Universidade do Minho
Departamento de Matemática
Lic. Ciências da Computação

Análise

Considere a função $f\colon \mathbb{R}^3 \to \mathbb{R}$ definida por $f(x,y,z) = x^2y + x + zy$, onde as variáveis x, y, z são dependentes de variáveis u e v por

$$x = v\text{sen}\, u, \quad y = u + v, \quad z = uv.$$

Considere a função composta

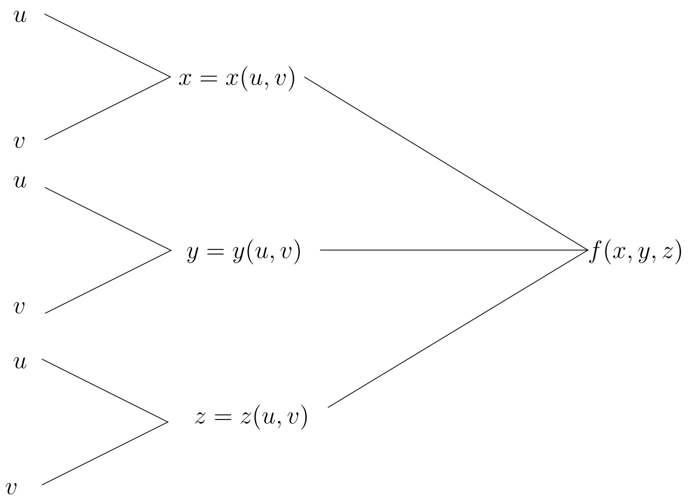
$$\begin{array}{rccl} h: & \mathbb{R}^2 & \to & \mathbb{R} \\ & (u,v) & \mapsto & f(x(u,v), y(u,v), z(u,v)) \end{array}$$

e calcule, usando a regra da cadeia:

(a) as derivadas parciais de h de $1^{\underline{a}}$ ordem;

(b) $\dfrac{\partial^2 h}{\partial u \partial v}(u,v)$.

Para calcular as derivadas parciais de $1^{\underline{a}}$ ordem de h tome atenção na construção em árvore que se segue:



Derivadas parciais de $1^{\underline{a}}$ ordem de h:

$$\begin{aligned} \frac{\partial h}{\partial u}(u,v) &= \frac{\partial x}{\partial u}(u,v) \cdot \frac{\partial f}{\partial x}(x,y,z) + \frac{\partial y}{\partial u}(u,v) \cdot \frac{\partial f}{\partial y}(x,y,z) + \frac{\partial z}{\partial u}(u,v) \cdot \frac{\partial f}{\partial z}(x,y,z) \\ &= v\cos u(2xy+1) + 1 \cdot (x^2 + z) + v.y \\ &= v\cos u(2v(u+v)\text{sen}\, u + 1) + v^2 \text{sen}^2 u + uv + v(u+v); \end{aligned}$$

$$\begin{aligned}
\frac{\partial h}{\partial v}(u,v) &= \frac{\partial x}{\partial v}(u,v)\cdot\frac{\partial f}{\partial x}(x,y,z)+\frac{\partial y}{\partial v}(u,v)\cdot\frac{\partial f}{\partial y}(x,y,z)+\frac{\partial z}{\partial v}(u,v)\cdot\frac{\partial f}{\partial z}(x,y,z)\\
&= \operatorname{sen} u(2xy+1)+1\cdot(x^2+z)+u.y\\
&= \operatorname{sen} u(2v(u+v)\operatorname{sen} u+1)+v^2\operatorname{sen}^2 u+uv+u(u+v).
\end{aligned}$$

Alínea **(b)**:

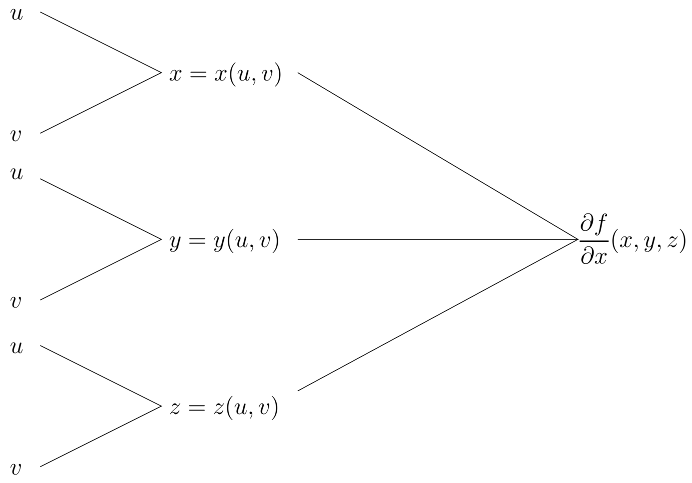
$$\begin{aligned}
\frac{\partial^2 h}{\partial u\partial v}(u,v) &= \frac{\partial}{\partial v}\left(\frac{\partial h}{\partial v}\right)\\
&= \frac{\partial}{\partial u}\left(\frac{\partial x}{\partial v}\cdot\frac{\partial f}{\partial x}+\frac{\partial y}{\partial v}\cdot\frac{\partial f}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial f}{\partial z}\right)\\
&= \underbrace{\frac{\partial}{\partial u}\left(\frac{\partial x}{\partial v}\cdot\frac{\partial f}{\partial x}\right)}_{(1)}+\underbrace{\frac{\partial}{\partial u}\left(\frac{\partial y}{\partial v}\cdot\frac{\partial f}{\partial y}\right)}_{(2)}+\underbrace{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\cdot\frac{\partial f}{\partial z}\right)}_{(3)}\\
&= (1)+(2)+(3)
\end{aligned}$$

Vamos agora calcular cada uma das 3 parcelas,[(1), (2), (3)], separadamente:

(1) Derivando o produto:

$$\begin{aligned}
\frac{\partial}{\partial u}\left(\frac{\partial x}{\partial v}\cdot\frac{\partial f}{\partial x}\right) &= \frac{\partial^2 x}{\partial u\partial v}\cdot\frac{\partial f}{\partial x}+\frac{\partial x}{\partial v}\cdot\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right)\\
&= \cos u(2xy+1)+\operatorname{sen} u\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right)\\
&= (1.a)\ \text{continua adiante}
\end{aligned}$$

Para calcular $\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right)$ é necessário usar novamente a regra da cadeia, agora com a seguinte árvore:



$$\begin{aligned}
\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial x}{\partial u}\cdot\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)+\frac{\partial y}{\partial u}\cdot\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)+\frac{\partial z}{\partial u}\cdot\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)\\
&= \frac{\partial x}{\partial u}\cdot\frac{\partial^2 f}{\partial x^2}+\frac{\partial y}{\partial u}\cdot\frac{\partial^2 f}{\partial y\partial x}+\frac{\partial z}{\partial u}\cdot\frac{\partial^2 f}{\partial z\partial x}\\
&= v\cos u\cdot(2y)+1\cdot(2x)+v\cdot 0
\end{aligned}$$

Consequentemente, substituindo em (1.a), obtém-se

$$\frac{\partial}{\partial u}\left(\frac{\partial x}{\partial v}\cdot\frac{\partial f}{\partial x}\right)=\cos u(2v\operatorname{sen} u(u+v)+1)+\operatorname{sen} u(2v\cos u(u+v)+2v\operatorname{sen} u)\quad(1)$$

(2) Derivando o produto:

$$\begin{aligned}
\frac{\partial}{\partial u}\left(\frac{\partial y}{\partial v}\cdot\frac{\partial f}{\partial y}\right) &= \frac{\partial^2 y}{\partial u\partial v}\cdot\frac{\partial f}{\partial y}+\frac{\partial y}{\partial v}\cdot\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right)\\
&= 0\cdot\frac{\partial f}{\partial y}+1\cdot\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right)\\
&= (2.a)\text{ continua adiante}
\end{aligned}$$

Para calcular $\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right)$ é necessário usar novamente a regra da cadeia. Aqui observe a árvore anterior, colocando $\frac{\partial f}{\partial y}$ no lugar de $\frac{\partial f}{\partial x}$.

$$\begin{aligned}
\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial x}{\partial u}\cdot\frac{\partial^2 f}{\partial x\partial y}+\frac{\partial y}{\partial u}\cdot\frac{\partial^2 f}{\partial y^2}+\frac{\partial z}{\partial u}\cdot\frac{\partial^2 f}{\partial z\partial y}\\
&= v\cos u\cdot(2x)+1\cdot 0+v\cdot 1
\end{aligned}$$

Consequentemente, substituindo em (2.a), obtém-se

$$\frac{\partial}{\partial u}\left(\frac{\partial y}{\partial v}\cdot\frac{\partial f}{\partial y}\right) = v^2 2\cos u \,\text{sen}\, u + v \tag{2}$$

(3) Derivando o produto:

$$\begin{aligned}
\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\cdot\frac{\partial f}{\partial z}\right) &= \frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial f}{\partial z} + \frac{\partial z}{\partial v}\cdot\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial z}\right) \\
&= 1\cdot y + u\cdot\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial z}\right) \\
&= (3.a) \text{ continua adiante}
\end{aligned}$$

Para calcular $\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial z}\right)$ é necessário usar novamente a regra da cadeia. Aqui observe a árvore acima com $\frac{\partial f}{\partial z}$ no lugar de $\frac{\partial f}{\partial x}$.

$$\begin{aligned}
\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial z}\right) &= \frac{\partial x}{\partial u}\cdot\frac{\partial^2 f}{\partial x\partial z} + \frac{\partial y}{\partial u}\cdot\frac{\partial^2 f}{\partial y\partial z} + \frac{\partial z}{\partial u}\cdot\frac{\partial^2 f}{\partial z^2} \\
&= v\cos u\cdot 0 + 1\cdot 1 + v\cdot 0
\end{aligned}$$

Consequentemente, substituindo em (3.a), obtém-se

$$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\cdot\frac{\partial f}{\partial z}\right) = 2u + v \tag{3}$$

Determinadas as derivadas (1), (2) e (3), finalmente obtém-se

$$\frac{\partial^2 h}{\partial u\partial v}(u,v) = 2v(2u+3v)\text{sen}\, u\cos u + \cos u + 2v\,\text{sen}^2 u + 2(u+v)$$