Global Stability of a Generalized Cohen-Grossberg Model with Unbounded Time-Varying Delays

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Boundedness of solutions

$$\dot{x}(t) = f(t, x_t)$$

Global stability

$$\dot{x}_i(t) = -k_i(x_i(t)) \left(b_i(x_i(t)) + \sum_{j=1}^{n} \sum_{p=1}^{P} h_{ij}^{(p)}(x_j(t - \tau_{ij}^{(p)}(t)) \right)$$

Application

Cohen-Grossberg neural network model with unbounded delays

In

José J. Oliveira, *Math. Comp. Model.* **50**(2009), 81-91

the case of neural networks with <u>bounded</u> distributed delays was treated, as well <u>bounded</u> discrete time-varying delays.

Notation

- $n \in \mathbb{N}$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$|x| = \max_{1 \le i \le n} |x_i|;$$

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- We consider the space of bounded and continuous functions $\varphi : (-\infty, 0] \to \mathbb{R}^n$

$$BC = BC((-\infty, 0]; \mathbb{R}^n),$$

with the norm $\|\varphi\| = \sup_{s \le 0} |\varphi(s)|$;

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with the norm $\|\varphi\| = \sup_{s \le 0} |\varphi(s)|$;

- $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is a **non-singular M-matrix** if $a_{ij} \le 0$, $i \ne j$ and $\mathrm{Re}\, \sigma(A) > 0$.

Boundedness of solutions

- FDE in BC

$$\dot{x}(t) = f(t, x_t), \qquad t \geq 0, \tag{1}$$

$f = (f_1, \dots, f_n) : [0, +\infty) \times BC \to \mathbb{R}^n$ continuous,
$x_t(s) = x(t+s)$, $s \in (-\infty, 0]$,

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- **Proposition 1**
  Assume that, $f = (f_1, \dots, f_n)$ satisfies
  (H) $\forall t \geq 0, \forall \varphi \in BC$:

$$\forall s \in (-\infty, 0), |\varphi(s)| < |\varphi(0)| \Rightarrow \varphi_i(0) f_i(t, \varphi) < 0,$$

for some $i \in \{1, \dots, n\}$ such that $|\varphi(s)| = |\varphi_i(0)|$.
Then all solution of (1) with initial condition on BC is defined and bounded on $[0, +\infty)$.

$$|x(t, 0, \varphi)| \leq \|\varphi\|$$

Proof of Proposition 1 (idea)

- $x(t) = x(t, 0, \varphi)$ solution on $[-\infty, a)$, $a > 0$, with $\varphi \in BC$
  $k := \sup_{s \leq 0} |\varphi(s)|$.

Proof of Proposition 1 (idea)

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- Suppose that $|x(t_1)| > k$ for some $t_1 > 0$ and define

$$T = \min\left\{ t \in [0, t_1] : |x(t)| = \max_{s \in [0, t_1]} |x(s)| \right\}.$$

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- We have $|x_T(s)| = |x(T + s)| < |x(T)|$, for $s < 0$. By **(H)** we conclude that,

$$x_i(T) f_i(T, x_T) < 0,$$

for some $i \in \{1, \ldots, n\}$ such that $|x_i(T)| = |x(T)|$. If $x_i(T) > 0$ (analogous if $x_i(T) < 0$), then $\dot{x}_i(T) < 0$.

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- $x_i(t) \leq |x(t)| < |x(T)| = x_i(T)$, $t \in [0, T)$,

$$\Rightarrow \dot{x}_i(T) \geq 0.$$

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$$\Rightarrow \dot{x}_i(T) \geq 0.$$

- Contradition. Thus $x(t)$ is defined and bounded on $[0, +\infty)$.

Global stability

- Consider the generalized Cohen-Grossberg neural network model with unbounded time-varying delays,

$$\dot{x}_i(t) = -k_i(x_i(t)) \left(b_i(x_i(t)) + \sum_{j=1}^{n} \sum_{p=1}^{P} h_{ij}^{(p)}(x_j(t - \tau_{ij}^{(p)}(t)) \right) \quad (2)$$

where $k_i, : \mathbb{R} \to (0, +\infty)$, $b_i, h_{ij}^{(p)} : \mathbb{R} \to \mathbb{R}$ and $\tau_{ij}^{(p)} : [0, +\infty) \to [0, +\infty)$ are continuous functions such that

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- **(A1)** $\exists \beta_i > 0, \forall u, v \in \mathbb{R}, u \neq v$:

$$(b_i(u) - b_i(v))/(u-v) \geq \beta_i;$$

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- **(A2)** $h_{ij}^{(p)}$ are Lipshitz functions with constant $l_{ij}^{(p)}$;
- **(A3)** $t - \tau_{ij}^{(p)}(t) \to +\infty$ as $t \to +\infty$.

- **Theorem 1**

  Assume **(A1)**, **(A2)**, **(A3)**. If the matrix

$$N := diag(\beta_1, \ldots, \beta_n) - [l_{ij}], \quad \text{where} \quad l_{ij} = \sum_{p=1}^{P} l_{ij}^{(p)},$$

  is a non-singular M-matrix, then there is a unique equilibrium point of (2), which is globally asymptotically stable.

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  is a non-singular M-matrix, then there is a unique equilibrium point of (2), which is globally asymptotically stable.
- **Proof (idea)**
  Existence and uniqueness of equilibrium point

$$\begin{array}{rccc} H: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & x & \mapsto & \left(b_i(x_i) + \sum_{j=1}^{n}\sum_{p=1}^{P} h_{ij}^{(p)}(x_j)\right)_{i=1}^{n} \end{array}$$

  is homeomorphism.
  Then there exists $x^* \in \mathbb{R}^n$, $H(x^*) = 0$, i.e. x^* is the equilibrium.

- N non-singular M-matrix $\Rightarrow$
  There is $d = (d_1, \dots, d_n) > 0$ such that $Nd > 0$, i.e.

$$\beta_i > d_i^{-1} \sum_{j=1}^{n} l_{ij} d_j, \qquad i = 1, \dots, n; \tag{3}$$

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- The change of variables

$$y_i(t) = d_i^{-1} x_i(t) - x_i^*$$

transforms (2) into

$$\dot{y}_i(t) = -\bar{k}_i(y_i(t)) \left[\bar{b}_i(y_i(t)) + \bar{h}_i(t, y_t) \right], \qquad i = 1, \ldots, n, \tag{4}$$

$$\bar{h}_i(t, \varphi) = d_i^{-1} \sum_{j=1}^{n} \sum_{p=1}^{P} h_{ij}^{(p)}(d_j(\varphi_j(-\tau_{ij}^{(p)}(t)) + x_j^*)), \qquad \varphi \in BC$$

$$\bar{b}_i(u) = d_i^{-1} b_i(d_i(u + x_i^*)), \qquad \bar{k}_i = k_i(d_i(u + x_i^*)), \qquad u \in \mathbb{R}$$

with $\bar{b}_i(0) + \bar{h}_i(t, 0) = 0$, $\forall t \geq 0, i = 1, \ldots, n$.

- Boundedness of solutions
  By **(A1)** and **(A2)**, we can conclude that

$$\bar{b}_i(\varphi_i(0)) + \bar{h}_i(t,\varphi) \geq \left(\beta_i - d_i^{-1}\sum_{j=1}^{n} l_{ij}d_j\right)\sup_{s\leq 0}|\varphi(s)| > 0,$$

for $\varphi \in BC$ such that $\varphi_i(0) = \sup_{s\leq 0}|\varphi(s)| > 0$. Hence $f = (f_1, \dots, f_n) : [0,+\infty) \times BC \to \mathbb{R}^n$, defined by

$$f_i(t,\varphi) = -\bar{k}_i(\varphi_i(0))\left[\bar{b}_i(\varphi_i(0)) + \bar{h}_i(t,\varphi)\right], \qquad i = 1,\dots,n,$$

satisfies **(H)** and, from Proposition 1, all solutions of (4) with initial conditions on BC are defined and bounded on $\mathbb{R}$.

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- Then $y(t)$ is defined and bounded on $\mathbb{R}$.

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$$v = \max_i\{v_i\}, \qquad u = \max_i\{u_i\},$$

$u, v \in \mathbb{R}$, $-v \leq u$.

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- We suppose $|v| \leq u$. ($|u| \leq v$ is similar)
  Let $i \in \{1, \ldots, n\}$ such that $u_i = u$.
- We can show that exists $(t_k)_{k\in\mathbb{N}}$ such that

$$t_k \nearrow +\infty, \qquad y_i(t_k) \to u, \qquad f_i(t_k, y_{t_k}) \to 0 \quad \text{as} \quad t \to +\infty.$$

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- As $t - \tau_{ij}^{(p)}(t) \to +\infty$, we have

$$\forall \varepsilon > 0, \exists k_0 \in \mathbb{N} : k \geq k_0 \Rightarrow |y(t_k)|, |y(t_k - \tau_{ij}^{(p)}(t_k))| < u_\varepsilon := u + \varepsilon$$

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- From **(A1)** and **(A2)**, we can obtain, for $k \geq k_0$,

$$\bar{b}_i(y_i(t_k)) + \bar{h}_i(t_k, y_{t_k})$$

$$\geq \beta_i y_i(t_k) - d_i^{-1} \sum_{j=1}^{n} l_{ij} d_j \sup_{k \geq k_0, \forall i,j,p} |y_j(t_k - \tau_{ij}^{(p)}(t_k))|$$

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- From **(A1)** and **(A2)**, we can obtain, for $k \geq k_0$,

$$\begin{aligned} &\bar{b}_i(y_i(t_k)) + \bar{h}_i(t_k, y_{t_k}) \\ &\geq \beta_i y_i(t_k) - d_i^{-1} \sum_{j=1}^{n} l_{ij} d_j \sup_{k \geq k_0, \forall i,j,p} |y_j(t_k - \tau_{ij}^{(p)}(t_k))| \\ &\geq \beta_i y_i(t_k) - d_i^{-1} \sum_{j=1}^{n} l_{ij} d_j u_\varepsilon. \end{aligned}$$

- Letting $k \to +\infty$ and $\varepsilon \to 0$, we get

$$\beta_i y_i(t_k) - d_i^{-1} \sum_{j=1}^{n} l_{ij} d_j u_\varepsilon \to \left(\beta_i - d_i^{-1} \sum_{j=1}^{n} l_{ij} d_j \right) u > 0.$$

- As $y_i(t)$ is bounded and $\bar{k}_i$ is continuous and positive,

$$\exists K > 0, \forall t \geq 0 : \bar{k}_i(y_i(t)) > K.$$

Thus

$$f_i(t_k, y_{t_k}) = -\bar{k}_i(y_i(t_k)) \left[\bar{b}_i(y_i(t_k)) + \bar{h}_i(t_k, y_{t_k}) \right] \nrightarrow 0 \text{ as } k \to +\infty$$

which is a contradiction.
Consequently $u = 0$, hence $v = 0$.

Cohen-Grossberg Model

Cohen-Grossberg neural network with unbounded discrete delays

$$\dot{x}_i(t) = -k_i(x_i(t)) \left[b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij} g_j(x_j(t)) - \sum_{j=1}^{n} d_{ij} f_j(x_j(t - \tau_{ij}(t))) \right] \quad (5)$$

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- $k_i : \mathbb{R} \to (0, +\infty)$, $\tau_{ij} : [0, +\infty) \to [0, +\infty)$ are continuous and $t - \tau_{ij}(t) \to +\infty$;

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- $b_i : \mathbb{R} \to \mathbb{R}$ are continuous satisfying **(A2)**;
- $N := diag(\beta_1, \dots, \beta_n) - [l_{ij}]$, where $l_{ij} = |c_{ij}|G_j + |d_{ij}|F_j$.

- **Corollary**

  If N is a non-singular M-matrix, then there is a unique equilibrium point of (5), which is globally asymptotically stable.

[1] T. Huang, A. Chan, Y. Huang, J. Cao, Stability of Cohen-Grossberg neural networks with time-varying delays, Neural Networks, 20 (2007) 868-873.

- **Corollary**
  If N is a non-singular M-matrix, then there is a unique equilibrium point of (5), which is globally asymptotically stable.
- **Proof**
  System (5) has the form (2) if
  $P = 2$, $h_{ij}^{(1)}(u) = -c_{ij}g_j(u)$ and $h_{ij}^{(2)}(u) = -d_{ij}f_j(u)$, $u \in \mathbb{R}$;
  $\tau_{ij}^{(1)}(t) = 0$, $\tau_{ij}^{(2)}(t) = \tau_{ij}(t)$, $t \geq 0$, $i, j = 1, \ldots, n$.

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- $h_{ij}^{(1)}, h_{ij}^{(2)}$ are Lipschitz functions with Lipschitz constant $l_{ij}^{(1)} = |c_{ij}| G_j$, $l_{ij}^{(2)} = |d_{ij}| F_j$ respectively.
  The result follows from Theorem 1.

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  $P = 2$, $h_{ij}^{(1)}(u) = -c_{ij}g_j(u)$ and $h_{ij}^{(2)}(u) = -d_{ij}f_j(u)$, $u \in \mathbb{R}$;
  $\tau_{ij}^{(1)}(t) = 0$, $\tau_{ij}^{(2)}(t) = \tau_{ij}(t)$, $t \geq 0$, $i, j = 1, \ldots, n$.
- $h_{ij}^{(1)}, h_{ij}^{(2)}$ are Lipschitz functions with Lipschitz constant $l_{ij}^{(1)} = |c_{ij}|G_j$, $l_{ij}^{(2)} = |d_{ij}|F_j$ respectively.
  The result follows from Theorem 1.
- In [1] the global stability was proved with additional conditions:
  b_i differentiable and $0 < \underline{k}_i \leq k_i(u) \leq \overline{k}_i$, $u \in \mathbb{R}$.

[1] T. Huang, A. Chan, Y. Huang, J. Cao, Stability of Cohen-Grossberg neural networks with time-varying delays, Neural Networks, 20 (2007) 868-873.

Thanks you

Obrigado