Convergence of asymptotic systems of non-autonomous Hopfield neural network models with infinite distributed delays

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Neural Network Models

*Pioneer Models:

- Cohen-Grossberg (1983)

$$x_i'(t) = -d_i(x_i(t))\left(b_i(x_i(t)) - \sum_{j=1}^{n} a_{ij}h_j(x_j(t))\right), \quad i = 1, \dots, n. \qquad (1)$$

- Hopfield (1984)

$$x_i'(t) = -b_i(x_i(t)) + \sum_{j=1}^{n} a_{ij}h_j(x_j(t)), \qquad i = 1, \dots, n. \qquad (2)$$

where, $n \in \mathbb{N}$ is the number of neurons;
d_i amplification functions; b_i controller functions;
h_j activation functions; $A = [a_{ij}]$ connection matrix.

*Some Hopfield neural network models in the literature:

- Xiao & Zhang (2007) and Yuan et al.(2008 and 2009)

$$x_i'(t) = -\beta_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij1}(t)h_j(x_j(t)) + \sum_{j=1}^{n} a_{ij2}(t)h_j(x_j(t-\tau_{ij}(t))) + I_i(t), \quad (3)$$

- Zhou et al. (2008)

$$x_i'(t) = -\beta_i(t)g_i(x_i(t)) + \sum_{j=1}^{n} a_{ij1}(t)h_j(x_j(t)) + \sum_{j=1}^{n} a_{ij2}(t)h_j(x_j(t-\tau_{ij}(t))) + I_i(t), \quad (4)$$

In both cases: $t \geq 0$ and $i = 1, \dots, n$, with $n \in \mathbb{N}$.

- Zhao (2004)

$$x_i'(t) = -\beta_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)h_j(x_j(t)) + \sum_{j=1}^{n} c_{ij}(t)f_j\left(\sigma_j \int_{-\infty}^{0} G_{ij}(-s)x_j(t+s)ds\right) + I_i(t), \quad (5)$$

- Zhu & Feng (2014)

$$x_i'(t) = -b_i(x_i(t)) + \sum_{j=1}^{n} a_{ij1}(t)h_{j1}(x_j(t)) + \sum_{j=1}^{n} a_{ij2}(t)h_{j2}(x_j(t-\tau_{ij}(t))) + \sum_{j=1}^{n} c_{ij}(t)\int_{-\infty}^{0} G_{ij}(-s)g_j(x_j(t+s))ds + I_i(t), \quad (6)$$

where $G_{ij} : [0,\infty) \to [0,\infty)$ are piecewise continuous and integrable such that

$$\int_0^{\infty} G_{ij}(u)du = 1 \text{ and } \int_0^{\infty} uG_{ij}(u)du < +\infty$$

*Generalized Hopfield neural network model

$$x_i'(t) = -b_i(t, x_i(t)) + \sum_{j=1}^{n}\sum_{p=1}^{P}\Bigg(a_{ijp}(t)h_{ijp}(x_j(t-\tau_{ijp}(t))) + c_{ijp}(t)f_{ijp}\left(\int_{-\infty}^{0} g_{ijp}(x_j(t+s))d\eta_{ijp}(s)\right)\Bigg) + I_i(t),\ t \geq 0, \quad (7)$$

where $n, P \in \mathbb{N}$ and, for $i, j = 1, \dots, n$, $p = 1, \dots, P$,
$b_i : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, $a_{ijp}, c_{ijp}, I_i : [0,\infty) \to \mathbb{R}$,
$h_{ijp}, f_{ijp}, g_{ijp} : \mathbb{R} \to \mathbb{R}$, and $\tau_{ijp} : [0,\infty) \to [0,\infty)$ are continuous,
$\eta_{ijp} : (-\infty, 0] \to \mathbb{R}$ are non-decreasing bounded and normalized

$$\eta_{ijp}(0) - \eta_{ijp}(-\infty) = 1.$$

*Initial Condition

$$x_{t_0} = \varphi, \qquad t_0 \geq 0,\ \varphi \in BC \quad (8)$$

where
$BC := \{\varphi \in C((-\infty, 0]; \mathbb{R}^n) : \varphi \text{ is bounded}\}$, $\|\varphi\| = \sup_{s\leq 0} |\varphi(s)|$.

- Consider a FDE with unbounded delay

$$x'(t) = f(t, x_t), \;\; t \geq 0$$

where $x_t : (-\infty, 0] \to \mathbb{R}^n$ is defined by $x_t(s) = x(t+s)$ for $s \leq 0$.

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- The admissible phase space [Hale and Kato (1978)]

$$UC_g = \left\{ \phi \in C((-\infty, 0]; \mathbb{R}^n) : \sup_{s \leq 0} \frac{|\phi(s)|}{g(s)} < \infty, \frac{\phi(s)}{g(s)} \text{ unif. cont.} \right\},$$

$$\|\phi\|_g = \sup_{s \leq 0} \frac{|\phi(s)|}{g(s)} \text{ with } |x| = |(x_1, \dots, x_n)| = \max_{1 \leq i \leq n} |x_i|$$

where:

(g1) $g : (-\infty, 0] \to [1, \infty)$ non-increasing, continuous, $g(0) = 1$;

(g2) $\displaystyle\lim_{u \to 0^-} \frac{g(s+u)}{g(s)} = 1$ uniformly on $(-\infty, 0]$;

(g3) $g(s) \to \infty$ as $s \to -\infty$.

- From T.Faria & J.J.Oliveira (2011), we have the following:
  Lemma: If, for some $\alpha > 0$,

$$\int_{-\infty}^{0} d\eta_{ijp}(s) < \alpha, \quad \forall i, j, p,$$

then there is a sequence $0 < r_m \nearrow \infty$ such that the function $g : (-\infty, 0] \to [1, \infty)$ defined by

(i) $g(s) = 1$ on $[-r_1, 0]$;
(ii) $g(-r_m) = m$, $m \in \mathbb{N}$;
(iii) g is continuous and piecewise linear (linear on intervals $[-r_{m+1}, -r_m]$),

satisfies **(g1)**, **(g2)**, **(g3)**, and

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- We consider IVP (7)-(8) in the phase space UC_g.
  Note that $BC \subseteq UC_g$.

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For each $i, j = 1, \ldots, n$, $p = 1, \ldots, P$

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- **(A1)** $\exists\ \beta_i : [0, \infty) \to (0, \infty)$, $\forall u, v \in \mathbb{R}\ u \neq v$:

$$(b_i(t, u) - b_i(t, v))/(u - v) \geq \beta_i(t), \qquad \forall t \geq 0;$$

  [In particular, for $b_i(t, u) = \beta_i(t)u$.]

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- **(A2)** $h_{ijp}, f_{ijp}, g_{ijp} : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with Lipschitz constants γ_{ijp}, μ_{ijp}, and σ_{ijp}, respectively;

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- **(A3)** $\lim\limits_{t\to\infty} \left(t - \tau_{ijp}(t)\right) = \infty$;

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- **(A2)** $h_{ijp}, f_{ijp}, g_{ijp} : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with Lipschitz constants γ_{ijp}, μ_{ijp}, and σ_{ijp}, respectively;
- **(A3)** $\lim\limits_{t\to\infty} \left(t - \tau_{ijp}(t)\right) = \infty$;
- **(A4)** There is $(d_1, \dots, d_n) > 0$ such that
$$\limsup_{t\to+\infty} \left(-\beta_i(t) + \sum_{j=1}^{n} \sum_{p=1}^{P} \frac{d_j}{d_i} (\gamma_{ijp}|a_{ijp}(t)| + \mu_{ijp}\sigma_{ijp}|c_{ijp}(t)|) \right) < 0.$$

- The scalar ODE's

$$x'(t) = -x(t) + \frac{2+t}{(t+1)^2}x^2(t), \qquad t \geq 0, \tag{9}$$

and

$$x'(t) = -x(t) + 0x^2(t), \qquad t \geq 0. \tag{10}$$

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- We say that the equation (10) is an *asymptotic equation* of (9) because

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- Note: The dynamic behavior of systems (9), (10) are totally different because:
  - ODE (9) has an unbounded solution, $x(t) = t + 1$;
  - the zero solution of (10) is globally exponentially stable.

The system

$$x_i'(t) = -\hat{b}_i(t, x_i(t)) + \sum_{j=1}^{n} \sum_{p=1}^{P} \Bigg(\hat{a}_{ijp}(t) h_{ijp}(x_j(t - \hat{\tau}_{ijp}(t))) + \\ + \hat{c}_{ijp}(t) f_{ijp} \left(\int_{-\infty}^{0} g_{ijp}(x_j(t+s)) d\eta_{ijp}(s) \right) \Bigg) + \hat{I}_i(t), \quad (11)$$

is an *asymptotic system* of (7) if
$\hat{b}_i(t,u)$, $\hat{a}_{ijp}(t)$, $\hat{c}_{ijp}(t)$, $\hat{\tau}_{ijp}(t)$, and $\hat{I}_i(t)$ are continuous such that $\hat{b}_i$ satisfies (A1) for some non-negative function $\hat{\beta}_i$ and

$$\begin{aligned} \lim_{t\to\infty} (\beta_i(t) - \hat{\beta}_i(t)) &= \lim_{t\to\infty} (b_i(t,u(t)) - \hat{b}_i(t,u(t))) = \lim_{t\to\infty} (a_{ijp}(t) - \hat{a}_{ijp}(t)) \\ &= \lim_{t\to\infty} (c_{ijp}(t) - \hat{c}_{ijp}(t)) = \lim_{t\to\infty} (\tau_{ijp}(t) - \hat{\tau}_{ijp}(t)) \\ &= \lim_{t\to\infty} (I_i(t) - \hat{I}_i(t)) = 0, \end{aligned}$$

for every bounded continuous function $u : \mathbb{R} \to \mathbb{R}$.

- **Idea:**

  To understand the behavior of the Hopfield neural model (7) by studying one of its asymptotic systems (11).

- **Lemma:** Assume (A1) and (A2).
  Then, each solution $x(t) = x(t, t_0, \varphi)$ of (7) is defined on $\mathbb{R}$.
  (where $t_0 \geq 0$ and $\varphi \in BC$)

  Proof: (omitted)
  - Generalized Gronwall's inequality
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  - Generalized Gronwall's inequality
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- The solutions of (11) with bounded initial conditions are also defined on $\mathbb{R}$.

Global convergence of asymptotic Hopfield systems

- Bounded coefficient functions
  (important to do applications, today)

- Unbounded coefficient functions

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- **Example:** The model

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  has an unbounded solution $x(t) = t + 1$ and the hypotheses (A1)-(A4) hold.
- We note that the coefficient functions are unbounded.

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Then all solutions of (7) with initial bounded condition are bounded.

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Then all solutions of (7) with initial bounded condition are bounded.
- **Proof**(idea): By (B), there is $M > 0$ such that

$$M \geq |b_i(t,0)| + |I_i(t)| + \sum_{j=1}^{n} \sum_{p=1}^{P} \Big(|a_{ijp}(t)|\,|h_{ijp}(0)| + |c_{ijp}(t)|\,|f_{ijp}(g_{ijp}(0))| \Big).$$

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- By contradiction, assume that $x(t,t_0,\varphi) = x(t) = (x_1(t),\dots,x_n(t))$ is unbounded and define $z(t) = (d_1^{-1}|x_1(t)|,\dots,d_n^{-1}|x_n(t)|)$.

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- By contradiction, assume that $x(t,t_0,\varphi) = x(t) = (x_1(t),\dots,x_n(t))$ is <u>unbounded</u> and define $z(t) = (d_1^{-1}|x_1(t)|,\dots,d_n^{-1}|x_n(t)|)$.
- Thus, for some i, there is a positive sequence $(t_k)_{k\in\mathbb{N}}$ such that some $t_k \nearrow \infty$, $0 < z_i(t_k) \nearrow \infty$,

$$z_i(t_k) = \|z_{t_k}\| \geq \|z_t\|, \quad \text{and} \quad z_i'(t_k) \geq 0, \quad \forall k\in\mathbb{N},\ \forall t \leq t_k.$$

For each $k \in \mathbb{N}$, we have

$$\begin{aligned}
z_i'(t_k) &= \operatorname{sign}(x_i(t_k)) d_i^{-1} x_i'(t_k) \\
&= -d_i^{-1} \operatorname{sign}(x_i(t_k)) \Big(b_i(t_k, x_i(t_k)) - b_i(t_k, 0) \Big) \\
&\quad + \operatorname{sign}(x_i(t_k)) d_i^{-1} \Big(-b_i(t_k, 0) + I_i(t_k) \Big) \\
&\quad + \operatorname{sign}(x_i(t_k)) d_i^{-1} \sum_{j=1}^{n} \sum_{p=1}^{P} \Big[a_{ijp}(t_k) \big(h_{ijp}(x_j(t_k - \tau_{ijp}(t_k))) - h_{ijp}(0) \big) \\
&\quad + c_{ijp}(t_k) \left(f_{ijp} \left(\int_{-\infty}^{0} g_{ijp}(x_j(t_k + s)) d\eta_{ijp}(s) \right) - f_{ijp}(g_{ijp}(0)) \right) \Big] \\
&\quad + \operatorname{sign}(x_i(t_k)) d_i^{-1} \sum_{j=1}^{n} \sum_{p=1}^{P} \Big(a_{ijp}(t_k) h_{ijp}(0) + c_{ijp}(t_k) f_{ijp}(g_{ijp}(0)) \Big).
\end{aligned}$$

- From (A1) and (A2) we obtain

$$
\begin{aligned}
z_i'(t_k) &\leq -\beta_i(t_k)z_i(t_k) + \sum_{j=1}^{n}\sum_{p=1}^{P}\frac{d_j}{d_i}\Bigg(|a_{ijp}(t_k)|\gamma_{ijp}z_j(t_k-\tau_{ijp}(t_k)) \\
&\quad + |c_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp}\|z_{j,t_k}\|\Bigg) + d_i^{-1}M \\
&\leq -\beta_i(t_k)z_i(t_k) + \sum_{j=1}^{n}\sum_{p=1}^{P}\frac{d_j}{d_i}\Bigg(|a_{ijp}(t_k)|\gamma_{ijp} + |c_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp}\Bigg)\|z_{t_k}\| + d_i^{-1}M \\
&\leq \Bigg(-\beta_i(t_k) + \sum_{j=1}^{n}\sum_{p=1}^{P}\frac{d_j}{d_i}\Bigg(|a_{ijp}(t_k)|\gamma_{ijp} + |c_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp}\Bigg)\Bigg)\|z_{t_k}\| + d_i^{-1}M
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&\quad +|c_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp}\|z_{j,t_k}\|\Big) + d_i^{-1}M \\
&\leq -\beta_i(t_k)z_i(t_k) + \sum_{j=1}^{n}\sum_{p=1}^{P}\frac{d_j}{d_i}\left(|a_{ijp}(t_k)|\gamma_{ijp} + |c_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp}\right)\|z_{t_k}\| + d_i^{-1}M \\
&\leq \left(-\beta_i(t_k) + \sum_{j=1}^{n}\sum_{p=1}^{P}\frac{d_j}{d_i}\left(|a_{ijp}(t_k)|\gamma_{ijp} + |c_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp}\right)\right)\|z_{t_k}\| + d_i^{-1}M
\end{aligned}$$

- Thus, from (A4) we have, for some $l < 0$,

$$z_i'(t_k) \leq l z_i(t_k) + d_i^{-1}M \to -\infty, \text{ as } k \to \infty,$$

which is a contradiction.

- **Theorem 1:** Assume (A1)-(A4) and (B). Then

$$\lim_{t\to\infty}|x_i(t)-\hat{x}_i(t)|=0, \qquad \forall i=1,\dots,n,$$

for all $x(t)=(x_1(t),\dots,x_n(t))$ solutions of (7) and $\hat{x}(t)=(\hat{x}_1(t),\dots,\hat{x}_n(t))$ solution of (11).

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- **Proof**(idea):
  - Let $x(t)$ and $\hat{x}(t)$ solutions of (7) and (11) respectively, with bounded initial conditions. Define

$$y(t) = (d_1^{-1}|x_1(t) - \hat{x}_1(t)|, \dots, d_n^{-1}|x_n(t) - \hat{x}_n(t)|).$$

  - The function $y(t)$ is bounded and define $\bar{y} := \sup_{t\in\mathbb{R}} |y(t)|$,

$$u_i := \limsup_{t\to\infty} y_i(t),\ \forall i, \qquad \text{and} \qquad u := \max_i\{u_i\} \in [0, \infty)$$

  - It remains to be proven that $u = 0$.

- Let $i \in \{1, \dots, n\}$ be such that $u_i = u$.
- Then, there is a positive sequence $t_k \nearrow \infty$ such that

$$y_i(t_k) \to u, \quad \text{and} \quad y_i'(t_k) \to 0, \quad \text{as} \quad k \to \infty. \tag{12}$$

- For the sake of contradiction, assume that $u > 0$.

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- For the sake of contradiction, assume that $u > 0$.
- Fix $0 < \delta < u$ and let $T = T(\delta) > 0$ such that $|y(t)| < u_\delta := u + \delta$ for $t \geq T$ and

$$\int_{-\infty}^{-T} d\eta_{ijp}(s) < \frac{\delta}{\bar{y}}, \quad \forall j, p.$$

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- As $t - \tau_{ijp}(t) \to \infty$, $\tau_{ijp}(t) - \hat{\tau}_{ijp}(t) \to 0$ as $t \to \infty$, and $y_i(t_k) \to u$ as $k \to \infty$,
then there is $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$,

$$t_k - \hat{\tau}_{ijp}(t_k) > T, \; t_k > 2T, \text{ and } y_i(t_k) > u_{-\delta} := u - \delta.$$

For $k > k_0$, from the hypotheses (A1) and (A2) we have

$$\begin{aligned}
y_i'(t_k) &= \operatorname{sign}(x_i(t_k) - \hat{x}_i(t_k)) d_i^{-1}(x_i'(t_k) - \hat{x}_i'(t_k)) = \cdots \\
&\leq -\hat{\beta}_i(t_k) y_i(t_k) + \sum_{j=1}^{n} \sum_{p=1}^{P} \frac{d_j}{d_i} \Bigg[|\hat{a}_{ijp}(t_k)| \gamma_{ijp} y_j(t_k - \hat{\tau}_{ijp}(t_k)) \\
&\quad + |\hat{c}_{ijp}(t_k)| \mu_{ijp} \sigma_{ijp} \int_{-\infty}^{0} y_j(t_k + s) d\eta_{ijp}(s) \Bigg] + \varepsilon_i(t_k),
\end{aligned}$$

where

$$\begin{aligned}
\varepsilon_i(t) &:= d_i^{-1} |b_i(t, x_i(t)) - \hat{b}_i(t, x_i(t))| \\
&\quad + \sum_{j=1}^{n} \sum_{p=1}^{P} d_i^{-1} \Bigg[|a_{ijp}(t) - \hat{a}_{ijp}(t)| \, |h_{ijp}(x_j(t - \tau_{ijp}(t)))| + \\
&\quad + |\hat{a}_{ijp}(t)| \gamma_{ijp} |x_j(t - \tau_{ijp}(t)) - x_j(t - \hat{\tau}_{ijp}(t))| + \\
&\quad + |c_{ijp}(t) - \hat{c}_{ijp}(t)| \left| f_{ijp}\left(\int_{-\infty}^{0} g_{ijp}(x_j(t+s)) d\eta_{ijp}(s) \right) \right| \Bigg] + d_i^{-1} |I_i(t) - \hat{I}_i(t)|.
\end{aligned}$$

As (11) is an asymptotic system of (7), then

$$\lim_{t \to \infty} \varepsilon_i(t) = 0.$$

Now, we have

$$\begin{aligned}
y_i'(t_k) \quad &\leq \quad -\hat{\beta}_i(t_k)y_i(t_k) + \sum_{j=1}^{n}\sum_{p=1}^{P}\frac{d_j}{d_i}\Bigg[|\hat{a}_{ijp}(t_k)|\gamma_{ijp}y_j(t_k - \hat{\tau}_{ijp}(t_k)) \\
&\quad +|\hat{c}_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp}\int_{-\infty}^{0} y_j(t_k+s)d\eta_{ijp}(s)\Bigg] + \varepsilon_i(t_k) \\
&\leq \quad \varepsilon_i(t_k) - \hat{\beta}_i(t_k)u_{-\delta} + \sum_{j=1}^{n}\sum_{p=1}^{P}\frac{d_j}{d_i}\Bigg[|\hat{a}_{ijp}(t_k)|\gamma_{ijp}u_\delta \\
&\quad +|\hat{c}_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp}\left(\int_{-\infty}^{-T} y_j(t_k+s)d\eta_{ijp}(s) + \int_{-T}^{0} y_j(t_k+s)d\eta_{ijp}(s)\right)\Bigg] \\
&\leq \quad \sum_{j=1}^{n}\sum_{p=1}^{P}\frac{d_j}{d_i}\Bigg[|\hat{a}_{ijp}(t_k)|\gamma_{ijp}u_\delta + |\hat{c}_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp}\left(\delta + u_\delta\int_{-T}^{0} d\eta_{ijp}(s)\right)\Bigg] \\
&\quad -\hat{\beta}_i(t_k)u_{-\delta} + \varepsilon_i(t_k) \\
&\leq \quad -\hat{\beta}_i(t_k)u_{-\delta} + \sum_{j=1}^{n}\sum_{p=1}^{P}\frac{d_j}{d_i}\Big(|\hat{a}_{ijp}(t_k)|\gamma_{ijp} + |\hat{c}_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp}\Big)u_{2\delta} + \varepsilon_i(t_k).
\end{aligned}$$

For $k > k_0$, we have

$$y_i'(t_k) \leq -\hat{\beta}_i(t_k)u_{-\delta} + \sum_{j=1}^{n}\sum_{p=1}^{P}\frac{d_j}{d_i}\left(|\hat{a}_{ijp}(t_k)|\gamma_{ijp} + |\hat{c}_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp}\right)u_{2\delta} + \varepsilon_i(t_k)$$

Letting $k \to \infty$ and $\delta \to 0$, we have $y_i'(t_k) \to 0$, thus (A4) implies

$$0 \leq \left(\limsup_{k\to+\infty}\left[-\hat{\beta}_i(t_k) + \sum_{j=1}^{n}\sum_{p=1}^{P}\frac{d_j}{d_i}\left(|\hat{a}_{ijp}(t_k)|\gamma_{ijp} + |\hat{c}_{ijp}(t_k)|\mu_{ijp}\sigma_{ijp}\right)\right]\right)u < 0,$$

which is a contradiction.
Consequently $u = 0$ and the proof is concluded.

- *Example 1*:
  Xiao & Zhang (2007), Zhou et al.(2008), Yuan et al.(2009)

$$x_i'(t) = -\beta_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij1}(t)h_j(x_j(t)) + \sum_{j=1}^{n} a_{ij2}(t)h_j(x_j(t-\tau_{ij}(t))) + I_i(t) \quad (3)$$

$$x_i'(t) = -\hat{\beta}_i(t)x_i(t) + \sum_{j=1}^{n} \hat{a}_{ij1}(t)h_j(x_j(t)) + \sum_{j=1}^{n} \hat{a}_{ij2}(t)h_j(x_j(t-\tau_{ij}(t))) + \hat{I}_i(t) \quad (13)$$

with $\lim\limits_{t\to\infty}(\beta_i(t) - \hat{\beta}_i(t)) = \lim\limits_{t\to\infty}(a_{ijp}(t) - \hat{a}_{ijp}(t)) = \lim\limits_{t\to\infty}(I_i(t) - \hat{I}_i(t)) = 0.$

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- **Corollary 1:** Assume (A2), (A3), (B), and $d=(d_1,\dots,d_n)>0$ such that

$$\limsup_{t\to+\infty}\left(-\beta_i(t) + \sum_{j=1}^{n}\frac{d_j}{d_i}\gamma_j(|a_{ij1}(t)| + a_{ij2}(t)|)\right) < 0 \qquad \forall i=1,\dots,n. \quad (14)$$

  Then

$$\lim_{t\to\infty}|x_i(t)-\hat{x}_i(t)| = 0, \qquad \forall i=1,\dots,n.$$

To obtain the global convergence of the models:

- In Xiao & Zhang (2007) assume that (3) has a periodic asymptotic system, i.e. model (13) is periodic, and for some $\eta > 0$,

$$-\beta_i(t) + \sum_{j=1}^{n} \frac{d_j}{d_i}\gamma_j(|a_{ij1}(t)| + |a_{ij2}(t)|) < -\eta, \qquad \forall i = 1, \dots, n;$$

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- In Zhou et al.(2008) assume the same hypotheses as in Xiao & Zhang (2007), but with $b_i(t,x) = \beta_i(t)g_i(x)$, where g_i satisfies (A1), instead of $b_i(t,x) = \beta_i(t)x$;

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- In Yuan et al.(2009), instead of (14), assume

$$\limsup_{t\to\infty}\left(\sum_{j=1}^{n} \frac{d_j\gamma_i(|a_{ji1}(t)| + a_{ji2}(t)|)}{d_i\beta_j(t)}\right) < 0 \qquad \forall i = 1, \dots, n \qquad (15)$$

with $\liminf_{t\to\infty} \beta_i(t) > 0$.
Conditions (15) and (14) in Corollary 1 are different.

- *Numerical example*:

$$\left\{\begin{array}{l} x_1'(t) = -(2+e^{-t})x_1(t) + (\cos e^t)x_1(t-1) + (\sin e^t)x_2(t-2) + e^{-t} \\ \\ x_2'(t) = -3x_2(t) + (\cos e^t)x_1(t-1) + 2(\sin e^t)x_2(t-2) + e^{-t} \end{array}\right. . \quad (16)$$

It is straightforward to check that the system

$$\left\{\begin{array}{l} x_1'(t) = -2x_1(t) + (\cos e^t)x_1(t-1) + (\sin e^t)x_2(t-2) \\ \\ x_2'(t) = -3x_2(t) + (\cos e^t)x_1(t-1) + 2(\sin e^t)x_2(t-2) \end{array}\right. \quad (17)$$

is an asymptotic system of (16).

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- Model (17) has the equilibrium solution $(x_1(t), x_2(t)) \equiv (0, 0)$;
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- Condition (14) holds, but condition (15) does not hold.

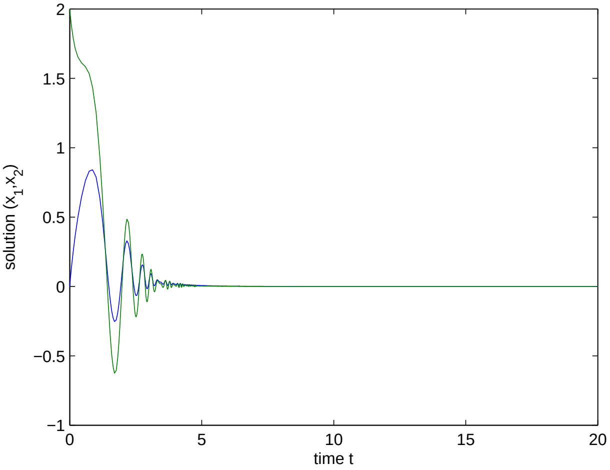

Figure: Solution $(x_1(t), x_2(t))$ of system (16) with initial condition $\varphi(s) = (\sin s, 2)$, $s < 0$.

- *Example 2*: Zhao (2004)

$$x_i'(t) = -\beta_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)h_j(x_j(t)) + \sum_{j=1}^{n} c_{ij}(t)f_j\left(\sigma_j\int_{-\infty}^{0} G_{ij}(-s)x_j(t+s)ds\right) + I_i(t) \qquad (5)$$

$$x_i'(t) = -\hat{\beta}_i(t)x_i(t) + \sum_{j=1}^{n} \hat{a}_{ij}(t)h_j(x_j(t)) + \sum_{j=1}^{n} \hat{c}_{ij}(t)f_j\left(\sigma_j\int_{-\infty}^{0} G_{ij}(-s)x_j(t+s)ds\right) + \hat{I}_i(t) \qquad (18)$$

with $\lim_{t\to\infty}(\beta_i(t) - \hat{\beta}_i(t)) = \lim_{t\to\infty}(a_{ij}(t) - \hat{a}_{ij}(t)) = \lim_{t\to\infty}(c_{ij}(t) - \hat{c}_{ij}(t)) = \lim_{t\to\infty}(I_i(t) - \hat{I}_i(t)) = 0.$

- **Theorem [Zhao (2004)]** For each $i, j = 1, \dots, n$
  (i) $\hat{\beta}_i, \hat{a}_{ij}, \hat{c}_{ij}, \hat{I}_i : [0, \infty) \to \mathbb{R}$ are continuous almost periodic and
  $$\underline{\hat{\beta}_i} = \inf_{t \geq 0} \hat{\beta}_i(t) > 0;$$
  (ii) $h_j, f_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with Lipschitz constants γ_j and μ_j respectively;
  (iii) $G_{ij} : [0, +\infty) \to [0, +\infty)$ is piecewise continuous and integrable with $\int_0^\infty G_{ij}(u)du = 1$.
  (iv) $\exists\ d = (d_1, \dots, d_n) > 0$ such that,
  $$-\underline{\hat{\beta}_i} d_i + \sum_{j=1}^{n} d_j \left(\gamma_j \overline{\hat{a}_{ij}} + \mu_j \sigma_j \overline{\hat{c}_{ij}}\right) < 0, \qquad \forall i \in \{1, \dots, n\}, \qquad (19)$$
  where $\overline{\hat{a}_{ij}} = \sup_{t \geq 0} |\hat{a}_{ij}(t)|$ and $\overline{\hat{c}_{ij}} = \sup_{t \geq 0} |\hat{c}_{ij}(t)|$.

  Then the system (18) has an almost periodic solution, $\bar{x}(t)$.

- With the same hypotheses Zhao(2004) obtained

$$\lim_{t\to\infty} |x_i(t) - \bar{x}_i(t)| = 0, \qquad \forall i = 1, \dots, n,$$

for all $x(t) = (x_1(t), \dots, x_n(t))$ solutions of (18).

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$$\lim_{t\to\infty} |x_i(t) - \bar{x}_i(t)| = 0, \qquad \forall i = 1, \dots, n,$$

for all $x(t) = (x_1(t), \dots, x_n(t))$ solutions of (18).

- From **Theorem 1**,
Corollary 2: Assume (i)-(iv), and $\beta_i, a_{ij}, c_{ij}, I_i : [0, +\infty) \to \mathbb{R}$, are continuous functions such that
$\lim_{t\to\infty} (\beta_i(t) - \hat{\beta}_i(t)) = \lim_{t\to\infty} (a_{ij}(t) - \hat{a}_{ij}(t)) = \lim_{t\to\infty} (c_{ij}(t) - \hat{c}_{ij}(t)) = \lim_{t\to\infty} (I_i(t) - \hat{I}_i(t)) = 0.$
Then

$$\lim_{t\to\infty} |x_i(t) - \bar{x}_i(t)| = 0, \qquad \forall i = 1, \dots, n$$

for all $x(t) = (x_1(t), \dots, x_n(t))$ solutions of (5) and $\bar{x}(t)$ the almost periodic solution (18).

- Here, we do not assume (B), but we consider $\tau_{ijp}(t) = \hat{\tau}_{ijp}(t)$.

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- **Theorem:** Assume (A1), (A2), and (A4).
  If (7) has a bounded solution, then all solutions of (7) and (11), with initial bounded conditions, are bounded.

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  If (7) has a bounded solution, then all solutions of (7) and (11), with initial bounded conditions, are bounded.
- **Theorem 2:** Assume (A1)-(A4).
  If (7) has a bounded solution, then

$$\lim_{t\to\infty} |x_i(t) - \hat{x}_i(t)| = 0, \qquad \forall i = 1, \dots, n.$$

  for all $x(t) = (x_1(t), \dots, x_n(t))$ and $\hat{x}(t) = (\hat{x}_1(t), \dots, \hat{x}_n(t))$ solutions of systems (7) and (11) respectively, with bounded initial conditions.

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Thank you