Ordinary Differential Equations (ODE's)

PDMA - 2021/2022

Exercises / Problems nº1

1. Identify the domain and solve the following differential equations:

   (a) $x'(t) = \sin t + \dfrac{x(t)}{t}$;

   (b) $x'(t) = 3x(t) + t^2$;

   (c) $x'(t) = -\dfrac{2tx(t)}{t^2 + 2x(t)}$;

   (d) $x'(t) = x(t)^2$;

   (e) $x'(t) = F(x(t))$, where $F(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$.

2. Consider the logistic equation

$$x'(t) = ax(t)\left(1 - \frac{x(t)}{k}\right), \quad t \in \mathbb{R},$$

   where $a, k \in]0, +\infty[$.

   (a) Identify its domain;

   (b) Solve the differential equation;

   (c) Identify its equilibrium points;

   (d) For each $x_0 \in \left\{\frac{1}{2}, 1, 2\right\}$, solve the IVP

$$\begin{cases} x'(t) = x(t)\left(1 - x(t)\right) \\ \\ x(1) = x_0 \end{cases}.$$

   Note that in this case, we have the logistic equation with $a = k = 1$;

   (e) Identify its phase space;

   (f) Draw the graphic of the equilibrium points and some solutions of the logistic equation;

   (g) Draw the phase curves of the logistic equation.

3. Consider the following equation

$$x'(t) = F(x(t)), \tag{1}$$

where $F : \mathbb{R} \to \mathbb{R}$ is defined by $F(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$.

   (a) Why is equation (1) not an ODE;
   (b) Justify why, for each $\delta > 0$, there is no function in $C^1(]2-\delta, 2+\delta[; \mathbb{R})$, $x :]2-\delta, 2+\delta[\to \mathbb{R}$, such that x is a solution of (1) and $x(2) = 0$;
   (c) Find a function $x : [-1, 1] \to \mathbb{R}$ in $C^1([-1, 1]; \mathbb{R})$ which is a solution of (1) and $x(0) = 2$.

4. Consider $U \subseteq \mathbb{R} \times \mathbb{R}^n$ an open set, $f : U \subseteq \mathbb{R}^{1+n} \to \mathbb{R}^n$ a continuous function, and $(t_0, x_0) \in U$.
   Prove that $x(t)$ is a solution of the IVP

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

   if and only if

$$x(t) = x_0 + \int_{t_0}^{t} F(s, x(s))ds, \qquad \forall t \in I_{t_0},$$

   where I_{t_0} is an open interval such that $t_0 \in I_{t_0}$.

5. Show that, if

$$\begin{array}{rccc} F: & \mathbb{R}^{1+d} & \to & \mathbb{R}^d \\ & (t, x) & \mapsto & F(t, x) \end{array}$$

   is a C^1 map, then F is locally lipschitz map on second variable.

6. Using the iterative process given by contraction principle map, find the solution of

$$\begin{cases} x'(t) = 2tx(t) \\ \\ x(0) = 1 \end{cases}, t \in \mathbb{R}$$

7. Consider the ODE $x'(t) = x(t)^2$.
   (a) Justify that the ODE has the solution uniqueness property.
   (b) For each $(t_0, x_0) \in \mathbb{R}^2$, find $x(\cdot, t_0, x_0)$.

8. Consider the ODE

$$\begin{cases} x_1'(t) = x_2(t) \\ \\ x_2'(t) = 4x_1(t) \end{cases}, \quad t \in \mathbb{R}.$$

   (a) Solve the ODE;
   (b) Identify its phase space;
   (c) Draw the phase curves of the ODE.

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Exercises / Problems nº2

1. Prove that: If
$$\begin{array}{rccc} f: & \mathbb{R}^{1+n} & \to & \mathbb{R}^n \\ & (t,x) & \mapsto & f(t,x) \end{array}$$
is a C^1 class funtion, than f is locally Lipschitz in x.

2. Prove that the PVI
$$\begin{cases} x'(t) = \sqrt[3]{x(t)^2} \\ x(3) = 0 \end{cases}$$
has several maximal defined on $\mathbb{R}$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ a $C^1(\mathbb{R})$ function and $x : I \to \mathbb{R}$ a bounded maximal solution of $x'(t) = f(x(t))$.
   (a) Prove that $I = \mathbb{R}$;
   (b) Prove that x is a strictly monotone function;
   (c) Prove that the image of x is an open interval $]a, b[$;
   (d) Prove that $a, b \in \mathbb{R}$ in (c) are fixed points of $x'(t) = f(x(t))$.

4. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ a continuous function such that $x \cdot f(x) \geq 0$ for all $x \in \mathbb{R}$. Prove that all the maximal solutions of the differential equation $x'(t) = f(x(t))$ are defined on $\mathbb{R}$.

5. Consider the differential equation
$$x'(t) = \operatorname{sen}\big(x(t)\big), \quad \forall t \in \mathbb{R}. \tag{1}$$
Without solving the differential equation:
   (a) Obtain all equilibrium points of (1);
   (b) Prove that all maximal solutions of (1) are defined on $\mathbb{R}$;
   (c) Prove that all solutions of (1) are bounded and monotone;
   (d) Draw e graph of some solutions of (1);
   (e) Draw the phase curves of (1).

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Exercises / Problems 3

1. Consider the following three conservative systems:

$$\text{A}\begin{cases} x'(t) = 1 - 2x(t)y(t) \\ y'(t) = y(t)^2 + 2x(t) \end{cases};\text{B}\begin{cases} x'(t) = -4y(t) \\ y'(t) = 2x(t) - 2 \end{cases};\text{C}\begin{cases} x'(t) = x(t)^2 - 2x(t) \\ y'(t) = 2y(t) - 2x(t)y(t) \end{cases}.$$

   (a) For each conservative system, find its integral;

   (b) For systems B and C, draw the phase curves;

   (c) For systems B and C, identify the stability of each equilibrium point.

2. Consider the ordinary differential equation

$$x''(t) = -3x(t)^2 + 4x(t) + 1 \tag{1}$$

   (a) Write the ODE (1) as a first order differential equation

   (b) Justify that (1) is a conservative equation and find an integral;

   (c) Draw the phase curves of (1);

   (d) Identify the stability of each equilibrium point.

3. Consider the family of ordinary differential equations

$$x''(t) = -3x(t)^2 + 4x(t) + \lambda, \tag{2}$$

   where $\lambda \in \mathbb{R}$. Study the phase curves of ODE (2) for different values of λ.

4. Consider the ODE that describes the oscillating pendulum

$$\theta''(t) = -\frac{g}{l}\sin(\theta(t)), \tag{3}$$

   where g is the gravity, and $\theta(t)$ is the angle, at time t, that an oscillating pendulum of length l makes with the vertical direction.

   (a) Find the integral of (3);

   (b) Draw the phase curves;

   (c) Identify, if they exist, periodic orbits; homoclinic orbits, and heteroclinic orbits.

5. Identify, if they exist, periodic orbits; homoclinic orbits, and heteroclinic orbits of each ODE:

$$\text{(a)}x''(t) = x(t)^3 - x(t); \quad \text{(b)}x''(t) = 4x(t)^3 - 4x(t)^2 - 8x(t).$$

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Exercises / Problems 4

1. Find a fundamental matrix solution for the linear ordinary differential equation

$$x'(t) = Ax(t),$$

where

(a) $A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$;

(b) $A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$;

(c) $A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$;

(d) $A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$;

(e) $A = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$;

(f) $A = \begin{bmatrix} 3 & 4 \\ 0 & -2 \end{bmatrix}$;

(g) $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$;

(h) $A = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$;

(i) $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

(j) $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$;

(k) $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$;

2. For each system in exercise 1, find the stable, unstable and the centre subspaces, E^s, E^u, and E^c.

3. For each system in exercise 1, draw its phase curve.

4. Let $x'(t) = Ax(t)$ a linear differential equation such that $dimE^s \neq 0$, $dimE^u \neq 1$, and $dimE^c = 0$. Show that, if $x_0 \notin (E^s \cup E^u)$, then the solution $x(t) = x(t, x_0)$ satisfies

$$\lim_{t \to \pm\infty} \|x(t)\| = +\infty.$$

5. Classify the equilibrium points, as sinks, sources, or saddles, of each nonlinear ODE's:

(a) $\begin{cases} x'(t) = -x(t)^2 - y(t)^2 - 1 \\ y'(t) = 2y(t) \end{cases}$;

(b) $\begin{cases} x'(t) = -2x(t) - x(t)y(t) \\ y'(t) = y(t) + x(t)^3 \end{cases}$;

(c) $\begin{cases} x'(t) = -2x(t) - 2x(t)y(t) \\ y'(t) = 2y(t) - x(t) + y(t)^2 \end{cases}$;

(d) $\begin{cases} x'(t) = -x(t) \\ y'(t) = -y(t) + x(t)^2 \\ z'(t) = z(t) + x(t)^2 \end{cases}$;

6. Study the stability of the equilibrium points of the ODE

$$\begin{cases} x'(t) = y(t) \\ y'(t) = x(t) + \alpha y(t) - x(t)^2 \end{cases},$$

where $\alpha \in \mathbb{R}$.