Global stability of a general neural network model with unbounded delays

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Neural Network Models

*Pioneer Models:

- Cohen-Grossberg (1983)

$$\dot{x}_i(t) = -a_i(x_i(t))\left(b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij} f_j(x_j(t))\right), \quad i = 1, \ldots, n. \qquad (1)$$

- Hopfield (1984)

$$\dot{x}_i(t) = -b_i(x_i(t)) + \sum_{j=1}^{n} c_{ij} f_j(x_j(t)), \qquad i = 1, \ldots, n. \qquad (2)$$

where
a_i amplification functions; b_i controller functions;
f_j activation functions; $C = [c_{ij}]$ conection matrix.

*Neural Network Models with **infinite** time-delay:

- Cohen-Grossberg type model

$$\dot{x}_i(t) = -a_i(x_i(t))\left(b_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{0} k_{ij}(-s) g_j(x_j(t+s))ds\right)$$

- Interval cellular neural network model

$$\dot{x}_i(t) = -b_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} \int_{-\infty}^{0} g_j(x_j(t+s)) d\eta_j(s) \qquad (4)$$

- Bidirectional associative memory neural network model

$$\begin{cases} \dot{x}_i(t) = -a_i(x_i(t))\left(b_i(x_i(t)) + \sum_{j=1}^{m} f_{ij}(y_j(t-\tau_{ij}))\right), i = 1, \dots, n, \\ \dot{y}_j(t) = -d_j(y_j(t))\left(c_j(y_j(t)) + \sum_{i=1}^{n} m_{ji} \int_{-\infty}^{0} k_{ji}(-s) g_{ji}(x_i(t+s))ds\right), j = 1, \dots, m, \end{cases}$$

*General Neural Network Model with infinite time-delay

$$\dot{x}_i(t) = -a_i(x_i(t))[b_i(x_i(t)) + f_i(x_t)], \qquad i = 1, \dots, n \tag{6}$$

where, for $t \geq 0$,

$$x_t(s) = x(t+s), \text{ for } s \leq 0, \text{i.e., } x_t = x_{|_{(-\infty,t]}}.$$

*Initial Condition

$$x_0 = \varphi, \qquad \varphi \in BC \tag{7}$$

where $BC := \{\varphi \in C((-\infty, 0]; \mathbb{R}^n) : \varphi \text{ is bounded}\}$

$$||\varphi||_\infty = \sup_{s \leq 0} |\varphi(s)|$$

***Phase Space** "strong fading memory"

$$UC_g = \left\{ \phi \in C((-\infty, 0]; \mathbb{R}^n) : \sup_{s\le 0} \frac{|\phi(s)|}{g(s)} < \infty, \frac{\phi(s)}{g(s)} \text{ unif. cont.} \right\},$$

$$\|\phi\|_g = \sup_{s\le 0} \frac{|\phi(s)|}{g(s)} \text{ with } |x| = |(x_1, \dots, x_n)| = \max_{1\le i\le n} |x_i|$$

where:

(g1) $g : (-\infty, 0] \to [1, +\infty)$ non-increasing, continuous, $g(0) = 1$;

(g2) $\lim_{u\to 0^-} \frac{g(s+u)}{g(s)} = 1$ uniformly on $(-\infty, 0]$;

(g3) $g(s) \to +\infty$ as $s \to -\infty$.

Example: $g(s) = e^{-\alpha s}$, $s \in (-\infty, 0]$, with $\alpha > 0$

BC_g subspace of bounded continuous functions, BC, equipped with the norm $\|\cdot\|_g$.

- FDE with ∞ delay in UC_g

$$\dot{x}(t) = f(t, x_t), \qquad t \geq 0 \tag{8}$$

$$x_t \in UC_g, \qquad x_t(s) = x(t+s), s \leq 0$$

with $f = (f_1, \ldots, f_n) : [0, +\infty) \times UC_g \to \mathbb{R}^n$ continuous

J.R. Haddock, W. Hornor, Funkcial Ekvac. 31 (1988) 349-361.

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- **Lemma A** [Haddock and Hornor (1988)] If $y : \mathbb{R} \to \mathbb{R}^n$ is such that $y_0 \in UC_g$, $y(t)$ is bounded and uniformly continuous on $[0, +\infty)$, and

$$\frac{|y(s)|}{g(s)} \to 0, \text{ as } s \to -\infty,$$

than the positive orbit $\{y_t : t \geq 0\}$ is precompact in UC_g.

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- **Lemma B**

  Assume that f transforms closed bounded sets of $(-\infty, 0] \times UC_g$ into bounded sets of $\mathbb{R}^n$.
  If
  (H) $\forall t > 0, \forall \varphi \in BC_g$:

  $$\forall s \in (-\infty, 0), |\varphi(s)| < |\varphi(0)| \Rightarrow \varphi_i(0) f_i(t, \varphi) < 0,$$

  for some $i \in \{1, \ldots, n\}$ such that $|\varphi(s)| = |\varphi_i(0)|$,

  then the solution $x(t) = x(t, 0, \varphi)$, $\varphi \in BC_g$, of (8) is defined and bounded on $[0, +\infty)$ and

  $$|x(t, 0, \varphi)| \leq ||\varphi||_\infty.$$

*Proof of Lemma B (idea)

- $x(t) = x(t, 0, \varphi)$ solution on $[-\infty, a)$, $a > 0$, with $\varphi \in BC_g$
  $k := \sup_{s\leq 0} |\varphi(s)|$.

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- $x(t) = x(t, 0, \varphi)$ solution on $[-\infty, a)$, $a > 0$, with $\varphi \in BC_g$
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- Suppose that $|x(t_1)| > k$ for some $t_1 > 0$ and define

$$T = \min\left\{t \in [0, t_1] : |x(t)| = \max_{s \in [0, t_1]} |x(s)|\right\}.$$

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$$T = \min \left\{ t \in [0, t_1] : |x(t)| = \max_{s \in [0, t_1]} |x(s)| \right\}.$$

- We have $|x_T(s)| = |x(T + s)| < |x(T)|$, for $s < 0$.
  By **(H)** we conclude that,

$$x_i(T) f_i(T, x_T) < 0,$$

  for some $i \in \{1, \ldots, n\}$ such that $|x_i(T)| = |x(T)|$. If $x_i(T) > 0$ (analogous if $x_i(T) < 0$), then $\dot{x}_i(T) < 0$.

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- $x_i(t) \le |x(t)| < |x(T)| = x_i(T)$, $t \in [0, T)$,

$$\Rightarrow \dot{x}_i(T) \ge 0.$$

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- $x_i(t) \le |x(t)| < |x(T)| = x_i(T)$, $t \in [0, T)$,

$$\Rightarrow \dot{x}_i(T) \ge 0.$$

- Contradition. Thus $x(t)$ is defined and bounded on $[0, +\infty)$.

- **Lemma B'**
  Assume that f transforms closed bounded sets $(-\infty, 0] \times UC_g$ into bounded sets of $\mathbb{R}^n$.
  If
  (H') $\forall t > 0, \forall \varphi \in UC_g$:

$$\forall s \in (-\infty, 0), \frac{|\varphi(s)|}{g(s)} < |\varphi(0)| \Rightarrow \varphi_i(0) f_i(t, \varphi) < 0,$$

  for some $i \in \{1, \dots, n\}$ such that $|\varphi(s)| = |\varphi_i(0)|$,

  then the solution $x(t) = x(t, 0, \varphi)$, $\varphi \in UC_g$, of (8) is defined and bounded on $[0, +\infty)$ and

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- **Remark: (H')$\Rightarrow$(H)**, and $||\varphi||_g \leq ||\varphi||_\infty$.

Global asymptotic stability

- Consider in UC_g,

$$\dot{x}_i(t) = -a_i(x_i(t))[b_i(x_i(t)) + f_i(x_t)] \qquad (6)$$

where $a_i : \mathbb{R} \to (0, +\infty)$, $b_i : \mathbb{R} \to \mathbb{R}$ and $f_i : UC_g \to \mathbb{R}$ are continuous functions such that

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- **(A1)** $\exists \beta_i > 0, \forall u, v \in \mathbb{R}, u \neq v$:

$$(b_i(u) - b_i(v))/(u - v) \geq \beta_i;$$

[In particular, for $b_i(u) = \beta_i u$.]

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- **(A2)** f_i is a Lipshitz function with Lipshitz constant l_i.

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- **Definition**

A equilibrium $x^* \in \mathbb{R}^n$ is said *global asymptotic stable* if it is stable and

$$x(t, 0, \varphi) \to x^* \text{ as } t \to \infty, \text{ for all } \varphi \in BC_g.$$

- **Theorem 1** (GAS)
  Assume **(A1)** and **(A2)**. If

$$\beta_i > l_i, \qquad \forall i \in \{1, \dots, n\},$$

  then there is a unique equilibrium point of (6), which is globally asymptotically stable.

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  Assume **(A1)** and **(A2)**. If

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  then there is a unique equilibrium point of (6), which is globally asymptotically stable.
- **Proof (idea)**
  Existence and uniqueness of equilibrium point

$$\begin{array}{rccc} H: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & x & \mapsto & (b_i(x_i) + f_i(x))_{i=1}^n \end{array}$$

  is homeomorphism.
  Then there exists $x^* \in \mathbb{R}^n$, $H(x^*) = 0$, i.e. x^* is the equilibrium.

- By translation, we may assume that $x^* = 0$, i.e.,

$$b_i(0) + f_i(0) = 0, \qquad \forall i = 1, \ldots, n$$

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- For $\varphi \in UC_g$ such that $||\varphi||_g = |\varphi(0)| = \varphi_i(0) > 0$ (analogous if $\varphi_i(0) < 0$),

$$\begin{aligned} b_i(\varphi_i(0)) + f_i(\varphi) &= [b_i(\varphi_i(0)) - b_i(0)] + [f_i(\varphi) - f_i(0)] \\ &\geq (\beta_i - l_i)||\varphi||_g > 0. \end{aligned}$$

(H') holds and from Lemma B' we conclude

- $x = 0$ is uniform stable
- all solutions are defined and bounded on $[0, +\infty)$

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  (H') holds and from Lemma B' we conclude
  - $x = 0$ is uniform stable
  - all solutions are defined and bounded on $[0, +\infty)$
- It remains to prove that $x = 0$ is global attractive.

- Let $x(t) = x(t, 0, \varphi)$ a solution of (6), with $\varphi \in BC_g$, and define

$$-v_i = \liminf_{t\to+\infty} x_i(t), \qquad u_i = \limsup_{t\to+\infty} x_i(t)$$

$$v = \max_i\{v_i\}, \qquad u = \max_i\{u_i\},$$

$u, v \in \mathbb{R}$, $-v \leq u$. We have to show that $\max(u, v) = 0$.

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- Suppose $|v| \le u$. ($|u| \le v$ is similar)
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$u, v \in \mathbb{R}$, $-v \le u$. We have to show that $\max(u,v) = 0$.
- Suppose $|v| \le u$. ($|u| \le v$ is similar)
Let $i \in \{1,\ldots,n\}$ such that $u_i = u$.
- By computations, we can show that exists $(t_k)_{k\in\mathbb{N}}$ such that
$$t_k \nearrow +\infty, \quad x_i(t_k) \to u, \quad \text{and} \quad b_i(x_i(t_k)) + f_i(x_{t_k}) \to 0.$$

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- As $x(t)$ is bounded, $\dot{x}(t)$ is also bounded, from Lemma A $\{x_t : t \geq 0\}$ is precompact in UC_g. Then

$$\exists \phi \in UC_g : \qquad x_{t_k} \to \phi \text{ on } UC_g,$$

with $\|\phi\|_g = \phi_i(0) = u$ and $b_i(\phi_i(0)) + f_i(\phi) = 0$.
Then $u = 0$. $\square$

Global exponential stability

- Consider in UC_g, with $g(s) = e^{-\alpha s}$ for some $\alpha > 0$,

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Assume:

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- **Definition**

  A equilibrium $x^* \in \mathbb{R}^n$ is said *global exponential stable* if there are $\varepsilon, M > 0$ such that

$$|x(t, 0, \varphi) - x^*| \leq Me^{-\varepsilon t}||\varphi - x^*||_\infty, \text{ for all } t \geq 0, \varphi \in BC_g.$$

- **Theorem 2** (GES)
  Assume **(A0)**, **(A1)**, and **(A2)**. If

  $$\beta_i > l_i, \qquad \forall i \in \{1, \dots, n\},$$

  then there is a unique equilibrium point of (6), which is globally exponentially stable.

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- **Proof (idea)**
  We may assume $x^* \equiv 0$, i.e. $b_i(0) + f_i(0) = 0$.
  $\beta_i > l_i \Rightarrow \varepsilon - \underline{a}(\beta_i - l_i) < 0$ for some $\varepsilon \in (0, \alpha)$
  Let $x(t, 0, \varphi)$ be a solution of (6) with $\varphi \in BC_g$.

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  $\beta_i > l_i \Rightarrow \varepsilon - \underline{a}(\beta_i - l_i) < 0$ for some $\varepsilon \in (0, \alpha)$
  Let $x(t, 0, \varphi)$ be a solution of (6) with $\varphi \in BC_g$.
- The change of variables

$$z(t) = e^{\varepsilon t} x(t)$$

  transforms (6) into

$$\dot{z}_i(t) = F_i(t, z_t), \qquad i = 1, \dots, n, \tag{9}$$

where

$$F_i(t,\phi) = \varepsilon\phi_i(0) - a_i(e^{-\varepsilon t}\phi_i(0))e^{\varepsilon t}\left[b_i(e^{-\varepsilon t}\phi_i(0)) + f_i(e^{-\varepsilon(t+\cdot)}\phi)\right]$$

- Let $\phi \in BC_g$ such that $|\phi(s)| < |\phi(0)|$, for $s \in (-\infty, 0)$. Consider $i \in \{1, \dots, n\}$ such that $|\phi_i(0)| = |\phi(0)|$.

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$$F_i(t,\phi) = \varepsilon\phi_i(0) - a_i(e^{-\varepsilon t}\phi_i(0))e^{\varepsilon t}\left[b_i(e^{-\varepsilon t}\phi_i(0)) + f_i(e^{-\varepsilon(t+\cdot)}\phi)\right]$$

- Let $\phi \in BC_g$ such that $|\phi(s)| < |\phi(0)|$, for $s \in (-\infty, 0)$. Consider $i \in \{1, \dots, n\}$ such that $|\phi_i(0)| = |\phi(0)|$.
- If $\phi_i(0) > 0$ ($\phi_i(0) < 0$ is analogous)
  From the hypotheses we conclude that

$$\begin{aligned} F_i(t,\phi) &\leq \varepsilon\phi_i(0) - \underline{a}e^{\varepsilon t}\,[b_i(e^{-\varepsilon t}\phi_i(0)) - b_i(0)+ \\ &\quad + f_i(e^{-\varepsilon(t+\cdot)}\phi) - f_i(0)] \\ &\leq \varepsilon\phi_i(0) - \underline{a}\left[\beta_i\phi_i(0) - l_i \sup_{s\leq 0} e^{(\alpha-\varepsilon)s}|\phi(s)|\right] \\ &\leq \phi_i(0)[\varepsilon - \underline{a}(\beta_i - l_i)] < 0. \end{aligned}$$

  Then $F = (F_1, \dots, F_n)$ satisfies **(H)**

where

$$F_i(t,\phi) = \varepsilon\phi_i(0) - a_i(e^{-\varepsilon t}\phi_i(0))e^{\varepsilon t}\left[b_i(e^{-\varepsilon t}\phi_i(0)) + f_i(e^{-\varepsilon(t+\cdot)}\phi)\right]$$

- Let $\phi \in BC_g$ such that $|\phi(s)| < |\phi(0)|$, for $s \in (-\infty, 0)$. Consider $i \in \{1,\ldots,n\}$ such that $|\phi_i(0)| = |\phi(0)|$.
- If $\phi_i(0) > 0$ ($\phi_i(0) < 0$ is analogous)
  From the hypotheses we conclude that

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  Then $F = (F_1,\ldots,F_n)$ satisfies **(H)**
- From Lemma B, $z(t)$ is defined on $[0,+\infty)$ and

$$|x(t,0,\varphi)| = |e^{-\varepsilon t}z(t,0,e^{\varepsilon\cdot}\varphi)| \leq e^{-\varepsilon t}||\varphi||_\infty.$$

Generalized Cohen-Grossberg Model

Cohen-Grossberg model with unbounded distributed delays (10)

$$\dot{x}_i(t) = -a_i(x_i(t))\left[b_i(x_i(t)) + \sum_{j=1}^{n}\sum_{p=1}^{P} f_{ij}^{(p)}\left(\int_{-\infty}^{0} g_{ij}^{(p)}(x_j(t+s))d\eta_{ij}^{(p)}(s)\right)\right] \quad (10$$

- $a_i : \mathbb{R} \to (0, +\infty)$, are continuous functions;

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- **Theorem 3**

  (a) If N is a non-singular M-matrix, then there is a unique equilibrium point of (10), which is globally asymptotically stable.

  (b) Assume, in addition, that a_i satisfy **(A0)** and there is $\gamma > 0$ such that each $\eta_{ij}^{(p)}$ satisfies

$$\exists \gamma > 0 : \quad \int_{-\infty}^{0} e^{-\gamma s} d\eta_{ij}^{(p)} < \infty.$$

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  If N is a non-singular M-matrix, then there is a unique equilibrium point of (10), which is globally exponentially stable.
- $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is a **non-singular M-matrix** if $a_{ij} \leq 0$, $i \neq j$ and Re $\sigma(A) > 0$.

- **Proof of (a)** (idea)
  N non-singular M-matrix $\Rightarrow \exists d = (d_1, \ldots, d_n) > 0$: $Nd > 0$
  $\Rightarrow \exists \delta > 0$:

$$\beta_i > d_i^{-1} \sum_{j=1}^{n} l_{ij}(1+\delta)d_j, \qquad i = 1, \ldots, n; \qquad (11)$$

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- By a technical Lemma, we can find $g : (-\infty, 0] \to [1, +\infty)$ satisfying **(g1)-(g3)** such that

$$\int_{-\infty}^{0} g(s)d\eta_{ij}^{(p)}(s) < 1 + \delta,$$

and we consider UC_g as the phase space of (10).

- The change of variables

$$y_i(t) = d_i^{-1} x_i(t)$$

transforms (10) into

$$\dot{y}_i(t) = -\bar{a}_i(y_i(t))\left[\bar{b}_i(y_i(t)) + \bar{f}_i(y_t)\right], \tag{12}$$

where

$$\bar{f}_i(\phi) = d_i^{-1} \sum_{j=1}^{n} \sum_{p=1}^{P} f_{ij}^{(p)} \left(\int_{-\infty}^{0} g_{ij}^{(p)}(d_j \phi_j(s)) d\eta_{ij}^{(p)}(s) \right), \phi \in UC_g$$

$$\bar{b}_i(u) = d_i^{-1} b_i(d_i(u)), \quad \bar{a}_i = a_i(d_i(u)), \quad u \in \mathbb{R}.$$

- After some computations,

$$|\bar{f}_i(\phi)-\bar{f}_i(\psi)| \leq \left(d_i^{-1} \sum_{j=1}^{n} l_{ij}(1+\delta)d_j \right) \|\phi-\psi\|_g, \qquad \phi, \psi \in UC_g,$$

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- The global exponential stability follows in the same way, considering UC_g with $g(s) = e^{-\alpha s}$, for some $\alpha \in (0,\gamma)$, such that

$$\int_{-\infty}^{0} e^{-\alpha s} d\eta_{ij}^{(p)}(s) < 1+\delta.$$

Example

Cohen-Grossberg neural network with unbounded distributed delays

$$\dot{x}_i(t) = -a_i(x_i(t))\left[b_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j\left(\int_{-\infty}^{0} k_{ij}(-s)x_j(t+s)ds\right)\right] \quad (13)$$

- $a_{ij} \in \mathbb{R}$, and $a_i : \mathbb{R} \to (0, +\infty)$, $b_i : \mathbb{R} \to \mathbb{R}$ are continuous functions and **(A1)** holds;

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- **Corollary**(GAS)
  If N is a non-singular M-matrix, then there is a unique equilibrium point of (13), which is globally asymptotically stable.

[1] L. Wang, Appl. Math. Comput. 160 (2005) 93-110.

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  System (10) reduces to (13) if $P = 1$, $f_{ij}^{(1)}(u) = a_{ij} f_j(u)$ and

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- In [1] assumed the additional conditions:
  f_j is bounded; $0 < \underline{a}_i \leq a_i(u) \leq \overline{a}_i$;
  The kernels functions satisfy $\int_0^\infty t k_{ij}(t) dt < \infty$
  $\underline{N} := B\underline{A} - \overline{A}[l_{ij}]$ is a non-singular M-matrix, where
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- **Corollary**(GES)
  Assume **(A0)** and that there is $\gamma > 0$ such that

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[2] W. Wu, B.T. Cui, X.Y. Lou, Math. Comput. Modelling, 47 (2008) 868-873.

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Thank you