Exponential stability of nonautonomous neural network models with unbounded distributed delays

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Neural Network Models

*Pioneer Models:

- Cohen-Grossberg (1983)

$$x_i'(t) = -a_i(x_i(t))\left(b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij} f_j(x_j(t))\right), \quad i = 1, \dots, n. \qquad (1)$$

- Hopfield (1984)

$$x_i'(t) = -b_i(x_i(t)) + \sum_{j=1}^{n} c_{ij} f_j(x_j(t)), \qquad i = 1, \dots, n. \qquad (2)$$

where
a_i amplification functions; b_i controller functions;
f_j activation functions; $C = [c_{ij}]$ conection matrix.

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$$x_i'(t) = -a_i(x_i(t))\left[b_i(t, x_i(t)) + \sum_{j=1}^{n} f_{ij}(t, x_{j,t})\right], \qquad t \geq 0 \tag{3}$$

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* **Phase Space:** For a convenient $\varepsilon > 0$,

$$UC_\varepsilon^n = \left\{ \phi \in C((-\infty, 0]; \mathbb{R}^n) : \sup_{s \leq 0} \frac{|\phi(s)|}{e^{-\varepsilon s}} < \infty, \frac{\phi(s)}{e^{-\varepsilon s}} \text{ unif. cont.} \right\},$$

$$\|\phi\|_\varepsilon = \sup_{s \leq 0} \frac{|\phi(s)|}{e^{-\varepsilon s}} \text{ with } |x| = |(x_1, \ldots, x_n)| = \max_{1 \leq i \leq n} |x_i|$$

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- $f_{ij} : [0, +\infty) \times UC_\varepsilon^1 \to \mathbb{R}$ are continuous functions.

* **Initial Condition**

$$x_0 = \varphi, \qquad \varphi \in BC_\varepsilon \tag{4}$$

where $BC_\varepsilon := \{\varphi \in C((-\infty, 0]; \mathbb{R}^n) : \varphi \text{ is bounded}\} \leq UC_\varepsilon$.

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* **Definition**
The model (3) is said *globally exponentially stable* if there are $\delta > 0$ and $M \geq 1$ such that,

$$|x(t, 0, \varphi_1) - x(t, 0, \varphi_2)| \leq Me^{-\delta t}\|\varphi_1 - \varphi_2\|_\infty,$$

for all $t \geq 0$, $\varphi_1, \varphi_2 \in BC_\varepsilon$.

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- **(A2)** $\exists\ \beta_i : [0, +\infty) \to (0, +\infty)$, $\forall u, v \in \mathbb{R}\ u \neq v$:

$$(b_i(t, u) - b_i(t, v))/(u - v) \geq \beta_i(t), \qquad \forall t \geq 0;$$

[In particular, for $b_i(t, u) = \beta_i(t)u$.]

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- **(A3)** $\exists \varepsilon > 0$, $\exists\ l_{ij} : [0, +\infty) \to (0, +\infty)$

$$|f_{ij}(t, \varphi) - f_{ij}(t, \psi)| \leq l_{ij}(t)||\varphi - \psi||_\varepsilon, \qquad \forall t \geq 0,\ \forall \varphi, \psi \in UC^1_\varepsilon;$$

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- **(A3)** $\exists \varepsilon > 0,\ \exists\ l_{ij} : [0, +\infty) \to (0, +\infty)$
$$|f_{ij}(t,\varphi) - f_{ij}(t,\psi)| \leq l_{ij}(t)||\varphi - \psi||_\varepsilon, \qquad \forall t \geq 0,\ \forall \varphi, \psi \in UC^1_\varepsilon;$$
- **(A4)** There exists $\lambda : \mathbb{R} \to (0, +\infty)$ a continuous function:
$$\underline{\rho}_i \beta_i(t) - e^{\int_0^t \lambda(s) - \varepsilon ds} \sum_{j=1}^{n} \overline{\rho}_j l_{ij}(t) > \lambda(t) \text{ and } \int_0^t \lambda(s)\, ds \geq \varepsilon t,\ \ \forall t \geq 0$$

Lemma: Each solution $x(t) = x(t, 0, \varphi)$ of (3) is defined on $\mathbb{R}$.

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- Assume $\alpha \in (0, +\infty)$.
- For $z(t) = (z_1(t), \dots, z_n(t)) := (\overline{\rho}_1^{-1}|x_1(t)|, \dots, \overline{\rho}_n^{-1}|x_n(t)|)$, there are $i \in \{1, \dots, n\}$ and $(t_k)_k \nearrow \alpha$ such that, $\forall k \in \mathbb{N}$,

$$z_i(t_k) \nearrow +\infty, \;\; z_i(t_k) \geq ||z_{t_k}||_\varepsilon > 0, \;\text{ and }\; z_i'(t_k) \geq 0.$$

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- $z_i'(t_k) = \overline{\rho}_i^{-1} \mathrm{sign}(x_i(t_k)) x_i'(t_k)$

$$= -\overline{\rho}_i^{-1} \mathrm{sign}(x_i(t_k)) a_i(x_i(t_k)) \Bigg[\Big(b_i(t_k, x_i(t_k)) - b_i(t_k, 0) \Big) + \sum_{j=1}^{n} \Big(f_{ij}(t_k, x_{j,t_k}) - f_{ij}(t_k, 0) \Big) + \Big(b_i(t_k, 0) + \sum_{j=1}^{n} f_{ij}(t_k, 0) \Big) \Bigg]$$

- By **(A2)**, **(A3)**, and **(A4)** we obtain

$$\begin{aligned} z_i'(t_k) &\leq -a_i(x_i(t_k))\left[\beta_i(t_k)z_i(t_k) - \left(\sum_{j=1}^{n}\frac{\overline{\rho}_j}{\overline{\rho}_i}l_{ij}(t_k)\right)\|z_{t_k}\|_\varepsilon - C_i\right] \\ &\leq -a_i(x_i(t_k))\left[\left(\beta_i(t_k) - \sum_{j=1}^{n}\frac{\overline{\rho}_j}{\overline{\rho}_i}l_{ij}(t_k)\right)z_i(t_k) - C_i\right] \\ &\leq -\underline{\rho}_i\left(\frac{\lambda(t_k)}{\overline{\rho}_i}z_i(t_k) - C_i\right) < 0, \text{ for large } k, \end{aligned}$$

where $C_i = \max_{t\in[0,\alpha]} \overline{\rho}_i^{-1}\left|b_i(t,0) + \sum_{j=1}^{n} f_{ij}(t,0)\right| \in \mathbb{R}$.

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- Contradiction, then $\alpha = +\infty$.

Global exponential stability

Theorem: Assume **(A1)-(A4)**
Then the system (3) is globally exponentially stable.

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  $x(t) = x(t,0,\varphi)$ and $y(t) = y(t,0,\psi)$ solutions, with $\varphi, \psi \in BC_\varepsilon$.
  Define, for $t \geq 0$, $\mathcal{V}(t) = (\mathcal{V}_1(t), \dots, \mathcal{V}_n(t))$ by

$$\mathcal{V}_i(t) = e^{\int_0^t \lambda(s)ds} \mathrm{sign}(x_i(t) - y_i(t)) \int_{y_i(t)}^{x_i(t)} \frac{1}{a_i(s)} ds.$$

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- By **(A1)**, $|x_i(t) - y_i(t)| \leq \overline{\rho}_i \mathcal{V}_i(t) e^{-\int_0^t \lambda(s)ds}$
- and $|x_i(t) - y_i(t)| \geq \underline{\rho}_i \mathcal{V}_i(t) e^{-\int_0^t \lambda(s)ds}$

- If $|\mathcal{V}(t)| \leq \max_j \left\{ \underline{\rho}_j^{-1} \right\} \|\varphi - \psi\|$, for all $t \geq 0$, then

$$|x(t)-y(t)| \min_j \left\{ \overline{\rho}_j^{-1} \right\} e^{\int_0^t \lambda(s)ds} \leq |\mathcal{V}(t)| \leq \max_j \left\{ \underline{\rho}_j^{-1} \right\} \|\varphi-\psi\|,$$

$$|x(t)-y(t)| \leq \frac{\max_j \left\{ \underline{\rho}_j^{-1} \right\}}{\min_j \left\{ \overline{\rho}_j^{-1} \right\}} e^{-\int_0^t \lambda(s)ds} \|\varphi-\psi\| \leq Me^{-\varepsilon t} \|\varphi-\psi\|,$$

with $M := \frac{\max_j \left\{ \underline{\rho}_j^{-1} \right\}}{\min_j \left\{ \overline{\rho}_j^{-1} \right\}}$. (recall $\int_0^t \lambda(s)ds \geq \varepsilon t$)

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- We need to prove: $|\mathcal{V}(t)| \leq \max_j \left\{ \underline{\rho}_j^{-1} \right\} \|\varphi - \psi\|, \forall t \geq 0$.

If not, as

$$\mathcal{V}_i(0) \leq \underline{\rho}_i^{-1} |x_i(0) - y_i(0)| \leq \max_j \left\{ \underline{\rho}_j^{-1} \right\} \|\varphi - \psi\|, \quad \forall i$$

then there is $t_1 > 0$ such that $|\mathcal{V}(t_1)| > \max_j \left\{ \underline{\rho}_j^{-1} \right\} \|\varphi - \psi\|$.

- Defining

$$T := \min\left\{ t \in [0, t_1] : |\mathcal{V}(t)| = \max_{s\in[0,t_1]} |\mathcal{V}(s)| \right\}$$

and choosing $i \in \{1, \dots, n\}$ such that $\mathcal{V}_i(T) = |\mathcal{V}(T)|$, then

$$\mathcal{V}_i(T) > 0, \quad \mathcal{V}_i'(T) \geq 0, \quad \text{and} \quad \mathcal{V}_i(T) > |\mathcal{V}(t)|, \quad \forall t < T.$$

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- By other hand, hypotheses **(A1)-(A3)** imply

$$\begin{aligned} \mathcal{V}_i'(T) &= \lambda(T)\mathcal{V}_i(T) - e^{\int_0^T \lambda(s)ds}\operatorname{sign}(x_i(T) - y_i(T)) \cdot \\ &\quad \left(\frac{1}{a_i(x_i(T))}x_i'(T) - \frac{1}{a_i(y_i(T))}y_i'(T)\right) \end{aligned}$$

From hypotheses **(A1)-(A3)**

$$\begin{aligned}\mathcal{V}_i'(T) &\leq \lambda(T)\mathcal{V}_i(T) - e^{\int_0^T \lambda(s)ds}\beta_i(T)|x_i(T) - y_i(T)| + \\ &\quad e^{\int_0^T \lambda(s)ds}\sum_{j=1}^n l_{ij}(T)\|x_{j,T} - y_{j,T}\|_\varepsilon \\ &\leq \lambda(T)\mathcal{V}_i(T) - \underline{\rho}_i\beta_i(T)\mathcal{V}_i(T) + e^{\int_0^T \lambda(s)ds}\sum_{j=1}^n l_{ij}(T)\overline{\rho}_j \cdot \\ &\quad \cdot \max\left\{\frac{\|\varphi - \psi\|}{\overline{\rho}_j e^{\varepsilon T}}, \sup_{-T<s\leq 0}\frac{\mathcal{V}_j(T+s)}{e^{\varepsilon T}}\right\}\end{aligned}$$

then, **(A4)** implies

$$\mathcal{V}_i'(T) \leq \left(\lambda(T) - \underline{\rho}_i\beta_i(T) + e^{\int_0^T \lambda(s)-\varepsilon ds}\sum_{j=1}^n \overline{\rho}_j l_{ij}(T)\right)\mathcal{V}_i(T) < 0,$$

which is a contradiction.

From hypotheses **(A1)-(A3)**

$$\begin{aligned}\mathcal{V}_i'(T) &\leq \lambda(T)\mathcal{V}_i(T) - e^{\int_0^T \lambda(s)ds}\beta_i(T)|x_i(T) - y_i(T)| + \\ &\quad e^{\int_0^T \lambda(s)ds}\sum_{j=1}^{n} l_{ij}(T)\|x_{j,T} - y_{j,T}\|_\varepsilon \\ &\leq \lambda(T)\mathcal{V}_i(T) - \underline{\rho}_i\beta_i(T)\mathcal{V}_i(T) + e^{\int_0^T \lambda(s)ds}\sum_{j=1}^{n} l_{ij}(T)\overline{\rho}_j \cdot \\ &\quad \cdot \max\left\{\frac{\|\varphi - \psi\|}{\overline{\rho}_j e^{\varepsilon T}}, \sup_{-T<s\leq 0}\frac{\mathcal{V}_j(T+s)}{e^{\varepsilon T}}\right\}\end{aligned}$$

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(A4*)

$$\underline{\rho}_i\beta_i(t) - e^{\int_0^t \lambda(s)ds} \sum_{j=1}^{n} \overline{\rho}_j l_{ij}(t) > \lambda(t) \text{ and } \int_0^t \lambda(s)\,ds \geq \varepsilon t,\ \forall t \geq 0.$$

Then the system (3) is globally exponentially stable.

Finite delay

Corollary 2: Assume **(A1)**, **(A2)** and

- **($\mathbf{A}_f$3)** For $f_{ij} : [0, +\infty) \times C_{ij} \to \mathbb{R}$, $\exists l_{ij} : [0, +\infty) \to (0, +\infty)$

$$|f_{ij}(t, \varphi) - f_{ij}(t, \psi)| \le l_{ij}(t)||\varphi - \psi||, \quad \forall t \ge 0, \ \forall \varphi, \psi \in C_{ij},$$

where $C_{ij} = C([-\tau_{ij}, 0]; \mathbb{R})$ with $\tau_{ij} > 0$, $\tau = \max_{ij} \tau_{ij}$;

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  $$|f_{ij}(t,\varphi)-f_{ij}(t,\psi)|\leq l_{ij}(t)||\varphi-\psi||,\qquad \forall t\geq 0,\ \forall\varphi,\psi\in C_{ij},$$

  where $C_{ij}=C([-\tau_{ij},0];\mathbb{R})$ with $\tau_{ij}>0$, $\tau=\max_{ij}\tau_{ij}$;

- **(A_f4)** There exists $\lambda:\mathbb{R}\to[-\tau,+\infty)$ a continuous function:

  $$\underline{\rho}_i\beta_i(t)-\sum_{j=1}^{n}e^{\int_{t-\tau_{ij}}^{t}\lambda(s)ds}\overline{\rho}_j l_{ij}(t)>\lambda(t) \text{ and } \int_0^t\lambda(s)\,ds\geq\varepsilon t,\ \forall t\geq 0$$

  then the system (3), is globally exponentially stable.

Corollary 3: Assume **(A1)**, **(A2)** and **(A_f3)**

- If $l_{ij}(t)$ are bounded and there exists $\alpha > 0$:

$$\underline{\rho}_i \beta_i(t) - \sum_{j=1}^{n} \overline{\rho}_j l_{ij}(t) > \alpha, \qquad \forall t \geq 0, \tag{5}$$

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Corollary 3: Assume **(A1)**, **(A2)** and **(A$_f$3)**

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* (Proof) For $l_{ij}(t) < L_{ij}$, from (5), we have

$$\underline{\rho}_i\beta_i(t) - \sum_{j=1}^{n} \overline{\rho}_j l_{ij}(t) \left(1 + \frac{\alpha}{2nL_{ij}\overline{\rho}_j}\right) > \frac{\alpha}{2}.$$

Taking $\varepsilon_{ij}^* = \frac{1}{\tau_{ij}} \log\left(1 + \frac{\alpha}{2nL_{ij}\overline{\rho}_j}\right) > 0$ and $\varepsilon = \min_{ij}\{\frac{\alpha}{2}, \varepsilon_{ij}^*\}$,

$$\underline{\rho}_i\beta_i(t) - \sum_{j=1}^{n} \overline{\rho}_j l_{ij}(t) e^{\varepsilon\tau_{ij}} > \varepsilon.$$

With $\lambda(t) = \varepsilon$, the condition **(A$_f$4)** holds.

In phase space $C_n = C([-\tau, 0], \mathbb{R}^n)$, assume that system

$$x_i'(t) = -a_i(x_i(t))\left[b_i(t, x_i(t)) + \sum_{j=1}^{n} f_{ij}(t, x_{j,t})\right], \qquad t \geq 0 \qquad (3)$$

is ω-periodic, $\omega > 0$, that is:

$$b_i(t, u) = b_i(t + \omega, u), \qquad \forall t \geq 0,\ \forall u \in \mathbb{R};$$

$$f_{ij}(t, \varphi) = f_{ij}(t + \omega, \varphi), \qquad \forall t \geq 0,\ \forall \varphi \in C_{ij}.$$

Theorem: Assume **(A1)**, **(A2)**, **(A$_f$3)** and

$$\underline{\rho}_i \beta_i(t) - \sum_{j=1}^{n} \overline{\rho}_j l_{ij}(t) > 0, \qquad \forall t \in [0, \omega].$$

Then (3) has a ω-periodic solution which is globally exponentially stable.

* (Proof) Show the existence of a periodic solution.
  From previous result

$$\|x_t(\varphi) - x_t(\bar{\varphi})\| \leq e^{-\varepsilon(t-\tau)}\|\varphi - \bar{\varphi}\|, \quad \forall t \geq \tau, \ \forall \varphi, \bar{\varphi} \in C_n.$$

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- Let $k \in \mathbb{N}$ such that $e^{-(k\omega-\tau)} \le \frac{1}{2}$ and define $P : C_n \to C_n$ by $P(\varphi) = x_\omega(\varphi)$.

$$\|P^k(\varphi) - P^k(\bar{\varphi})\| = \|x_{k\omega}(\varphi) - x_{k\omega}(\bar{\varphi})\| \le \frac{1}{2}\|\varphi - \bar{\varphi}\|,$$

P^k is a contraction map on Banach space C_n. Thus, P^k has a unique fixed point $\varphi^* \in C_n$: $P^k(\varphi^*) = \varphi^*$.

* (Proof) Show the existence of a periodic solution.
From previous result

$$\|x_t(\varphi) - x_t(\bar{\varphi})\| \leq e^{-\varepsilon(t-\tau)}\|\varphi - \bar{\varphi}\|, \quad \forall t \geq \tau, \ \forall \varphi, \bar{\varphi} \in C_n.$$

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- As $P^k(P(\varphi^*)) = P(P^k(\varphi^*)) = P(\varphi^*)$, then

$$P(\varphi^*) = \varphi^* \Leftrightarrow x_\omega(\varphi^*) = \varphi^*$$

and $x(t, 0, \varphi^*)$ is the periodic solution of (3).

Hopfield neural network model [1]

$$x_i'(t) = -b_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) + I_i(t) \quad (6)$$

[1] Q. Zhang, X. Wei, J. Xu, *Chaos Solitons & Fractals* 39 (2009) 1152-1157.

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- $b_i, a_{ij}, b_{ij} : [0,+\infty) \to \mathbb{R}$, $\tau_{ij}(t) \geq 0$ are continuous;
- $f_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with constant l_j;

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- $b_i, a_{ij}, b_{ij} : [0, +\infty) \to \mathbb{R}$, $\tau_{ij}(t) \geq 0$ are continuous;
- $f_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with constant l_j;
- $b_i(t) - \sum_{j=1}^{n} l_j \left(|a_{ij}(t)| + |b_{ij}(t)| e^{\int_{t-\tau_{ij}}^{t} \lambda(s)ds} \right) > \lambda(t)$

  and $\int_0^t \lambda(s)ds \geq \varepsilon t$, for some $\varepsilon > 0$ and some function $\lambda(t)$.
  Then system (6) is globally exponentially stable.

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  and $\int_0^t \lambda(s)ds \geq \varepsilon t$, for some $\varepsilon > 0$ and some function $\lambda(t)$.
  Then system (6) is globally exponentially stable.
- In [1], a different set of conditions is assumed to get the same conclusion.

[1] Q. Zhang, X. Wei, J. Xu, *Chaos Solitons & Fractals* 39 (2009) 1152-1157.

For the periodic model:

$$x_i'(t) = -b_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) + I_i(t) \quad (7)$$

[2] M. Tan, Y. Tan, *Appl. Math. Model.* 33 (2009) 373-385.

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$$x_i'(t) = -b_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) + I_i(t) \quad (7)$$

- $b_i, a_{ij}, b_{ij} : [0,+\infty) \to \mathbb{R}$, $\tau_{ij}(t) \geq 0$ are ω-periodic continuous;
- $f_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with constant l_j;

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- $f_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with constant l_j;
- $b_i(t) - \sum_{j=1}^{n} l_j(|a_{ij}(t)| + |b_{ij}(t)|) > 0, \quad \forall t \in [0,\omega].$

  Then (7) has a global exponential stable ω-periodic solution.

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For the periodic model:

$$x_i'(t) = -b_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) + I_i(t) \quad (7)$$

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- $$b_i(t) - \sum_{j=1}^{n} l_j(|a_{ij}(t)| + |b_{ij}(t)|) > 0, \qquad \forall t \in [0,\omega].$$
  Then (7) has a global exponential stable ω-periodic solution.
- In [2] assumed the additional hypothesis

$$b_i(t) - \sum_{i=1}^{n} l_j(|a_{ij}(t)| + |b_{ij}(t)|) > 0, \qquad \forall t \in [0,\omega],$$

[2] M. Tan, Y. Tan, *Appl. Math. Model.* 33 (2009) 373-385.

Cohen-Grossberg BAM neural network model [3]

$$x_i'(t) = -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{m} a_{ij}(t) f_j(y_j(t-\tau_{ij}(t))) - I_i(t)\right]$$

$$y_j'(t) = -c_j(y_j(t))\left[d_j(y_j(t)) - \sum_{i=1}^{n} c_{ji}(t) g_i(x_i(t-\sigma_{ji}(t))) - I_j(t)\right] \qquad (8)$$

$i = 1, \ldots, n$ and $j = 1, \ldots, m$.

Cohen-Grossberg BAM neural network model [3]

$$
\begin{aligned}
x_i'(t) &= -a_i(x_i(t))\left[b_i(x_i(t)) - \sum_{j=1}^{m} a_{ij}(t) f_j(y_j(t-\tau_{ij}(t))) - I_i(t)\right] \\
y_j'(t) &= -c_j(y_j(t))\left[d_j(y_j(t)) - \sum_{i=1}^{n} c_{ji}(t) g_i(x_i(t-\sigma_{ji}(t))) - I_j(t)\right]
\end{aligned}
\qquad (8)
$$

$i = 1, \dots, n$ and $j = 1, \dots, m$.

- $\exists \underline{a}_i, \overline{a}_i, \underline{c}_j > 0, \overline{c}_j, \forall t \in [0, +\infty)$:

$$0 < \underline{a}_i \leq a_i(t) \leq \overline{a}_i, \qquad 0 < \underline{c}_j \leq c_j(t) \leq \overline{c}_j;$$

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\end{aligned}
\tag{8}
$$

$i = 1, \dots, n$ and $j = 1, \dots, m$.

- $\exists \underline{a}_i, \overline{a}_i, \underline{c}_j > 0, \overline{c}_j, \forall t \in [0, +\infty)$:

$$0 < \underline{a}_i \leq a_i(t) \leq \overline{a}_i, \qquad 0 < \underline{c}_j \leq c_j(t) \leq \overline{c}_j;$$

- $\exists \beta_i, \delta_j > 0, \forall u \in \mathbb{R}$:

$$\frac{b_i(u) - b_i(v)}{u - v} \geq \beta_i, \qquad \frac{d_j(u) - d_j(v)}{u - v} \geq \delta_j;$$

- $f_j, g_i : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with Lipschitz constants F_j and G_i;

[3] K. Liu, Z. Zhang, L. Wang, *Abstract and Applied Analysis* ID805846 (2012).

- $f_j, g_i : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with Lipschitz constants F_j and G_i;
- $\tau_{ij}(t) \geq 0$, $\sigma_{ji}(t) \geq 0$, $a_{ij}(t)$, $c_{ji}(t)$, are ω-periodic continuous.

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- $\tau_{ij}(t) \geq 0$, $\sigma_{ji}(t) \geq 0$, $a_{ij}(t)$, $c_{ji}(t)$, are ω-periodic continuous.
- $$\begin{cases} \underline{a}_i\beta_i - \sum_{j=1}^{m} \overline{c}_j F_j |a_{ij}(t)| > 0, \\ \\ \underline{c}_j\delta_j - \sum_{i=1}^{n} \overline{a}_i G_i |c_{ji}(t)| > 0, \end{cases} \quad \forall t \in [0, \omega]. \tag{9}$$

  Then (8) has a global exponential stable periodic solution.

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- $f_j, g_i : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with Lipschitz constants F_j and G_i;
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-

$$\begin{cases} \underline{a}_i \beta_i - \sum_{j=1}^{m} \overline{c}_j F_j |a_{ij}(t)| > 0, \\ \underline{c}_j \delta_j - \sum_{i=1}^{n} \overline{a}_i G_i |c_{ji}(t)| > 0, \end{cases} \quad \forall t \in [0, \omega]. \tag{9}$$

Then (8) has a global exponential stable periodic solution.

- In [3] a strong hypothesis than (9) is assumed to get the same conclusion.

[3] K. Liu, Z. Zhang, L. Wang, *Abstract and Applied Analysis* ID805846 (2012).

Thank you

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