Global Stability for Delayed Neural Network Models

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1. Boundedness of solutions

$$\dot{y}(t) = f(t, y_t)$$

2. Global stability

$$\dot{y}_i(t) = f_i(y_t), \qquad i = 1, \dots, n$$
$$\dot{x}_i(t) = -\rho_i(x_i(t))(b_i(x_i(t)) + f_i(x_t)), \qquad i = 1, \dots, n$$

3. Exponential stability

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4. Neural Network Models

$$\dot{x}_i(t) = -\rho_i(x_i(t))\left(b_i(x_i(t)) + \sum_{j=1}^{n} f_{ij}(x_{j,t})\right), i = 1, \dots, n$$

Notation

- $n \in \mathbb{N}$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$,

$$|x|_\infty = \max_{1 \le i \le n} |x_i|;$$

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- $\tau \in \mathbb{R}^+$, $C_n := C([-\tau, 0]; \mathbb{R}^n)$

$$\|\varphi\|_\infty = \sup_{\theta \in [-\tau, 0]} |\varphi(\theta)|_\infty;$$

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$$\|\varphi\|_\infty = \sup_{\theta \in [-\tau, 0]} |\varphi(\theta)|_\infty;$$

- $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is a **non-singular M-matrix** if $a_{ij} \le 0$, $i \ne j$ and $\mathrm{Re}\, \sigma(A) > 0$.

1. Boundedness of solutions

- FDE in C_n

$$\dot{y}(t) = f(t, y_t), \qquad t \geq t_0, \tag{1}$$

$t_0 \in \mathbb{R}$,
$f = (f_1, \dots, f_n) : [t_0, +\infty) \times C_n \to \mathbb{R}^n$ continuous,
$y_t \in C_n$, $y_t(\theta) = y(t + \theta)$, $\theta \in [-\tau, 0]$.

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- $y(t) = y(t, t_0, \varphi)$ denote the solution of (1) such that $y_{t_0} = \varphi \in C_n$.

- **Proposition 1**
  Assume that, $f = (f_1, \dots, f_n)$ satisfies
  (H) $\forall t \geq t_0, \forall \varphi \in C_n, \forall i \in \{1, \dots, n\}$,

$$\|\varphi\|_\infty = |\varphi(0)|_\infty = |\varphi_i(0)| > 0 \Rightarrow \varphi_i(0) f_i(t, \varphi) < 0;$$

  Then, the solution $y(t) = y(t, t_0, \varphi)$ of (1) is defined and bounded on $[t_0, +\infty)$, and

$$|y(t)|_\infty \leq \|\varphi\|_\infty, \quad \text{for} \quad t \geq t_0.$$

- **Proof** [1].

[1] T. Faria, J. J. Oliveira, Local and global stability for Lotka-Volterra systems with distributed delays and instantaneous negative feedbacks, *J. Differential Equations* 244 (2008) 1049-1079.

2. Global stability

- FDE in C_n

$$\dot{y}_i(t) = f_i(y_t), \qquad t \geq 0, \; i = 1, \dots, n, \tag{2}$$

$f_i : C_n \to \mathbb{R}$ are continuous, for $i \in \{1, \dots, n\}$
Hypotheses:

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- **(H1)**$\Rightarrow$ $y \equiv 0$ is the unique equilibrium of (2).

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  Let $y(t) = (y_1(t), \dots, y_n(t))$ be a solution of (2).
- **Proposition 1** $\Rightarrow y(t)$ defined and bounded on $[-\tau, +\infty)$

$$-v_i = \liminf_{t \to +\infty} y_i(t), \qquad u_i = \limsup_{t \to +\infty} y_i(t)$$

$$v = \max_i \{v_i\}, \qquad u = \max_i \{u_i\},$$

$u, v \in \mathbb{R}$, $-v \leq u$.

- We have to show $\max(u, v) = 0$.

- We suppose $|v| \leq u$. ($|u| \leq v$ is similar)
  Let $i \in \{1, \dots, n\}$ such that $u_i = u$.

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- We can show that exists $(t_k)_{k \in \mathbb{N}}$ such that

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- **(H1)**+**(H2)** $\Rightarrow \dot{y}(t)$ is bounded $\Rightarrow \{y_{t_k}\}$ is bounded and equicontinuous $\Rightarrow \exists \varphi \in C_n$

$$y_{t_k} \to \varphi \text{ on } C_n,$$

with $\|\varphi\|_\infty \leq u$, $\varphi_i(0) = u$ and $f_i(\varphi) = 0$.

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- **(H1)** $\Rightarrow u = 0$. $\square$

- Applications to general neural network models

$$\dot{x}_i(t) = -\rho_i(x_i(t))[b_i(x_i(t)) + f_i(x_t)], \qquad t \geq 0, \; i = 1, \dots, n, \quad (3)$$

where $\rho_i : \mathbb{R} \to (0, +\infty)$, $b_i : \mathbb{R} \to \mathbb{R}$ and $f = (f_1, \dots, f_n) : C_n \to \mathbb{R}^n$ are continuous.

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$f = (f_1, \dots, f_n) : C_n \to \mathbb{R}^n$ are continuous.

Hypotheses:

- **(A1)** $\exists \beta_i > 0, \forall u, v \in \mathbb{R}, u \neq v$:

$$(b_i(u) - b_i(v))/(u - v) \geq \beta_i;$$

[In particular, for $b_i(u) = \beta_i u$.]

- **(A2)** $f_i : C_n \to \mathbb{R}$ are Lipschitz functions with Lipschitz constants l_i.

- **Theorem 2**
  Assume **(A1)** and **(A2)**.
  If $\beta_i > l_i, \forall i$, then (3) has an equilibrium point $x^* \in \mathbb{R}^n$, which is globally asymptotically stable.

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- **Proof (idea)**
  Existence of equilibrium point

$$\begin{array}{rccl} H: & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & x & \longmapsto & (b_1(x_1) + f_1(x), \dots, b_n(x_n) + f_n(x)) \end{array}$$

  is a homeomorphism.
  Then there exists $x^* \in \mathbb{R}^n$, $H(x^*) = 0$, i.e. x^* is an equilibrium.

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- By translation, we may suppose $x^* = 0$, $b_i(0) + f_i(0) = 0, \forall i$.
- $\beta_i > l_i, \forall i \Rightarrow$**(H1)** and **(H2)**
  From Theorem 1, we have the result.□

3. Exponential stability

- Consider again the FDE,

$$\dot{x}_i(t) = -\rho_i(x_i(t))[b_i(x_i(t))+f_i(x_t)], \qquad t \geq 0,\ i = 1, \dots, n \quad (3)$$

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- **Definition**
  An equilibrium $x^* \in \mathbb{R}$ of (3) is said to be *globally exponentially stable* if there are $\varepsilon, M > 0$ such that

$$|x(t, 0, \varphi) - x^*|_\infty \leq Me^{-\varepsilon t}\|\varphi - x^*\|_\infty, \qquad t \geq 0, \ \varphi \in C_n.$$

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- **Theorem 3**
Suppose that $0 < r \leq \rho_i(x)$, $\forall x \in \mathbb{R}$, $i = 1, \dots, n$
and assume **(A1)** and **(A2)** with $\beta_i > l_i$, $i = 1, \dots, n$.
Then there is a unique equilibrium x^* of (3) which is globally exponentially stable.

$$\dot{x}_i(t) = -\rho_i(x_i(t))(b_i(x_i(t)) + f_i(x_t)), \qquad i = 1, \dots, n$$

- Proof (idea)
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- The change of variables

$$z(t) = e^{\varepsilon t} x(t),$$

for $\varepsilon > 0$ small enough, transform (3) into

$$\dot{z}(t) = g(t, z_t), \tag{4}$$

$g = (g_1, \dots, g_n)$ with

$$g_i(t, \varphi) = \varepsilon \varphi_i(0) - \rho_i(t, e^{-\varepsilon(t+\cdot)}\varphi) e^{\varepsilon t} \left[b_i(e^{-\varepsilon t}\varphi_i(0)) + f_i(t, e^{-\varepsilon(t+\cdot)}\varphi) \right]$$

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- **(A1)**+**(A2)**$\Rightarrow$**(H)**
- From Proposition 1 $|z(t)|_\infty \le \|z_0\|_\infty$, $t \ge 0$, and

$$|x(t, 0, \varphi)|_\infty = |e^{-\varepsilon t} z(t, 0, e^{\varepsilon \cdot}\varphi)|_\infty \le e^{-\varepsilon t}\|\varphi\|_\infty . \Box$$

4. Neural Network Models

Cohen-Grossberg neural network with distributed delays

$$\dot{x}_i(t) = -\rho_i(x_i(t)) \left[b_i(x_i(t)) + \sum_{j=1}^{n} f_{ij}(x_{j,t}) \right], \qquad (5)$$

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- $\rho_i : \mathbb{R} \to (0, +\infty)$ are continuous;
- $f_{ij} : C_1 \to \mathbb{R}$ are Lipschitzian with

$$|f_{ij}(\varphi) - f_{ij}(\psi)| \leq l_{ij}\|\varphi - \psi\|, \qquad \varphi, \psi \in C_1 = C([-\tau, 0]; \mathbb{R});$$

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- $b_i : \mathbb{R} \to \mathbb{R}$ are continuous satisfying **(A1)**;
- $N := diag(\beta_1, \ldots, \beta_n) - [l_{ij}]$.

- **Theorem 4**
  If N is a non-singular M-matrix, then there is a unique equilibrium of (5), which is globally asymptotically stable. If in addition $0 < r \leq \rho_i(x)$, $\forall x \in \mathbb{R}$, $i = 1, \ldots, n$, then the equilibrium of (5) is globally exponentially stable.

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- **Proof (idea)**
  N non-singular M-matrix $\Rightarrow$
  Exists $d = (d_1, \ldots, d_n) > 0$ such that $Nd > 0$, i.e.

$$\beta_i > d_i^{-1} \sum_{j=1}^{n} l_{ij} d_j, \qquad i = 1, \ldots, n; \tag{6}$$

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- The change of variables

$$y_i(t) = d_i^{-1} x_i(t)$$

- transform (5) into

$$\dot{y}_i(t) = -\bar{\rho}_i(y_i(t))[\bar{b}_i(y_i(t)) + \bar{f}_i(y_t)], \qquad i = 1, \dots, n,$$

$$\bar{f}_i(\varphi) = d_i^{-1} \sum_{j=1}^{n} f_{ij}(d_j \varphi_j), \qquad \varphi \in C_n$$

$$\bar{b}_i(u) = d_i^{-1} b_i(d_i u), \qquad \bar{\rho}_i = \rho_i(d_i u), \qquad u \in \mathbb{R}$$

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$$\bar{b}_i(u) = d_i^{-1}b_i(d_iu), \qquad \bar{\rho}_i = \rho_i(d_iu), \qquad u \in \mathbb{R}$$

- $\bar{f}_i$ satisfies **(A2)**, $i \in \{1, \dots, n\}$:

$$|\bar{f}_i(\varphi) - \bar{f}_i(\psi)| \leq \left(d_i^{-1}\sum_{j=1}^{n} l_{ij}d_j\right)\|\varphi - \psi\|_\infty$$

and $\bar{b}_i$ satisfies **(A1)** with, by (6),

$$\bar{\beta}_i = \beta_i > \bar{l}_i := d_i^{-1}\sum_{j=1}^{n} l_{ij}d_j,$$

and the result follows from Theorems 2 and 3.□

The Cohen-Grossberg model with time-varing delays

$$\dot{x}_i(t) = -\rho_i(x_i(t)) \left[b_i(x_i(t)) + \sum_{j=1}^{n} \sum_{p=1}^{P} h_{ijp}(x_j(t - \tau_{ijp}(t))) \right] \quad (7)$$

has an equilibrium globally asymptotically (exponentially) stable if

- $\tau_{ijp} : [0, +\infty) \to [0, +\infty)$ are continuous with $\tau_{ijp}(t) \leq \tau$;

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- $b_i : \mathbb{R} \to \mathbb{R}$ are continuous satisfying **(A1)**, i.e., $\exists \beta_i > 0, \forall u, v \in \mathbb{R}, u \neq v$:
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- $b_i : \mathbb{R} \to \mathbb{R}$ are continuous satisfying **(A1)**, i.e., $\exists \beta_i > 0, \forall u, v \in \mathbb{R}, u \neq v$:
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- h_{ijp} are Lipschitz functions with constant l_{ijp};

The Cohen-Grossberg model with time-varing delays

$$\dot{x}_i(t) = -\rho_i(x_i(t)) \left[b_i(x_i(t)) + \sum_{j=1}^{n} \sum_{p=1}^{P} h_{ijp}(x_j(t - \tau_{ijp}(t))) \right] \quad (7)$$

has an equilibrium globally asymptotically (exponentially) stable if

- $\tau_{ijp} : [0, +\infty) \to [0, +\infty)$ are continuous with $\tau_{ijp}(t) \leq \tau$;
- $\rho_i : \mathbb{R} \to (0, +\infty)$ are continuous (with $\rho_i(x) \geq r > 0$);
- $b_i : \mathbb{R} \to \mathbb{R}$ are continuous satisfying **(A1)**, i.e., $\exists \beta_i > 0, \forall u, v \in \mathbb{R}, u \neq v$:
$$(b_i(u) - b_i(v))/(u - v) \geq \beta_i;$$
- h_{ijp} are Lipschitz functions with constant l_{ijp};
- N is a non-singular M-matrix, where

$$N = diag(\beta_1, \ldots, \beta_n) - [l_{ij}], \quad \text{with} \quad l_{ij} := \sum_{p=1}^{P} l_{ijp}.$$

In H. Jiang et al.[2], proved the exponential stability assuming:

- $\tau_{ijp} : [0, +\infty) \to [0, +\infty)$ are continuous with $\tau_{ijp}(t) \leq \tau$;
- $\rho_i(x)$ are locally Lipschitzian and $0 < r \leq \rho_i(x) \leq R < \infty$;
- $b_i \in C^1(\mathbb{R}, \mathbb{R})$ with $b_i'(x) \geq \beta_i > 0$;
- $h_{ijp}(x) = c_{ijp} g_{ijp}(x)$, with $c_{ijp} \in \mathbb{R}$ and g_{ijp} Lipschitz functions with constant μ_{ijp};
- $\exists \alpha_{ijp}, \gamma_{ijp} \in \mathbb{R}, \omega_i > 0, r > 1, \sigma > 0$ such that, $\forall i$

$$r\omega_i \underline{k_i} \beta_i - (r-1)\sum_{j=1}^{n}\sum_{p=1}^{P} \omega_j \overline{k_i} \mu_{ijp}^{\frac{r-\gamma_{ijp}}{r-1}} |c_{ijp}|^{\frac{r-\alpha_{ijp}}{r-1}} - \sum_{j=1}^{n}\sum_{p=1}^{P} \omega_j \overline{k_i} \mu_{ijp}^{\gamma_{ijp}} |c_{ijp}|_{ijp}^{\alpha} > \sigma. \qquad (8)$$

Instead of (8), we only assume N non-singular M-matrix.

[2] H. Jiang, J. Cao and Z. Teng, Dynamics of Cohen-Grossberg neural networks with time-varing delays, *Phys. Lett. A* 354 (2006) 414-422.

- Example:

$$\dot{x}_1(t) = -(8 + 2\sin x_1(t))[7x_1(t) - \tanh x_1(t) - 2\tanh x_2(t) - \tanh(x_1(t - \tfrac{1}{3}\sin t - 1)) - \tanh(x_2(t - \tfrac{1}{4}e^{-\sin t} - 1)) + 2]$$

$$\dot{x}_2(t) = -(5 + 2\cos x_2(t))[10x_2(t) - 2\tanh x_1(t) - \tanh x_2(t) - \tanh(x_1(t - \tfrac{1}{4}e^{-\sin t} - 1)) - 2\tanh(x_2(t - \tfrac{1}{3}\sin t - 1)) + 3]$$

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- $N = diag(7, 10) - \begin{pmatrix} 2 & 3 \\ 3 & 3 \end{pmatrix}$ is a non-singular M-matrix.

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- $N = diag(7, 10) - \begin{pmatrix} 2 & 3 \\ 3 & 3 \end{pmatrix}$ is a non-singular M-matrix.
- If (8) holds, then there are $\omega_1, \omega_2 > 0$ such that

$$\begin{cases} 22\omega_1 - 30\omega_2 > 0 \\ -21\omega_1 + 9\omega_2 > 0 \end{cases},$$

which is impossible.

Static NNM with S-type distributed delays

$$\dot{x}_i(t) = -\rho_i(x_i(t)) \left[b_i(x_i(t)) + f_i \left(\sum_{j=1}^{n} \omega_{ij} \int_{-\tau}^{0} x_j(t+\theta) d\eta_{ij}(\theta) + J_i \right) \right] \quad (9)$$

- $\tau > 0, J_i, \omega_{ij} \in \mathbb{R}$;
- $\rho_i : \mathbb{R} \to (0, +\infty)$ continuous (with $\rho_i(x) \geq r > 0$);
- $b_i : \mathbb{R} \to \mathbb{R}$ continuous satisfying **(A1)**;
- $f_i : \mathbb{R} \to \mathbb{R}$ Lipschitz with Lipschitz constant l_i;
- $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ are normalized bounded variation functions;

$$N := diag(\beta_1, \dots, \beta_n) - [l_i|\omega_{ij}|]$$

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$$N := diag(\beta_1, \dots, \beta_n) - [l_i|\omega_{ij}|]$$

- **Theorem 5**
  If N is a non-singular M-matrix, then there is an equilibrium point of (9) which is globally asymptotically (exponentially) stable.

- Example:

$$\dot{x}_i(t) = -b_i x_i(t) + f_i \left(\sum_{j=1}^{n} \omega_{ij} \int_{-\tau}^{0} x_j(t+\theta) d\eta_{ij}(\theta) + J_i \right) \qquad (10)$$

- $b_i > 0$, $J_i, \omega_{ij} \in \mathbb{R}$;
- $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ are normalized bounded variation functions;
- $f_i : \mathbb{R} \to \mathbb{R}$ Lipschitz with Lipschitz constant l_i;

$$N = diag(b_1, \ldots, b_n) - [l_i|\omega_{ij}|].$$

- Example:

$$\dot{x}_i(t) = -b_i x_i(t) + f_i\left(\sum_{j=1}^{n} \omega_{ij} \int_{-\tau}^{0} x_j(t+\theta) d\eta_{ij}(\theta) + J_i\right) \qquad (10)$$

- $b_i > 0$, $J_i, \omega_{ij} \in \mathbb{R}$;
- $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ are normalized bounded variation functions;
- $f_i : \mathbb{R} \to \mathbb{R}$ Lipschitz with Lipschitz constant l_i;

$$N = diag(b_1, \ldots, b_n) - [l_i|\omega_{ij}|].$$

- **Corollary**
  If N is a non-singular M-matrix, then there is an equilibrium point of (10) which is globally exponentially stable.

- Example:

$$\dot{x}_i(t) = -b_i x_i(t) + f_i\left(\sum_{j=1}^{n} \omega_{ij} \int_{-\tau}^{0} x_j(t+\theta) d\eta_{ij}(\theta) + J_i\right) \qquad (10)$$

- $b_i > 0$, $J_i, \omega_{ij} \in \mathbb{R}$;
- $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ are normalized bounded variation functions;
- $f_i : \mathbb{R} \to \mathbb{R}$ Lipschitz with Lipschitz constant l_i;

$$N = diag(b_1, \ldots, b_n) - [l_i|\omega_{ij}|].$$

- **Corollary**

  If N is a non-singular M-matrix, then there is an equilibrium point of (10) which is globally exponentially stable.
- **Remark**

  In 2006, M. Wang and L. Wang proved the global **asymptotic** stability with η_{ij} normalizing and **nondecreasing bounded variation functions** on $[-\tau, 0]$.

Thanks you

Obrigado