Exponential stability for impulsive functional differential equations with infinite delay

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Impulsive ordinary differential equations

- IVP of impulsive ordinary differential equations:

$$\begin{cases} x'(t) = f(t, x(t)), & 0 \leq t \neq t_k \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1, 2, \cdots, \\ x(0) = x_0 \in \mathbb{R}^n, & \end{cases} \tag{1}$$

where

- $\Delta x(t_k) := x(t_k^+) - x(t_k^-)$;
- $f : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $I_k : \mathbb{R}^n \to \mathbb{R}^n$ are cont. functions;
- $(t_k)_{k\in\mathbb{N}} \nearrow +\infty$ as $k \to +\infty$.

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  - $(t_k)_{k \in \mathbb{N}} \nearrow +\infty$ as $k \to +\infty$.
- A function $x : [0, d] \to \mathbb{R}^n$ is a **solution** of (1) if
  - it is continuous on $[0, d] \setminus \{t_k : k \in \mathbb{N}\}$,
  - $x(t_k^-)$ and $x(t_k^+)$ exist with $x(t_k^-) = x(t_k)$
  - satisfies (1).

Impulsive delay differential equations

- Impulsive functional differential equations:

$$\begin{cases} x'(t) = f(t, x_t), & 0 \le t \neq t_k \\ \Delta x(t_k) = I_k(x_{t_k}), & k = 1, 2, \cdots \end{cases} \quad (2)$$

where

- $x_t(s) = x(t+s)$, for $s \in (-\infty, 0]$;
- $f : [0, +\infty) \times \mathcal{PS} \to \mathbb{R}^n$ and $I_k : \mathcal{PS} \to \mathbb{R}^n$ are continuous.

With $\mathcal{PS}$ a convenient Phase Space of functions $\phi : (-\infty, 0] \to \mathbb{R}^n$.

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- We consider Bounded Initial Conditions:

$$x_0 = \phi \in B\mathcal{PS}, \text{(bounded functions on } \mathcal{PS}\text{)}. \quad (3)$$

*Phase Space

- $[\gamma, \beta]$ compact interval of $\mathbb{R}$, $PC([\gamma, \beta]; \mathbb{R}^n)$ space of functions $\phi : [\gamma, \beta] \to \mathbb{R}^n$ continuous except for a finite points s, $\phi(s^-), \phi(s^+)$ exist, and $\phi(s^-) = \phi(s)$;

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- $R([\gamma, \beta]; \mathbb{R}^n) = \overline{PC([\gamma, \beta]; \mathbb{R}^n)}$ on the space of bounded functions with sup norm;
- $PC := PC((-\infty, \beta]; \mathbb{R}^n) = \left\{\phi : (-\infty, 0] \to \mathbb{R}^n | \phi|_{[\gamma,\beta]} \in R([\gamma, \beta]; \mathbb{R}^n), \forall [\gamma, \beta] \subseteq (-\infty, 0]\right\}$;

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- For $\alpha > 0$,

$$PC_\alpha := \left\{\phi \in PC : \sup_{s\leq 0} |\phi(s)| e^{\alpha s} < \infty\right\}$$

$$\|\phi\|_\alpha = \sup_{s\leq 0} |\phi(s)| e^{\alpha s}, \text{ with } |x| = |(x_1, \dots, x_n)| = \max_{1\leq i\leq n} |x_i|$$

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- $\mathcal{PS} = PC_\alpha$

$$BPC_\alpha = \{\phi \in PC_\alpha : \phi \text{ bounded } \}.$$

***Theorem** (Existence of solutions)
If

- $t \to f(t, x_t)$ measurable on $(0, +\infty]$, for $x : \mathbb{R} \to \mathbb{R}^n$ bounded;
- $\exists p, q : [0, +\infty) \to [0, \infty)$ continuous with q non-decreasing, $q(u)u > 0$, and $\int^\infty \frac{1}{q} = \infty$ such that

$$|f(t, \psi)| \leq p(t)q(\|\psi\|), \qquad t \geq 0, \psi \in BPC_\alpha;$$

- $I_k(X)$ is bounded for all $X \subseteq BPC_\alpha$ bounded.

then, the PVI (2)-(3) has a solution $x(t)$ defined on $[0, +\infty)$.

*Impulsive general neural network model

$$\begin{cases} x_i'(t) = -a_i(x_i(t))[b_i(x_i(t)) + f(t, x_t)], & 0 \leq t \neq t_k, \\ \Delta(x_i(t_k)) = I_{ik}(x_i(t_k^-)), \ i = 1, \cdots, n, & k = 1, 2, \cdots \end{cases} \quad (4)$$

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(A1) $a_i : \mathbb{R} \to (0, +\infty)$, continuous and $a_i(u) \geq \underline{a_i} > 0, \forall u$;

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$$\frac{b_i(u) - b_i(v)}{u - v} \ge \beta_i > 0, \qquad \forall u \ne v;$$

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(A5) $|\hat{I}_{ik}(u) - \hat{I}_{ik}(v)| \leq \hat{\gamma}_{ik} |u - v|$ with $\hat{I}_{ik}(u) = u + I_{ik}(u)$.

* **Proposition:** Assume **(A2)**, **(A3)**, and **(A4)**.
Then (4) with initial bounded condition

$$x_0 = \phi \in BPC_\alpha \tag{5}$$

has a solution $x(t)$ defined on $[0, +\infty)$.

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- **Proposition:** Assume **(A2)**, **(A3)**, and **(A4)**.
  If $t \to f_i(t, x)$ are constant, for each $x \in \mathbb{R}^n$, then
  $\exists^1 x^* = (x_1^*, \dots, x_n^*) \in \mathbb{R}^n$ equilibrium point of non-impulsive model (4).

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* **Assumption:** We assume that

$$I_{ik}(x_i^*) = 0, \qquad i = 1, \ldots, n, \ \forall k.$$

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  and x^* is called the equilibrium point of (4).
* **Definition:** The equilibrium point $x^* \in \mathbb{R}^n$ is said *global exponential stable* if exists $M, \epsilon > 0$:

$$|x(t, 0, \phi) - x^*| \leq M e^{-\epsilon t} \|\phi - x^*\|_\alpha, \qquad \forall t \geq 0, \phi \in BPC_\alpha.$$

- **Lemma:**
  Assume **(A2)**, **(A3)**, and $x^* = (0, \ldots, 0)$ equilibrium of (4).
  Let $x : (-\infty, b] \to \mathbb{R}^n$, $b > a$, a solution of non-impulsive equation of (4) on $[a, b]$, with $x_a \in PC_\alpha$.
  If exists $c > 0$ and $\epsilon \in (0, \alpha]$, with $\epsilon < \min_i\{\underline{a_i}(\beta_i - l_i)\}$, such that

$$|x(t)| \leq ce^{-\epsilon(t-a)}, \quad \text{for} \quad t \leq a,$$

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$$|x(t)| \leq ce^{-\epsilon(t-a)}, \quad \text{for} \quad t \leq b. \tag{6}$$

- *Proof* (idea)
  By contradition, assume that (6) does not hold.
  Then there are $\delta > 0$, $m \in \{1, \dots, n\}$, $t^* \in (a, b]$:

$$|x_m(t^*)| = (c + \delta)e^{-\epsilon(t^* - a)} \quad \text{and} \quad |x_i(t)| < (c + \delta)e^{-\epsilon(t-a)},$$

  for all $t < t^*$ and $i = 1, \dots, n$.

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  for all $t < t^*$ and $i = 1, \dots, n$.
- Consider the function $y(t) := (c+\delta)e^{-\epsilon(t-a)}$, $t \in [a, b]$.
  Assuming $x_m(t^*) > 0$ (analogous if $x_m(t^*) < 0$), we have

  $$x_m'(t^*) \geq y'(t^*).$$

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  Assuming $x_m(t^*) > 0$ (analogous if $x_m(t^*) < 0$), we have

$$x'_m(t^*) \geq y'(t^*).$$

- On the other hand, using hypotheses, we have

$$\begin{aligned} x'_m(t^*) &= -a_m(x_m(t^*))[b_m(x_m(t^*)) + f_m(t^*, y_{t^*})] \\ &\leq -\underline{a_m}[\beta_m x_m(t^*) - l_m \|x_{t^*}\|_\alpha] \\ &\leq -\underline{a_m}[\beta_m y(t^*) - l_m \sup_{s\leq 0}(c+\delta)e^{-\epsilon(t^*+s-a)+\alpha s}] \\ &\leq -\underline{a_m}(\beta_m - l_m)y(t^*) < -\epsilon z(t^*) = y'(t^*) \end{aligned}$$

Theorem

- Assume hypotheses **(A1)-(A5)**, and

  (A6) for some $k_0 \in \mathbb{N}$ and $\hat{\gamma}_k := \max_i \hat{\gamma}_{ik}$,

$$\eta := \sup_{k \geq k_0} \left(\frac{\log(\max\{1, \hat{\gamma}_k\})}{t_k - t_{k-1}} \right) < \alpha < \min_i \{\underline{a_i}(\beta_i - l_i)\}. \qquad (7)$$

  If x^* is the equilibrium point of (4), then it is globally exponentially stable.

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  If x^* is the equilibrium point of (4), then it is globally exponentially stable.
- **Notes:**
  - The situation $\hat{\gamma}_k > 1$, for large k, is especially relevant.
  - It is frequent in the literature:

$$I_{ik}(u) = -\alpha_{ik}(u - x_i^*), \qquad \text{with } 0 < \alpha_{ik} < 2,$$

    which implies

$$|x_i(t_k^+) - x_i^*| < |x_i(t_k) - x_i^*|.$$

- *Proof* (idea)
  Suppose that $x^* = (0, \dots, 0)$ and write $\eta_k = \max\{1, \hat{\gamma}_k\}$.
  Let $x(t)$ solution of (4) defined on $\mathbb{R}$, with $x_0 \in PC_\alpha$.

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- Consequently, we have

$$|x(t)| \leq \|x_0\|_\alpha e^{-\alpha t}, \quad \text{for} \quad t \in (-\infty, 0],$$

and, by Lemma, we have

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- For some i,

$$|x(t_1^+)| = |x_i(t_1^+)| = |\hat{I}_{i1}(x_i(t_1))| \leq \hat{\gamma}_{i1}|x_i(t_1)| \leq \eta_1 \|x_0\|_\alpha e^{-\alpha t_1}.$$

Thus

$$|x(t)| \leq \eta_1 \|x_0\|_\alpha e^{-\alpha t_1} e^{-\alpha(t-t_1)}, \quad \text{for} \quad t \in (-\infty, t_1^+],$$

and, again by the Lemma,

$$|x(t)| \leq \eta_1 \|x_0\|_\alpha e^{-\alpha t}, \quad \text{for} \quad t \in (-\infty, t_2].$$

- Iterating the process, we have

$$|x(t)| \leq \eta_1\eta_2\ldots\eta_{k-1}\|x_0\|_\alpha e^{-\alpha t}, \qquad \text{for } t\in(t_{k-1},t_k],\;\; k=1,2,\cdots.$$

From **(A6)** we have $\eta_k \leq e^{\eta(t_k-t_{k-1})}$, for all $k\geq k_0$.
Thus, for $t\in(t_{k-1},t_k]$ and $k>k_0$,

$$\begin{aligned}|x(t)| &\leq \eta_1\eta_2\ldots\eta_{k_0-1}\|x_0\|_\alpha e^{\eta t_{k-1}}e^{-\alpha t}\\ &\leq \eta_1\eta_2\ldots\eta_{k_0-1}\|x_0\|_\alpha e^{-(\alpha-\eta)t},\end{aligned}$$

which implies

$$|x(t)-x^*| \leq \eta_1\eta_2\ldots\eta_{k_0-1}\|x_0-x^*\|_\alpha e^{-(\alpha-\eta)t}, \qquad t\geq 0.$$

Impulsive general Cohen-Grossberg neural network

$$
\begin{aligned}
x_i'(t) = -a_i(x_i(t))\Bigg[b_i(x_i(t)) - \sum_{j=1}^{n}\sum_{p=1}^{P}\Bigg(h_{ij}^{(p)}(x_j(t-\tau_{ij}^{(p)}(t)))\\
+f_{ij}^{(p)}\left(\int_{-\infty}^{0} g_{ij}^{(p)}(x_j(t+s))d\eta_{ij}^{(p)}(s)\right)\Bigg)\Bigg], \ 0 \geq t \neq t_k,\\
\Delta(x_i(t_k)) = I_{ik}(x_i(t_k^-)),\ i=1,\dots,n,\ k\in\mathbb{N}, \qquad (8)
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- $a_i : \mathbb{R} \to (0, +\infty)$, are continuous satisfying **(A1)**;

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- $a_i : \mathbb{R} \to (0, +\infty)$, are continuous satisfying **(A1)**;
- $b_i : \mathbb{R} \to \mathbb{R}$ are continuous satisfying **(A2)**;
- $h_{ij}^{(p)}, f_{ij}^{(p)}, g_{ij}^{(p)} : \mathbb{R} \to \mathbb{R}$ are Lipschitzian with constant $\zeta_{ij}^{(p)}$, $\mu_{ij}^{(p)}$, $\sigma_{ij}^{(p)}$;

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- $a_i : \mathbb{R} \to (0, +\infty)$, are continuous satisfying **(A1)**;
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- $h_{ij}^{(p)}, f_{ij}^{(p)}, g_{ij}^{(p)} : \mathbb{R} \to \mathbb{R}$ are Lipschitzian with constant $\zeta_{ij}^{(p)}$, $\mu_{ij}^{(p)}$, $\sigma_{ij}^{(p)}$;
- $\tau_{ij}^{(p)} : [0, \infty) \to [0, \infty)$ are continuous with $\tau_{ij}^{(p)}(t) \leq \tau_{ij} \leq \tau$.

- **Corollary**

Consider (8) under the hypotheses above.
In addiction, consider $\eta_{ij}^{(p)} : (-\infty, 0] \to \mathbb{R}$ non-decreasing bounded functions so that

$$\eta_{ij}^{(p)}(0) - \eta_{ij}^{(p)}(-\infty) = 1, \text{ and } \int_{-\infty}^{0} e^{-\gamma s} d\eta_{ij}^{(p)} < \infty,$$

for some $\gamma > \eta$.
If

$$M = diag\left(\beta_1 - \frac{\eta}{\underline{a_1}}, \dots, \beta_n - \frac{\eta}{\underline{a_n}}\right) - [n_{ij}]$$

is a M-matrix, then there is a unique equilibrium point x^* of (8) which is globally exponential stable.

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- $$n_{ij} = \sum_{p=1}^{P} \left(\zeta_{ij}^{(p)} e^{\eta \tau_{ij}^{(p)}} + \mu_{ij}^{(p)} \sigma_{ij}^{(p)} \int_{-\infty}^{0} e^{-\eta s} d\eta_{ij}^{(p)}(s) \right)$$

Thank you