}f_2(x_2(t-\tau))\end{aligned} \tag{6}$$

where, for $i = 1, 2$, $\tau, b_i > 0$, $a_{ij} \in \mathbb{R}$, $a_{ii} \neq 0$, and $f_i : \mathbb{R} \to \mathbb{R}$ such that $|f_i(u) - f_i(v)| \leq |u - v|$, $u, v \in \mathbb{R}$.

- If $N = diag(b_1, b_2) - [|a_{ij}|]$ is a non-singular M-matrix, i.e

$$b_1 > a_{11} \text{ and } (b_1 - a_{11})(b_2 - a_{22}) > |a_{12}a_{21}|, \tag{7}$$

then, from Proposition 3, (6) has an equilibrium x^*, which is globally exponentially stable.

- For the situation $a_{12}a_{21} < 0$, define

$$f(\varphi) = A \begin{pmatrix} f_1(\varphi_1(-\tau)) \\ f_2(\varphi_2(-\tau)) \end{pmatrix}, \text{ for } A = [a_{ij}],\ \varphi = (\varphi_1, \varphi_2) \in C_2.$$

- For the situation $a_{12}a_{21} < 0$, define

$$f(\varphi) = A \begin{pmatrix} f_1(\varphi_1(-\tau)) \\ f_2(\varphi_2(-\tau)) \end{pmatrix}, \text{ for } A = [a_{ij}],\ \varphi = (\varphi_1, \varphi_2) \in C_2.$$

- Choosing $d = (d_1, d_2) > 0$ such that $d_1 a_{11} a_{12} + d_2 a_{21} a_{22} = 0$,

$$|f(\varphi) - f(\psi)|_{2,d} \leq \left(\max\{\frac{a_{11}}{a_{22}}, \frac{a_{22}}{a_{11}}\} \det A \right)^{1/2} \|\varphi - \psi\|_{2,d}$$

- For the situation $a_{12}a_{21} < 0$, define

$$f(\varphi) = A\begin{pmatrix} f_1(\varphi_1(-\tau)) \\ f_2(\varphi_2(-\tau)) \end{pmatrix}, \text{ for } A = [a_{ij}],\ \varphi = (\varphi_1, \varphi_2) \in C_2.$$

- Choosing $d = (d_1, d_2) > 0$ such that $d_1 a_{11} a_{12} + d_2 a_{21} a_{22} = 0$,

$$|f(\varphi) - f(\psi)|_{2,d} \leq \left(\max\{\frac{a_{11}}{a_{22}}, \frac{a_{22}}{a_{11}}\} \det A\right)^{1/2} \|\varphi - \psi\|_{2,d}$$

- From Proposition 4, if

$$\left(\max\{\frac{a_{11}}{a_{22}}, \frac{a_{22}}{a_{11}}\} \det A\right)^{1/2} < \min\{b_1, b_2\} \tag{8}$$

then (6) has an equilibrium, x^*, globally exponentially stable.

- For the situation $a_{12}a_{21} < 0$, define

$$f(\varphi) = A\begin{pmatrix} f_1(\varphi_1(-\tau)) \\ f_2(\varphi_2(-\tau)) \end{pmatrix}, \text{ for } A = [a_{ij}],\ \varphi = (\varphi_1, \varphi_2) \in C_2.$$

- Choosing $d = (d_1, d_2) > 0$ such that $d_1 a_{11} a_{12} + d_2 a_{21} a_{22} = 0$,

$$|f(\varphi) - f(\psi)|_{2,d} \leq \left(\max\{\frac{a_{11}}{a_{22}}, \frac{a_{22}}{a_{11}}\} \det A\right)^{1/2} \|\varphi - \psi\|_{2,d}$$

- From Proposition 4, if

$$\left(\max\{\frac{a_{11}}{a_{22}}, \frac{a_{22}}{a_{11}}\} \det A\right)^{1/2} < \min\{b_1, b_2\} \tag{8}$$

then (6) has an equilibrium, x^*, globally exponentially stable.

- (7) $\not\Rightarrow$ (8) and (8) $\not\Rightarrow$ (7)

Example 2: The Cohen-Grossberg model with time-varing delays

$$\dot{x}_i(t) = -\rho_i(x_i(t))\left[b_i(x_i(t)) + \sum_{j=1}^{n}\sum_{p=1}^{P} h_{ijp}(x_j(t - \tau_{ijp}(t)))\right] \quad (9)$$

has an equilibrium point globally exponentially stable if

- $\tau_{ijp} : [0, +\infty) \to [0, +\infty)$ are continuous with $\tau_{ijp}(t) \le \tau$;

Example 2: The Cohen-Grossberg model with time-varing delays

$$\dot{x}_i(t) = -\rho_i(x_i(t)) \left[b_i(x_i(t)) + \sum_{j=1}^{n} \sum_{p=1}^{P} h_{ijp}(x_j(t - \tau_{ijp}(t))) \right] \quad (9)$$

has an equilibrium point globally exponentially stable if

- $\tau_{ijp} : [0, +\infty) \to [0, +\infty)$ are continuous with $\tau_{ijp}(t) \leq \tau$;
- $\rho_i : \mathbb{R} \to (0, +\infty)$ are continuous with $\rho_i(x) \geq r > 0$;

Example 2: The Cohen-Grossberg model with time-varing delays

$$\dot{x}_i(t) = -\rho_i(x_i(t))\left[b_i(x_i(t)) + \sum_{j=1}^{n}\sum_{p=1}^{P} h_{ijp}(x_j(t - \tau_{ijp}(t)))\right] \quad (9)$$

has an equilibrium point globally exponentially stable if

- $\tau_{ijp} : [0, +\infty) \to [0, +\infty)$ are continuous with $\tau_{ijp}(t) \le \tau$;
- $\rho_i : \mathbb{R} \to (0, +\infty)$ are continuous with $\rho_i(x) \ge r > 0$;
- $b_i : \mathbb{R} \to \mathbb{R}$ are continuous satisfying **(A1)**, i.e.,
  $\exists \beta_i > 0, \forall u, v \in \mathbb{R}, u \neq v$:

$$(b_i(u) - b_i(v))/(u - v) \ge \beta_i;$$

Example 2: The Cohen-Grossberg model with time-varing delays

$$\dot{x}_i(t) = -\rho_i(x_i(t))\left[b_i(x_i(t)) + \sum_{j=1}^{n}\sum_{p=1}^{P} h_{ijp}(x_j(t-\tau_{ijp}(t)))\right] \qquad (9)$$

has an equilibrium point globally exponentially stable if

- $\tau_{ijp} : [0,+\infty) \to [0,+\infty)$ are continuous with $\tau_{ijp}(t) \le \tau$;
- $\rho_i : \mathbb{R} \to (0,+\infty)$ are continuous with $\rho_i(x) \ge r > 0$;
- $b_i : \mathbb{R} \to \mathbb{R}$ are continuous satisfying **(A1)**, i.e., $\exists \beta_i > 0, \forall u, v \in \mathbb{R}, u \ne v$:

$$(b_i(u) - b_i(v))/(u - v) \ge \beta_i;$$

- h_{ijp} are Lipschitz functions with constant l_{ijp};

Example 2: The Cohen-Grossberg model with time-varing delays

$$\dot{x}_i(t) = -\rho_i(x_i(t))\left[b_i(x_i(t)) + \sum_{j=1}^{n}\sum_{p=1}^{P} h_{ijp}(x_j(t-\tau_{ijp}(t)))\right] \qquad (9)$$

has an equilibrium point globally exponentially stable if

- $\tau_{ijp} : [0, +\infty) \to [0, +\infty)$ are continuous with $\tau_{ijp}(t) \leq \tau$;
- $\rho_i : \mathbb{R} \to (0, +\infty)$ are continuous with $\rho_i(x) \geq r > 0$;
- $b_i : \mathbb{R} \to \mathbb{R}$ are continuous satisfying **(A1)**, i.e., $\exists \beta_i > 0, \forall u, v \in \mathbb{R}, u \neq v$:
$$(b_i(u) - b_i(v))/(u - v) \geq \beta_i;$$
- h_{ijp} are Lipschitz functions with constant l_{ijp};
- N is a non-singular M-matrix, where

$$N = diag(\beta_1, \dots, \beta_n) - [l_{ij}], \quad \text{with} \quad l_{ij} := \sum_{p=1}^{P} l_{ijp}.$$

In H. Jiang et al.[2], the same result was proved assuming:

- $\tau_{ijp} : [0, +\infty) \to [0, +\infty)$ are continuous with $\tau_{ijp}(t) \leq \tau$;
- $\rho_i(x)$ are locally Lipschitzian and $0 < r \leq \rho_i(x) \leq R < \infty$;
- $b_i \in C^1(\mathbb{R}, \mathbb{R})$ with $b_i'(x) \geq \beta_i > 0$;
- $h_{ijp}(x) = c_{ijp} g_{ijp}(x)$, with $c_{ijp} \in \mathbb{R}$ and g_{ijp} Lipschitz functions with constant μ_{ijp};
- $\exists \alpha_{ijp}, \gamma_{ijp} \in \mathbb{R}, \omega_i > 0, r > 1, \sigma > 0$ such that, $\forall i$

$$r\omega_i \underline{k_i} \beta_i - (r-1)\sum_{j=1}^{n}\sum_{p=1}^{P} \omega_j \overline{k_i} \mu_{ijp}^{\frac{r-\gamma_{ijp}}{r-1}} |c_{ijp}|^{\frac{r-\alpha_{ijp}}{r-1}} - \sum_{j=1}^{n}\sum_{p=1}^{P} \omega_j \overline{k_i} \mu_{ijp}^{\gamma_{ijp}} |c_{ijp}|^{\alpha_{ijp}} > \sigma. \qquad (10)$$

[2] H. Jiang, J. Cao and Z. Teng, Dynamics of Cohen-Grossberg neural networks with time-varing delays, *Phys. Lett. A* 354 (2006) 414-422.

- Example 3:

$$\dot{x}_1(t) = -(8 + 2\sin x_1(t))[7x_1(t) - \tanh x_1(t) - 2\tanh x_2(t) \\ - \tanh(x_1(t - \tfrac{1}{3}\sin t - 1)) - \tanh(x_2(t - \tfrac{1}{4}e^{-\sin t} - 1)) + 2]$$

$$\dot{x}_2(t) = -(5 + 2\cos x_2(t))[10x_2(t) - 2\tanh x_1(t) - \tanh x_2(t) \\ - \tanh(x_1(t - \tfrac{1}{4}e^{-\sin t} - 1)) - 2\tanh(x_2(t - \tfrac{1}{3}\sin t - 1)) + 3]$$

- Example 3:

$$\begin{aligned}\dot{x}_1(t) = &-(8 + 2\sin x_1(t))[7x_1(t) - \tanh x_1(t) - 2\tanh x_2(t) \\ &- \tanh(x_1(t - \tfrac{1}{3}\sin t - 1)) - \tanh(x_2(t - \tfrac{1}{4}e^{-\sin t} - 1)) + 2]\end{aligned}$$

$$\begin{aligned}\dot{x}_2(t) = &-(5 + 2\cos x_2(t))[10x_2(t) - 2\tanh x_1(t) - \tanh x_2(t) \\ &- \tanh(x_1(t - \tfrac{1}{4}e^{-\sin t} - 1)) - 2\tanh(x_2(t - \tfrac{1}{3}\sin t - 1)) + 3]\end{aligned}$$

- $N = diag(7, 10) - \begin{pmatrix} 2 & 3 \\ 3 & 3 \end{pmatrix}$ is a non-singular M-matrix.

- Example 3:

$$\dot{x}_1(t) = -(8 + 2\sin x_1(t))[7x_1(t) - \tanh x_1(t) - 2\tanh x_2(t) - \tanh(x_1(t - \tfrac{1}{3}\sin t - 1)) - \tanh(x_2(t - \tfrac{1}{4}e^{-\sin t} - 1)) + 2]$$

$$\dot{x}_2(t) = -(5 + 2\cos x_2(t))[10x_2(t) - 2\tanh x_1(t) - \tanh x_2(t) - \tanh(x_1(t - \tfrac{1}{4}e^{-\sin t} - 1)) - 2\tanh(x_2(t - \tfrac{1}{3}\sin t - 1)) + 3]$$

- $N = diag(7, 10) - \begin{pmatrix} 2 & 3 \\ 3 & 3 \end{pmatrix}$ is a non-singular M-matrix.
- If (10) holds, then there are $\omega_1, \omega_2 > 0$ such that

$$\begin{cases} 22\omega_1 - 30\omega_2 > 0 \\ -21\omega_1 + 9\omega_2 > 0 \end{cases},$$

which is impossible.

Thanks you

Obrigado