

Estabilidade exponencial em modelos de redes neuronais com atrasos

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Neural Network Models

*Pioneer Models:

- Cohen-Grossberg (1983)

$$x_i'(t) = -a_i(x_i(t)) \left(b_i(x_i(t)) - \sum_{j=1}^n c_{ij} f_j(x_j(t)) \right), \quad i = 1, \dots, n. \quad (1)$$

- Hopfield (1984)

$$x_i'(t) = -b_i(x_i(t)) + \sum_{j=1}^n c_{ij} f_j(x_j(t)), \quad i = 1, \dots, n. \quad (2)$$

where

a_i amplification functions; b_i controller functions;
 f_j activation functions; $C = [c_{ij}]$ connection matrix.

* **Nonautonomous system of delay differential equations**

$$x_i'(t) = -a_i(t, x_t) [b_i(t, x_i(t)) + f_i(t, x_t)], \quad t \geq 0, i = 1, \dots, n \quad (3)$$

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* **Phase Space:** $C_n := C([-\tau, 0]; \mathbb{R}^n)$ for $\tau > 0$,

$$\|\phi\| = \sup_{s \in [-\tau, 0]} |\phi(s)| \quad \text{with } |x| = |(x_1, \dots, x_n)| = \max_{1 \leq i \leq n} |x_i|$$

$$x_t \in C_n, \quad x_t(s) = x(t + s), \quad s \in [-\tau, 0]$$

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$$x_t \in C_n, \quad x_t(s) = x(t + s), \quad s \in [-\tau, 0]$$

- ▶ $a_i : [0, +\infty) \times C_n \rightarrow (0, +\infty)$ are continuous functions;
- ▶ $b_i : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions;
- ▶ $f_i : [0, +\infty) \times C_n \rightarrow \mathbb{R}$ are continuous functions.

* **Initial Condition**

$$x_0 = \bar{\varphi}, \quad \bar{\varphi} \in C_n \quad (4)$$

* **Definition**

The solution $x(t, 0, \bar{\varphi})$ of (3)-(4) is said *globally exponentially stable* if $\exists \delta > 0$ and $M \geq 1$

$$|x(t, 0, \varphi) - x(t, 0, \bar{\varphi})| \leq Me^{-\delta t} \|\varphi - \bar{\varphi}\|,$$

for all $t \geq 0$, $\varphi \in C_n$.

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The model (3) is said *globally exponentially stable* if $\exists \delta > 0$ and $M \geq 1$

$$|x(t, 0, \varphi_1) - x(t, 0, \varphi_2)| \leq Me^{-\delta t} \|\varphi_1 - \varphi_2\|,$$

for all $t \geq 0$, $\varphi_1, \varphi_2 \in C_n$.

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- ▶ **(A1)** $\exists x^* = (x_1^*, \dots, x_n^*) \in \mathbb{R}^n$ a equilibrium point of (3);
- ▶ **(A2)** $\exists \underline{\rho}_i > 0$:

$$0 < \underline{\rho}_i \leq \inf\{a_i(t, \varphi) : t \geq 0, \varphi \in C_n\};$$

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- ▶ **(A3)** $\exists \beta_i : [0, +\infty) \rightarrow (0, +\infty), \forall u, v \in \mathbb{R} \ u \neq v$:

$$(b_i(t, u) - b_i(t, v))/(u - v) \geq \beta_i(t), \quad \forall t \geq 0;$$

[In particular, for $b_i(t, u) = \beta_i(t)u$.]

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- ▶ **(A4)** $\exists l_i : [0, +\infty) \rightarrow (0, +\infty)$

$$|f_i(t, \varphi) - f_i(t, \psi)| \leq l_i(t) \|\varphi - \psi\|, \quad \forall t \geq 0, \forall \varphi, \psi \in C_n;$$

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- ▶ **(A5)** $\exists \varepsilon > 0$ and $\lambda : \mathbb{R} \rightarrow (0, +\infty)$ a continuous function:

$$\underline{\rho}_i \left(\beta_i(t) - l_i(t) e^{\int_{t-\tau}^t \lambda(s) ds} \right) > \lambda(t) \text{ and } \int_0^t \lambda(s) ds \geq \varepsilon t, \forall t \geq 0.$$

- In C_n , consider the FDE

$$x'(t) = f(t, x_t), \quad t \geq 0 \quad (5)$$

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* **Lemma:[1]** If

(H) $\forall t > 0, \forall \varphi \in C_n$:

$$\forall s \in [-\tau, 0), |\varphi(s)| < |\varphi(0)| \Rightarrow \varphi_i(0)f_i(t, \varphi) < 0,$$

for some $i \in \{1, \dots, n\}$ such that $|\varphi(s)| = |\varphi_i(0)|$,

then the solution $x(t) = x(t, 0, \varphi)$, $\varphi \in C_n$, of (5) is defined and bounded on $[-\tau, +\infty)$ and

$$|x(t, 0, \varphi)| \leq \|\varphi\|.$$

*Proof of Lemma (idea)

- ▶ $x(t) = x(t, 0, \varphi)$ solution on $[-\tau, \alpha)$, $\alpha > 0$, with $\varphi \in C_n$
- ▶ Suppose that $|x(t_1)| > \|\varphi\|$ for some $t_1 > 0$ and define

$$T = \min \left\{ t \in [0, t_1] : |x(t)| = \max_{s \in [0, t_1]} |x(s)| \right\}.$$

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- ▶ We have $|x_T(s)| = |x(T + s)| < |x(T)|$, for $s \in [-\tau, 0)$.
By **(H)** we conclude that,

$$x_i(T)f_i(T, x_T) < 0,$$

for some $i \in \{1, \dots, n\}$ such that $|x_i(T)| = |x(T)|$. If $x_i(T) > 0$ (analogous if $x_i(T) < 0$), then $x'_i(T) < 0$.

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- ▶ $x_i(t) \leq |x(t)| < |x(T)| = x_i(T)$, $t \in [0, T)$,

$$\Rightarrow x'_i(T) \geq 0.$$

- ▶ Contradiction. Thus $x(t)$ is defined and bounded on $[0, +\infty)$.

Global exponential stability

Theorem 1: Assume **(A1)-(A5)**

Then the equilibrium of (3) is globally exponentially stable.

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* Proof of Theorem (idea)

- The change of variables $z(t) = e^{\int_0^t \lambda(u) du} x(t)$ transforms (3) into

$$z'_i(t) = g_i(t, z_t), \quad t \geq 0 \quad (6)$$

with

$$g_i(t, \varphi) = \lambda(t)\varphi_i(0) - a_i(t, \psi(t))e^{\int_0^t \lambda(u) du} [b_i(t, \psi(t); (0)) + f_i(t, \psi(t))]$$

$$\text{and } \psi(t)(s) = e^{-\int_0^{t+s} \lambda(u) du} \varphi(s), \quad s \in [-\tau, 0]$$

- $\psi(t) \in C_n$

- Take $t \geq 0$ and $\varphi \in C_n$ such that

$$|\varphi(s)| < |\varphi(0)| = \|\varphi\| = \varphi_i(0) > 0, \quad \forall s \in [-\tau, 0).$$

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- From the hypotheses and assuming $x^* = 0$

$$\begin{aligned} g_i(t, \varphi) &= \lambda(t)\varphi_i(0) - a_i(t, \psi(t))e^{\int_0^t \lambda(u) du} [b_i(t, \psi(t)_i(0)) - b_i(t, 0) \\ &\quad + f_i(t, \psi(t)) - f_i(t, 0)] \\ &\leq \lambda(t)\varphi_i(0) - a_i(t, \psi(t))e^{\int_0^t \lambda(u) du} \cdot \left[\beta_i(t)e^{-\int_0^t \lambda(u) du} \varphi_i(0) - l_i(t)e^{-\int_0^{t-\tau} \lambda(u) du} \|\varphi\| \right] \\ &\leq \varphi_i(0) \left[\lambda(t) - \underline{\rho}_i \left(\beta_i(t) - l_i(t)e^{\int_{t-\tau}^t \lambda(u) du} \right) \right] < 0 \end{aligned}$$

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- Then **(H)** holds and, from the Lemma,

$$\begin{aligned} |x(t, 0, \varphi)| &= \left| e^{-\int_0^t \lambda(u) du} z(t, 0, \psi(0)) \right| \\ &\leq e^{-\varepsilon t} \left| z \left(t, 0, e^{-\int_0^t \lambda(u) du} \varphi \right) \right| \leq e^{-\varepsilon t} \|\varphi\|. \end{aligned}$$

Corollary 1: Assume $a_i(t, \varphi) = 1$ and **(A3)**-**(A5)**

Then the system

$$x_i'(t) = -b_i(t, x_i(t)) + f_i(t, x_t), \quad t \geq 0, \quad (7)$$

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- Let $\bar{x}(t) = x(t, 0, \bar{\varphi})$. The change of variables $z(t) = x(t) - \bar{x}(t)$ transforms (7) into

$$z_i'(t) = -\bar{b}_i(t, z_i(t)) + \bar{f}_i(t, z_t), \quad t \geq 0 \quad (8)$$

with

$$\bar{b}_i(t, u) = b_i(t, u + \bar{x}_i(t)) \text{ and } \bar{f}_i(t, \varphi) = f_i(t, \varphi + \bar{x}_t) + b_i(t, \bar{x}_i(t)) - f_i(t, \bar{x}_t)$$

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- Zero is an equilibrium of (8) and from the Theorem 1

$$|x(t) - \bar{x}(t)| = |z(t)| \leq e^{-\varepsilon t} \|z_0\| = e^{-\varepsilon t} \|\varphi - \bar{\varphi}\|, \quad \forall t \geq 0.$$

Corollary 2: Assume **(A3)** and **(A4)**

- ▶ If $l_i(t)$ are bounded and there exists $\alpha > 0$:

$$\beta_i(t) - l_i(t) > \alpha, \quad \forall t \geq 0, \quad (9)$$

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- * (Proof) For $l_i(t) < L_i$, from (9), we have

$$\beta_i(t) - l_i(t) \left(1 + \frac{\alpha}{2L_i}\right) > \frac{\alpha}{2}.$$

Taking $\varepsilon_i^* = \frac{1}{\tau} \log \left(1 + \frac{\alpha}{2L_i}\right) > 0$ and $\varepsilon = \min_i \left\{\frac{\alpha}{2}, \varepsilon_i^*\right\}$,

$$\beta_i(t) - l_i(t)e^{\varepsilon\tau} > \varepsilon.$$

With $\lambda(t) = \varepsilon$, the condition **(A5)** holds.

Periodic systems

Assume that the system

$$x_i'(t) = -b_i(t, x_i(t)) + f_i(t, x_t), \quad t \geq 0 \quad (7)$$

is ω -periodic, $\omega > 0$, that is:

$$b_i(t, u) = b_i(t + \omega, u), \quad \forall t \geq 0, \forall u \in \mathbb{R};$$

$$f_i(t, \varphi) = f_i(t + \omega, \varphi), \quad \forall t \geq 0, \forall \varphi \in C_n.$$

Theorem 2: Assume **(A3)**, **(A4)**, and there exists $\alpha > 0$:

$$\beta_i(t) - l_i(t) > \alpha, \quad \forall t \in [0, \omega].$$

Then (7) has a ω -periodic solution which is globally exponentially stable.

- * (Proof) Show the existence of a periodic solution.
From Corollary 2

$$\|x_t(\varphi) - x_t(\bar{\varphi})\| \leq e^{-\varepsilon(t-\tau)} \|\varphi - \bar{\varphi}\|, \quad \forall t \geq \tau, \quad \forall \varphi, \bar{\varphi} \in C_n.$$

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- Let $k \in \mathbb{N}$ such that $e^{-(k\omega-\tau)} \leq \frac{1}{2}$ and define $P : C_n \rightarrow C_n$ by $P(\varphi) = x_{k\omega}(\varphi)$.

$$\|P^k(\varphi) - P^k(\bar{\varphi})\| = \|x_{k\omega}(\varphi) - x_{k\omega}(\bar{\varphi})\| \leq \frac{1}{2} \|\varphi - \bar{\varphi}\|,$$

P^k is a contraction map on Banach space C_n . Thus, P^k has a unique fixed point $\varphi^* \in C_n$: $P^k(\varphi^*) = \varphi^*$.

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P^k is a contraction map on Banach space C_n . Thus, P^k has a unique fixed point $\varphi^* \in C_n$: $P^k(\varphi^*) = \varphi^*$.

- As $P^k(P(\varphi^*)) = P(P^k(\varphi^*)) = P(\varphi^*)$, then

$$P(\varphi^*) = \varphi^* \Leftrightarrow x_\omega(\varphi^*) = \varphi^*$$

and $x(t, 0, \varphi^*)$ is the periodic solution of (7).

Hopfield neural network model [2]

$$x_i'(t) = -b_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^n b_{ij}(t)f_j(x_j(t - \tau_{ij}(t))) + l_i(t) \quad (10)$$

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- ▶ $b_i, a_{ij}, b_{ij}, l_i : [0, +\infty) \rightarrow \mathbb{R}, \tau_{ij}(t) \geq 0$ are continuous;
- ▶ $f_j : \mathbb{R} \rightarrow \mathbb{R}$ are Lipschitz functions with constant l_j ;
- ▶ $b_i(t) - \sum_{j=1}^n l_j \left(|a_{ij}(t)| + |b_{ij}(t)| e^{\int_{t-\tau_{ij}}^t \lambda(s) ds} \right) > \lambda(t), \quad \forall i$

and $\int_0^t \lambda(s) ds \geq \varepsilon t$, for some $\varepsilon > 0$ and some function $\lambda(t)$.
Then system (10) is globally exponentially stable.

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and $\int_0^t \lambda(s) ds \geq \varepsilon t$, for some $\varepsilon > 0$ and some function $\lambda(t)$.
Then system (10) is globally exponentially stable.
- ▶ In [2], a different hypotheses set is assumed to get the same conclusion.

For the periodic model:

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- ▶ $b_i, a_{ij}, b_{ij}, l_i : [0, +\infty) \rightarrow \mathbb{R}$, $\tau_{ij}(t) \geq 0$ are ω -periodic continuous;
- ▶ $f_j : \mathbb{R} \rightarrow \mathbb{R}$ are Lipschitz functions with constant l_j ;
- ▶ $b_i(t) - \sum_{j=1}^n l_j(|a_{ij}(t)| + |b_{ij}(t)|) > 0$, $\forall i, \forall t \in [0, \omega]$.

Then (11) has a global exponential stable ω -periodic solution.

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- ▶ $b_i, a_{ij}, b_{ij}, l_i : [0, +\infty) \rightarrow \mathbb{R}, \tau_{ij}(t) \geq 0$ are ω -periodic continuous;
- ▶ $f_j : \mathbb{R} \rightarrow \mathbb{R}$ are Lipschitz functions with constant l_j ;
- ▶ $b_i(t) - \sum_{j=1}^n l_j(|a_{ij}(t)| + |b_{ij}(t)|) > 0, \quad \forall i, \forall t \in [0, \omega]$.

Then (11) has a global exponential stable ω -periodic solution.

- ▶ In [3] assumed the additional hypothesis

$$b_j(t) - \sum_{i=1}^n l_j(|a_{ij}(t)| + |b_{ij}(t)|) > 0, \quad \forall j, \forall t \in [0, \omega],$$

Thank you

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