Global asymptotic stability of the periodic solution for a periodic model of hematopoiesis with impulses

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Delay differential equations

- For $\tau \in \mathbb{R}^+$ and $m \in \mathbb{N}$, consider

$$\mathcal{C} := C([-\tau,0];\mathbb{R}^m) = \left\{ \varphi : [-\tau,0] \to \mathbb{R}^m \,\middle|\, \varphi \text{ is continuous } \right\}$$

with the norm

$$\|\varphi\| = \sup_{\theta\in[-\tau,0]} |\varphi(\theta)|_{\mathbb{R}^m}.$$

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- The real space $(\mathcal{C}, \|\cdot\|)$ is a Banach space.

- For $D \subseteq \mathbb{R} \times \mathcal{C}$ and $f : D \to \mathbb{R}^m$ continuous, we call a delay differential equation to the equation

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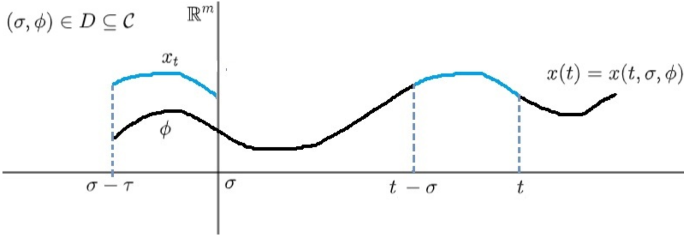
- Let $\sigma \in \mathbb{R}$, $b \in (\sigma, +\infty]$, and $x : [\sigma - \tau, b] \to \mathbb{R}^m$ a continuous function.
  For $t \in [\sigma, b]$, the function $x_t \in \mathcal{C}$ is defined by

$$x_t(\theta) = x(t + \theta), \qquad \forall \theta \in [-\tau, 0].$$



For $(\sigma, \phi) \in D$, consider the initial value problem

$$\begin{cases} x'(t) = f(t, x_t) \\ x_\sigma = \phi \end{cases} \tag{1}$$

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- If $x(t)$ is a maximal solution of (1), then, for each compact $W \subseteq D$, there is $t_W \in \mathbb{R}$ such that $(t, t_W) \notin D$ for $t \geq t_W$.

Biological models

In what follows, we only consider the scalar case ($m = 1$) and non-negative time ($t \geq 0$).

- Scalar biological models

$$x'(t) = -a(t)x(t) + f(t, x_t), \qquad t \geq 0,$$

  where:
  - $a : [0, \infty) \to \mathbb{R}^+$ is a periodic functions;
  - $\forall \phi \in \mathcal{C}$, $t \mapsto f(t, \phi)$ is a periodic functions.
- Mortality: $a(t)x(t)$
- Birth: $f(t, x_t)$

Impulsive biological models

- Scalar impulsive delay differential equation

$$\begin{cases} x'(t) = -a(t)x(t) + f(t, x_t), & 0 \leq t \neq t_k, \\ x(t_k^+) = x(t_k) + I_k(x(t_k)), & k = 1, 2, \dots \end{cases}, \qquad (2)$$

where

- $(t_k)_{k \in \mathbb{N}}$ such that $0 < t_k \nearrow +\infty$;
- $I_k : \mathbb{R} \to \mathbb{R}$ continuous;
- $a : [0, \infty) \to (0, \infty)$ continuous;
- $f : [0, \infty) \times PC \to [0, \infty)$ with some regularities.

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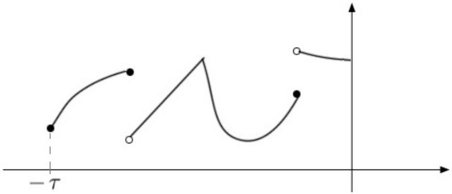
  - $a : [0, \infty) \to (0, \infty)$ continuous;
  - $f : [0, \infty) \times PC \to [0, \infty)$ with some regularities.
- Note that:

$$x(t_k^+) - x(t_k) = I_k(x(t_k)), \quad \forall k \in \mathbb{N}.$$

- $PC = \left\{ \varphi : [-\tau, 0] \to \mathbb{R} \,\middle|\, \varphi \text{ is piecewise continuous} \right\}$



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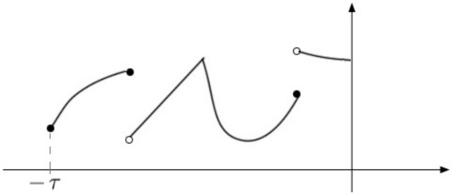
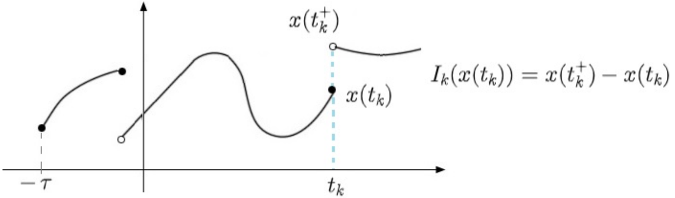


- For $x(t)$ a solution of (2)



The key step in the study of stability of many impulsive models.
For $x(t)$ a solution of (2) on $[0,\infty)$, define

$$\begin{aligned} y(t) &= \prod_{k:0\leq t_k<t} \frac{x(t_k)}{x(t_k)+I_k(x(t_k))} x(t) \\ &= \prod_{k:0\leq t_k<t} J_k(x(t_k))x(t), \end{aligned}$$

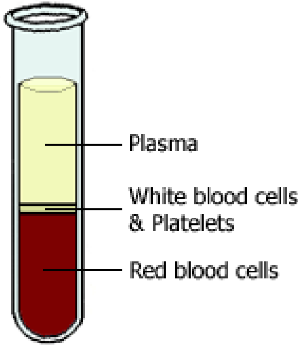
where $J_k(u)=\frac{u}{u+I_k(u)}$, $\forall k\in\mathbb{N}$, $u\neq 0$.
The function $y(t)$ is continuous and it is solution of

$$y'(t) = -a(t)y(t) + \prod_{k:0\leq t_k<t} J_k(x(t_k))f(t,x_t), \quad 0\leq t\neq t_k.$$

Processo hematopoiese

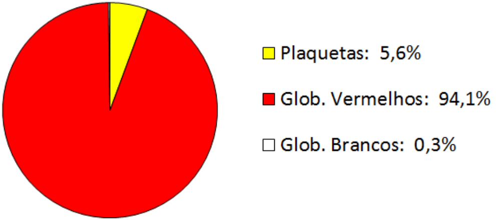
Processo de produção, multiplicação e especialização das células do sangue na medula óssea.

Constituintes do sangue



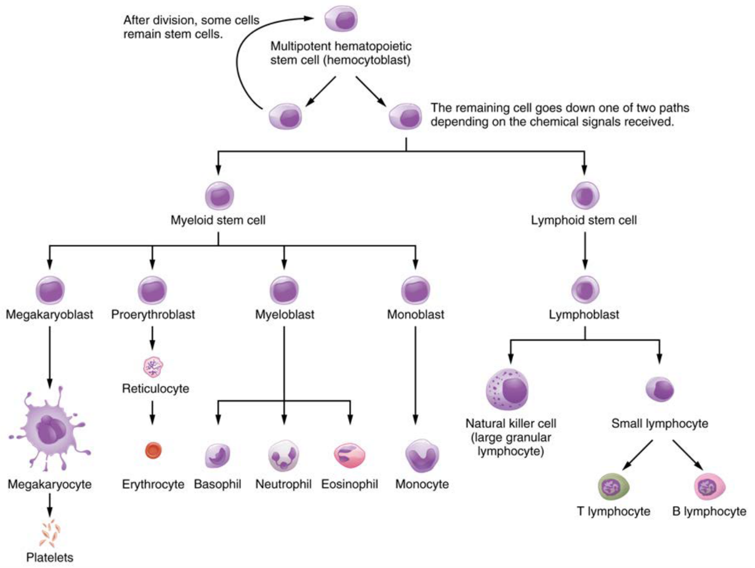
O sangue é composto por 55 % plasma e 45% células sanguíneas.

Tipo de células	células/$1\mu l$ [2]
thrombocytes (Plaquetas)	15×10^4 - 40×10^4
erythrocytes (Glob. vermelhos)	homem 43×10^5 - 59×10^5 mulher 35×10^5 - 55×10^5
leukocytes (Glob. brancos)	4500 - 11000



$$1\mu l = 1mm^3$$

[2] L. Dean, *Blood Group and Red Cell Antigens, National Center for Biotechnology Information (US)*, 2005.


After division, some cells remain stem cells.
Multipotent hematopoietic stem cell (hemocytoblast)
The remaining cell goes down one of two paths depending on the chemical signals received.
Myeloid stem cell
Lymphoid stem cell
Megakaryoblast
Proerythroblast
Myeloblast
Monoblast
Lymphoblast
Reticulocyte
Natural killer cell (large granular lymphocyte)
Small lymphocyte
Megakaryocyte
Erythrocyte
Basophil
Neutrophil
Eosinophil
Monocyte
T lymphocyte
B lymphocyte
Platelets

Tempo de gestação na medula óssea

Tipo de células	Tempo de gestação
thrombocytes (Plaquetas)	$\simeq$7 dias [3]
erythrocytes (Glob. vermelhos)	$\simeq$6 dias [4]
neutrophils (60% dos Glob. brancos)	$\simeq$15 dias [5]

[3] G.P. Langlois, M. Craig, A.R. Humphries et al., *Normal and pathological dynamics of platelets in humans*, J. Math. Biol. 75 (2017), 1411–1462.

[4] J. Bélair, M. C. Mackey, J. M. Mahaffy, *Age-structured and two-delay models for erythropoiesis*, Math. Biosci. 128 (1995), 317–346.

[5] Y. Yan, J. Sugie, *Existence regions of positive periodic solutions for a discrete hematopoiesis model with unimodal production functions* Appl. Math. Model. 68 (2019), 152–168.

Hematopoiesis models

Mackey and Glass [1], proposed the following models to describe the hematopoiesis process (the process of production, multiplication, and specialization of blood cells in the bone marrow):

- Hematopoieses with monotone prodution rate

$$z'(t) = -\gamma z(t) + \frac{F_0 \eta^n}{\eta^n + z(t-\tau)^n}, \qquad n > 0; \tag{3}$$

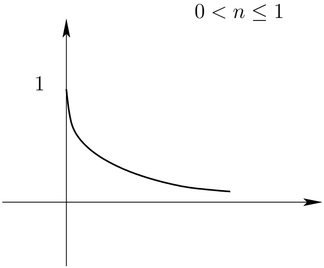
- Hematopoiesis with unimodal prodution rate

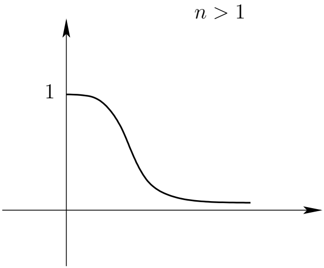
$$z'(t) = -\gamma z(t) + \frac{F_0 \eta^n z(t-\tau)}{\eta^n + z(t-\tau)^n}, \qquad n > 1; \tag{4}$$

$z(t)$ density of cells at time t; τ time delay; γ destruction rate;
F_0 maximal prodution rate (only for (3)); η a shape parameter.

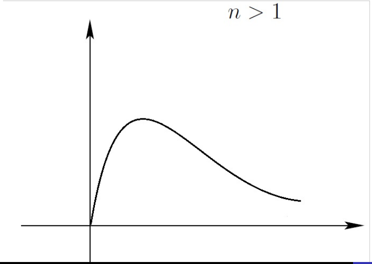
[1] M.C.Mackey, L. Glass, Science 197 (1977) 287-289.

- Monotone prodution rate





- Unimodal prodution rate



Hematopoiesis model with several delays

$$y'(t) = -a(t)y(t) + \sum_{i=1}^{m} \frac{\beta_i(t)}{1 + y(t - \tau_i(t))^n}, \qquad t \geq 0$$

Notation: $\tau(t) = \max_i \tau_i(t)$ and $\overline{\tau} = \sup_t \tau(t)$

Hematopoiesis model with linear impulses

For $(t_k)_k$ an increasing sequence such that $t_k \to \infty$, we consider

$$\begin{cases} y'(t) = -a(t)y(t) + \sum_{i=1}^{m} \dfrac{\beta_i(t)}{1 + y(t - \tau_i(t))^n}, & 0 \leq t \neq t_k, \\ y(t_k^+) = (1 + b_k)y(t_k), & k \in \mathbb{N} \end{cases} \tag{5}$$

The impulsive function is linear: $I_k(u) = b_k u$. In fact,

$$y(t_k^+) = (1 + b_k)y(t_k) \Leftrightarrow y(t_k^+) = y(t_k) + b_k y(t_k)$$

$$PC_0^+ = \left\{ \varphi \in PC : \varphi(\theta) \geq 0 \text{ for } \theta \in [-\overline{\tau}, 0),\ \varphi(0) > 0 \right\}$$

- **Periodic Hematopoiesis model with linear impulses**

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- **(H1)** $a, \beta_i \in C(\mathbb{R}; (0,\infty))$ and $\tau_i \in C(\mathbb{R}; [0,\infty))$ are ω-periodic, for some $\omega > 0$;

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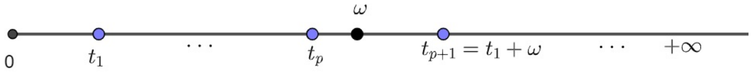
- **(H2)** $\exists p \in \mathbb{N}$ such that $0 < t_1 < \cdots < t_p < \omega$ and

$$t_{k+p} = t_k + \omega, \qquad b_{k+p} = b_k, \qquad k \in \mathbb{N};$$



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- **(H3)** $1 + b_k > 0$, $\forall k \in \mathbb{N}$;

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- **(H3)** $1 + b_k > 0$, $\forall k \in \mathbb{N}$;
- **(H4)** $\prod_{k=1}^{p}(1+b_k) < e^{\int_0^\omega a(t)dt}$

Goal

To establish sufficient conditions for global asymptotic stability (GAS) of a positive ω-periodic solution of (5).

Existence of periodic solution

- **Theorem 2** Faria & Oliveira [3]:
  Assume **(H1)-(H4)**.
  Then system (5) has at least one positive ω-periodic solution.

[3] T. Faria and J.J. Oliveira, *Existence of positive periodic solution for scalar delay differential equations with and without impulses*, J. Dyn. Differ. Equ., 31 (2019), 1223-1245.

Existence of periodic solution

- **Theorem 2** Faria & Oliveira [3]:
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- In what follows, we fix $y^*(t)$ a positive ω-periodic solution of system (5).

[3] T. Faria and J.J. Oliveira, *Existence of positive periodic solution for scalar delay differential equations with and without impulses*, J. Dyn. Differ. Equ., 31 (2019), 1223-1245.

Global asymptotic stability

In [4,5], global stability criteria were obtained for the general impulsive model

$$\begin{cases} x'(t) = -a(t)x(t) + \sum_{i=1}^{m} f_i(t, x(t-\tau_i(t))), & t \neq t_k, \\ x(t_k^+) = (1+b_k)x(t_k), & k \in \mathbb{N} \end{cases}, \qquad (6)$$

where, for each i, $f_i : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is piecewise continuous and continuous in the second variable.

- [4] T. Faria and J.J. Oliveira, *On stability for impulsive delay differential equations and applications to a periodic Lasota-Wazewska model*, Disc. Cont. Dyn. Systems Series B, 21 (2016), 2451-2472.
- [5] T. Faria and J.J. Oliveira, *A note on stability of impulsive scalar delay differential equations*, Electron. J. Qual. Theory Differ. Equ., Paper No. 69 (2016), 1-14.

- **Theorem 5** Faria & Oliveira, [4,5]:
Assume **(H2)-(H3)** and $a(t) \not\equiv 0$ ω-periodic continuous.
The zero solution of (6) is globally asymptotically stable if

(A1) (**Yorke Condition**)

(A2) ($\frac{3}{2}$-**Condition**)

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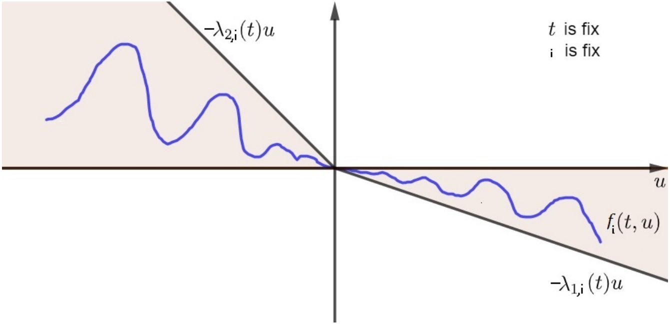
Assume **(H2)-(H3)** and $a(t) \not\equiv 0$ ω-periodic continuous.
The zero solution of (6) is GAS if

(A1) (**Yorke Condition**) There are $\lambda_{1,i}, \lambda_{2,i} : [0, \infty) \to [0, \infty)$ piecewise continuous such that, for $t \geq 0$ and $u \in \mathbb{R}$,

$$-\lambda_{1,i}(t) \max\{u, 0\} \leq f_i(t, u) \leq \lambda_{2,i}(t) \max\{-u, 0\};$$



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  The zero solution of (6) is GAS if

(A1) (**Yorke Condition**)

(A2) ($\frac{3}{2}$-**Condition**) There is $T > 0$ such that

$$\alpha_1^* \alpha_2^* < 1 \text{ or } \alpha_1 \alpha_2 < \frac{9}{4} \tag{7}$$

where $\alpha_j^* = \sup_{t \geq T} \alpha_j^*(t)$, $\alpha_j = \sup_{t \geq T} \alpha_j^*(t)\, e^{\int_{t-\tau(t)}^{t} a(u)du}$ $(j = 1, 2)$,

$$\alpha_j^*(t) = \int_{t-\tau(t)}^{t} \sum_{i=1}^{m} \lambda_{j,i}(s) B_i(s)\, e^{-\int_s^t a(u)du}\, ds, \qquad j = 1, 2.$$

$$\text{with } B_i(s) = \prod_{k: t-\tau_i(t) \leq t_k < t} (1 + b_k)^{-1}, \qquad i = 1, \ldots, m.$$

Proof of the main results (idea)

- We translate the positive ω-periodic solution of (5), $y^*(t)$, to the origin with the change $x(t) = y(t) - y^*(t)$.

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where:

- for $t \geq 0$ and $u \geq -y^*(t - \tau_i(t))$, $f_i(t, u) = \beta_i(t)g_i(t, u)$ with

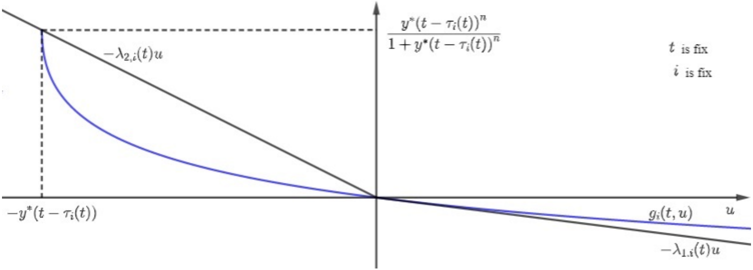
$$g_i(t, u) = \frac{1}{1 + [u + y^*(t - \tau_i(t))]^n} - \frac{1}{1 + y^*(t - \tau_i(t))^n}, \quad (8)$$

$$S = \left\{\varphi \in PC : \varphi(\theta) \geq -y^*(\theta) \text{ for } \theta \in [-\overline{\tau}, 0),\ \varphi(0) > -y^*(0)\right\}$$

Let $n \in (0,1]$.
Considering $g_i(t,u)$ defined in (8), we have

$$\frac{\partial g_i}{\partial u}(t,u) < 0 \text{ and } \frac{\partial^2 g_i}{\partial u^2}(t,u) > 0, \qquad \forall u > -y^*(t-\tau_i(t)), \forall t \geq 0$$

with $\frac{\partial g_i}{\partial u}(t,0) = -\frac{ny^*(t-\tau_i(t))^{n-1}}{[1+y^*(t-\tau_i(t))^n]^2}$.



$$\lambda_{1,i}(t) = \frac{ny^*(t-\tau_i(t))^{n-1}}{[1+y^*(t-\tau_i(t))^n]^2}, \qquad \lambda_{2,i}(t) = \frac{y^*(t-\tau_i(t))^{n-1}}{1+y^*(t-\tau_i(t))^n}.$$

- **Theorem 3**: Assume **(H1)-(H4)** and $n \in (0,1]$.
  The periodic solution $y^*(t)$ of (5) is GAS, in the set of positive solutions, if there is $T > 0$ such that

$$\alpha_1^* \alpha_2^* < 1 \quad \text{or} \quad \alpha_1 \alpha_2 < \frac{9}{2},$$

where $\alpha_j^* = \sup_{t \geq T} \alpha_j^*(t)$, $\alpha_j = \sup_{t \geq T} \alpha_j^*(t)\, e^{\int_{t-\tau(t)}^t a(u)du}$ $(j = 1, 2)$,

and

$$\alpha_1^*(t) = \int_{t-\tau(t)}^t \sum_{i=1}^m \beta_i(s) \frac{n y^*(s-\tau_i(s))^{n-1}}{[1 + y^*(s-\tau_i(s))^n]^2} B_i(s)\, e^{-\int_s^t a(u)\,du}\, ds$$

$$\alpha_2^*(t) = \int_{t-\tau(t)}^t \sum_{i=1}^m \beta_i(s) \frac{y^*(s-\tau_i(s))^{n-1}}{1 + y^*(s-\tau_i(s))^n} B_i(s)\, e^{-\int_s^t a(u)\,du}\, ds$$

$$\text{with } B_i(s) = \prod_{k: t-\tau_i(t) \leq t_k < t} (1 + b_k)^{-1}, \qquad i = 1, \dots, m.$$

Let $n \in (1, \infty)$.
In this case, $g_i(t, u)$ defined in (8) verifies

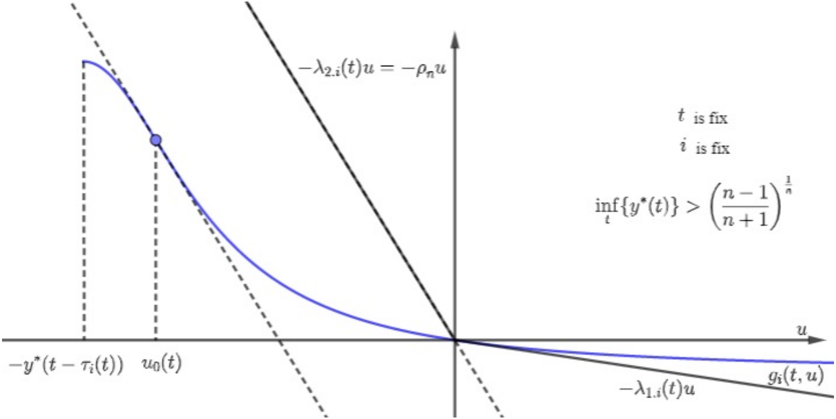
$$\frac{\partial g_i}{\partial u}(t, u) < 0, \qquad \forall u > -y^*(t - \tau_i(t)),\ \forall t \geq 0,$$

and

$$\begin{cases} \dfrac{\partial^2 g_i}{\partial u^2}(t, u) > 0 \text{ for } u > u_0(t), \\ \\ \dfrac{\partial^2 g_i}{\partial u^2}(t, u) < 0 \text{ for } u \in (-y^*(t - \tau_i(t)), u_0(t)), \end{cases}$$

where $u_0(t) := -y^*(t - \tau_i(t)) + \left(\frac{n-1}{n+1}\right)^{\frac{1}{n}}$ is the unique inflection point of $u \mapsto g_i(t, u)$.

We have $\frac{\partial g_i}{\partial u}(t, u_0(t)) = -\rho_n = -\frac{(n+1)^2}{4n}\left(\frac{n-1}{n+1}\right)^{\frac{n-1}{n}}$.


$-\lambda_{2.i}(t)u = -\rho_n u$
t is fix
i is fix
$\inf_t\{y^*(t)\} > \left(\frac{n-1}{n+1}\right)^{\frac{1}{n}}$
u
$-y^*(t-\tau_i(t))$
$u_0(t)$
$g_i(t,u)$
$-\lambda_{1.i}(t)u$

Theorem 4: Assume **(H1)-(H4)** and $n > 1$.
The periodic solution $y^*(t)$ of (5) is GAS (in PC_0^+) if, for some $T > 0$, one of the following conditions holds:

(i) $\left(\alpha_1\gamma < \frac{9}{4} \text{ or } \alpha_1^*\gamma^* < 1\right)$ and $\inf_t\{y^*(t)\} \geq \left(\frac{n-1}{n+1}\right)^{\frac{1}{n}}$;

(ii) $\left(\alpha_1\gamma < \frac{9}{4} \text{ or } \alpha_1^*\gamma^* < 1\right)$ and $\sup_t\{y^*(t)\} \leq \left(\frac{n-1}{n+1}\right)^{\frac{1}{n}}$;

(iii) $\gamma < \frac{3}{2}$ or $\gamma^* < 1$,

where $\gamma^* = \sup_{t \geq T} \gamma^*(t)$, $\gamma = \sup_{t \geq T} \gamma^*(t)\, e^{\int_{t-\tau(t)}^t a(u)du}$, with

$$\gamma^*(t) = \rho_n \int_{t-\tau(t)}^t \sum_{i=1}^m \beta_i(s) B_i(s)\, e^{-\int_s^t a(u)\,du}\, ds,$$

with $\rho_n = \frac{(n+1)^2}{4n}\left(\frac{n-1}{n+1}\right)^{\frac{n-1}{n}}$, $B_i(s)$, α_1, and α_1^* as above.

In case that $y^*(t)$ is unknown, we have the estimate

$$\mathfrak{m} \leq y^*(t) \leq \mathfrak{M}, \qquad t \geq 0,$$

where

$$\mathfrak{M} = \min\left\{ M\beta\overline{B}, M\overline{B}(e^{A(\omega)} - 1)\, e^{A(\omega)} \left(\max_{t\in[0,\omega]} \frac{\sum_{i=1}^m \beta_i(t)}{a(t)} \right) \right\}$$

$$\mathfrak{m} = \frac{e^{-A(\omega)}\, M\underline{B}}{1 + \mathfrak{M}^n} \max\left\{ \beta, (e^{A(\omega)} - 1) \left(\min_{t\in[0,\omega]} \frac{\sum_{i=1}^m \beta_i(t)}{a(t)} \right) \right\}$$

with $\beta = \int_0^\omega \sum_{i=1}^m \beta_i(s)ds$, $A(\omega) = \int_0^\omega a(u)du$,
$M = \left(\prod_{k=1}^p (1 + b_k)^{-1} - e^{-A(\omega)} \right)^{-1}$,
$\overline{B} = \max\left\{ 1, \prod_{k=j}^{j+l} (1 + b_k)^{-1} : j = 1, \ldots, p, l = 0, \ldots, p-1 \right\}$, and
$\underline{B} = \min\left\{ 1, \prod_{k=j}^{j+l} (1 + b_k)^{-1} : j = 1, \ldots, p, l = 0, \ldots, p-1 \right\}$.

- One delay multiple of the period ($m = 1$, $\tau(t) = q\omega$, $q \in \mathbb{N}$)

$$\begin{cases} y'(t) = -a(t)y(t) + \dfrac{\beta(t)}{1 + y(t - q\omega)^n}, & 0 \le t \ne t_k, \\ y(t_k^+) = (1 + b_k)y(t_k), & k \in \mathbb{N} \end{cases} \tag{9}$$

- One delay multiple of the period ($m = 1$, $\tau(t) = q\omega$, $q \in \mathbb{N}$)

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- **Theorem 5**: Assume **(H2)-(H4)** and $a, \beta : [0, \infty) \to (0, \infty)$ ω-periodic continuous functions.
  Then (9) has a positive periodic solution which is GAS if:
  case $n > 1$

$$\rho_n B_0^q \sup_{t \geq T} \int_{t-q\omega}^{t} \beta(s)\, e^{-\int_s^t a(u)du}\, ds < \max\left\{1, \frac{3}{2}\, e^{-q\int_0^\omega a(u)du}\right\},$$

or, case $0 < n \leq 1$,

$$\sqrt{n} B_0^q \sup_{t \geq T} \int_{t-q\omega}^{t} \beta(s)\, e^{-\int_s^t a(u)du}\, ds < \mathfrak{m} \max\left\{1, \frac{3}{2}\, e^{-q\int_0^\omega a(u)du}\right\}$$

where $B_0 = \prod_{k=1}^{p} (b_k + 1)^{-1}$.

- **Remark:** Saker and Alzabut [6] proved the existence of a positive periodic solution of (9) and its GAS assuming (H1), (H3), $n \in \mathbb{N}$, the function

$$t \mapsto \prod_{k:t_k \in [0,t)} (1 + b_k) \text{ is } \omega\text{-periodic}, \tag{10}$$

and the "3/2-type condition"

$$\rho_n q \int_0^\omega \beta(s) ds < \frac{3}{2} \, e^{-q \int_0^\omega a(u) du} \, . \tag{11}$$

[6] S.H. Saker and JO. Alzabut, *On the impulsive delay hematopoiesis model with periodic coefficients*, Rocky Mountain J. Math. 39 (2009) 1657-1688.

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- Condition (10) is stronger than (H2)+(H3).

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- Condition (10) is stronger than (H2)+(H3).
- Condition (11) is stronger than

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Thus Theorem 5 improves the stability criterion in [6].

[6] S.H. Saker and JO. Alzabut, *On the impulsive delay hematopoiesis model with periodic coefficients*, Rocky Mountain J. Math. 39 (2009) 1657-1688.

- No impulsive case ($b_k = 0, \forall k \in \mathbb{N}$)

$$y'(t) = -a(t)y(t) + \sum_{i=1}^{m} \frac{\beta_i(t)}{1 + y(t - \tau_i(t))^n}, \ t \geq 0, \qquad (12)$$

[7] G. Liu, J. Yan and F. Zhang, J. Math. Anal. Appl. 334 (2007), 157-171.

- No impulsive case ($b_k = 0$, $\forall k \in \mathbb{N}$)

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- **Theorem 5**: Consider $n > 1$ and assume **(H2)**. If

$$\rho_n \sup_{t\in[0,\omega]} \int_{t-\tau(t)}^{t} \sum_{i=1}^{m} \beta_i(s)\, \mathrm{e}^{-\int_s^t a(u)du}\, ds < \max\left\{1, \frac{3}{2}\,\mathrm{e}^{-\mathcal{A}}\right\},$$

where $\mathcal{A} = \sup_{t\in[0,\omega]} \int_{t-\tau(t)}^{t} a(u)du$, then there is a positive ω-periodic solution of (12) which is GAS.

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where $\mathcal{A} = \sup_{t\in[0,\omega]} \int_{t-\tau(t)}^{t} a(u)du$, then there is a positive ω-periodic solution of (12) which is GAS.

- Liu et al [7] proved the existence of a positive periodic solution of (12) and its GAS assuming (H1), $n > 1$, and

$$(n-1)^{\frac{n-1}{n}} \frac{e^{A(\omega)}}{e^{A(\omega)} - 1} \int_0^{\omega} \sum_{i=1}^{m} \beta_i(s)ds \leq 1.$$

[7] G. Liu, J. Yan and F. Zhang, J. Math. Anal. Appl. 334 (2007), 157-171.

Numerical example

Consider the 1-periodic model

$$y'(t) = -\left(1+\frac{1}{2}\cos(2\pi t)\right)y(t)+\frac{\eta_1\left(1+\frac{1}{2}\cos(2\pi t)\right)}{1+y(t-6-\cos(2\pi t))^n}$$
$$+\frac{\eta_2\left(1+\frac{1}{2}\sin(2\pi t)\right)}{1+y(t-7-\cos(2\pi t))^n}+\frac{\eta_3\left(1+\frac{1}{2}\cos(2\pi t)\right)}{1+y(t-15-\cos(2\pi t))^n},$$

where η_1, η_2, η_3 are positive real numbers.

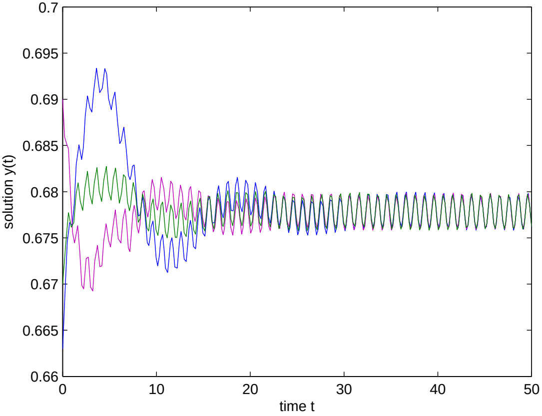

Figure: Numerical simulation of three solutions where $\eta_1 = 1.1$, $\eta_2 = 0.03$, $\eta_3 = 0.001$ and $n = 1.03$, with initial condition $\varphi(\theta) = 0.67$, $\varphi(\theta) = 0.65(1 + 0.02\cos(\theta))$, and $\varphi(\theta) = 0.69(1 + 0.02\sin(\theta))$, for $\theta \in [-16, 0]$, respectively.

Thank you

The presented results are published in

[8] T. Faria and J.J. Oliveira, *Global asymptotic stability for a periodic delay hematopoiesis model with impulses*, Applied Mathematical Modelling 79 (2020) 843-864.