Estabilidade global de equações diferenciais retardadas escalares impulsivas. Aplicações a modelos biológicos periódicos

José J. Oliveira[a] and Teresa Faria[b]

4 de Dezembro de 2018

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Equações diferenciais retardadas escalares

- Espaço de fase. Dado $\tau > 0$,

$$C := C([-\tau, 0]; \mathbb{R}), \qquad (1)$$

equipado com a norma $\|\varphi\| = \sup_{\theta \in [-\tau, 0]} |\varphi(\theta)|$.

- Equação diferencial retardada

$$x'(t) = F(t, x_t), \qquad t \geq 0 \qquad (2)$$

onde $x_t \in C$ definido por

$$x_t(\theta) = x(t + \theta), \qquad \theta \in [-\tau, 0].$$

- Um equilíbrio $\alpha \in \mathcal{C} \subseteq C$ de (2) diz-se *globalmente atractivo* em $\mathcal{C}$ se, para quaisquer $\varphi \in C$ e $t_0 \geq 0$,

$$x(t, t_0, \varphi) \to \alpha, \text{ quando } t \to +\infty.$$

Equação Logística

- Para $a, k \in \mathbb{R}^+$,

$$x'(t) = ax(t)\left(1 - \frac{1}{k}x(t-\tau)\right), \qquad t \geq 0. \tag{3}$$

- **Teorema** [E.M.Wright, 1955]
  Se $a\tau \leq \frac{3}{2}$, então

$$x(t, t_0, \varphi) \to k, \text{ quando } t \to +\infty,$$

  para $\varphi \in \mathcal{C}_0 = \{\varphi \in C : \varphi(0) > 0 \text{ e } \varphi(\theta) \geq 0 \text{ para } \theta \in [-\tau, 0)\}$.

- **Conjectura** [E.M.Wright, 1955]
  Se $a\tau < \frac{\pi}{2}$, então

$$x(t, t_0, \varphi) \to k, \text{ quando } t \to +\infty,$$

  para $\varphi \in \mathcal{C}_0$.

- **Lemma B'**
  Assume that f transforms closed bounded sets $(-\infty, 0] \times UC_g$ into bounded sets of $\mathbb{R}^n$.
  If
  (H') $\forall t > 0, \forall \varphi \in UC_g$:

$$\forall s \in (-\infty, 0), \frac{|\varphi(s)|}{g(s)} < |\varphi(0)| \Rightarrow \varphi_i(0) f_i(t, \varphi) < 0,$$

  for some $i \in \{1, \ldots, n\}$ such that $|\varphi(s)| = |\varphi_i(0)|$,

  then the solution $x(t) = x(t, 0, \varphi)$, $\varphi \in UC_g$, of (**??**) is defined and bounded on $[0, +\infty)$ and

$$|x(t, 0, \varphi)| \leq ||\varphi||_g.$$

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- **Remark: (H')⇒(H)**, and $||\varphi||_g \leq ||\varphi||_\infty$.

Global asymptotic stability

- Consider in UC_g,

$$\dot{x}_i(t) = -a_i(x_i(t))[b_i(x_i(t)) + f_i(x_t)] \qquad (6)$$

where $a_i : \mathbb{R} \to (0, +\infty)$, $b_i : \mathbb{R} \to \mathbb{R}$ and $f_i : UC_g \to \mathbb{R}$ are continuous functions such that

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- **(A1)** $\exists \beta_i > 0, \forall u, v \in \mathbb{R}, u \neq v$:

$$(b_i(u) - b_i(v))/(u - v) \geq \beta_i;$$

[In particular, for $b_i(u) = \beta_i u$.]

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- **(A2)** f_i is a Lipshitz function with Lipshitz constant l_i.

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- **Definition**

A equilibrium $x^* \in \mathbb{R}^n$ is said *global asymptotic stable* if it is stable and

$$x(t, 0, \varphi) \to x^* \text{ as } t \to \infty, \text{ for all } \varphi \in BC_g.$$

- **Theorem 1** (GAS)
  Assume **(A1)** and **(A2)**. If

$$\beta_i > l_i, \qquad \forall i \in \{1, \dots, n\},$$

then there is a unique equilibrium point of (**??**), which is globally asymptotically stable.

- **Theorem 1** (GAS)
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- **Proof (idea)**
  Existence and uniqueness of equilibrium point

$$\begin{array}{rccc} H: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & x & \mapsto & (b_i(x_i) + f_i(x))_{i=1}^n \end{array}$$

  is homeomorphism.
  Then there exists $x^* \in \mathbb{R}^n$, $H(x^*) = 0$, i.e. x^* is the equilibrium.

- By translation, we may assume that $x^* = 0$, i.e.,

$$b_i(0) + f_i(0) = 0, \qquad \forall i = 1, \dots, n$$

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- For $\varphi \in UC_g$ such that $||\varphi||_g = |\varphi(0)| = \varphi_i(0) > 0$ (analogous if $\varphi_i(0) < 0$),

$$\begin{aligned} b_i(\varphi_i(0)) + f_i(\varphi) &= [b_i(\varphi_i(0)) - b_i(0)] + [f_i(\varphi) - f_i(0)] \\ &\geq (\beta_i - l_i)||\varphi||_g > 0. \end{aligned}$$

(H') holds and from Lemma B' we conclude

  - $x = 0$ is uniform stable
  - all solutions are defined and bounded on $[0, +\infty)$

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  (H') holds and from Lemma B' we conclude
  - $x = 0$ is uniform stable
  - all solutions are defined and bounded on $[0, +\infty)$
- It remains to prove that $x = 0$ is global attractive.

- Let $x(t) = x(t, 0, \varphi)$ a solution of (**??**), with $\varphi \in BC_g$, and define

$$-v_i = \liminf_{t\to+\infty} x_i(t), \qquad u_i = \limsup_{t\to+\infty} x_i(t)$$

$$v = \max_i\{v_i\}, \qquad u = \max_i\{u_i\},$$

$u, v \in \mathbb{R}$, $-v \leq u$. We have to show that $\max(u, v) = 0$.

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Let $i \in \{1, \ldots, n\}$ such that $u_i = u$.
- By computations, we can show that exists $(t_k)_{k\in\mathbb{N}}$ such that
$$t_k \nearrow +\infty, \qquad x_i(t_k) \to u, \qquad \text{and} \qquad b_i(x_i(t_k)) + f_i(x_{t_k}) \to 0.$$

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$$t_k \nearrow +\infty, \quad x_i(t_k) \to u, \quad \text{and} \quad b_i(x_i(t_k)) + f_i(x_{t_k}) \to 0.$$
- As $x(t)$ is bounded, $\dot{x}(t)$ is also bounded, from Lemma A $\{x_t : t \geq 0\}$ is precompact in UC_g. Then
$$\exists \phi \in UC_g : \qquad x_{t_k} \to \phi \text{ on } UC_g,$$
with $\|\phi\|_g = \phi_i(0) = u$ and $b_i(\phi_i(0)) + f_i(\phi) = 0$.
Then $u = 0$. $\square$

Global exponential stability

- Consider in UC_g, with $g(s) = e^{-\alpha s}$ for some $\alpha > 0$,

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Assume:

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Assume:

- **(A0)** $\underline{a} := \inf\{a_i(y) : y \in \mathbb{R}, 1 \le i \le n\} > 0$;

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- **Definition**

A equilibrium $x^* \in \mathbb{R}^n$ is said *global exponential stable* if there are $\varepsilon, M > 0$ such that

$$|x(t, 0, \varphi) - x^*| \leq Me^{-\varepsilon t}||\varphi - x^*||_\infty, \text{ for all } t \geq 0, \varphi \in BC_g.$$

- **Theorem 2** (GES)
  Assume **(A0)**, **(A1)**, and **(A2)**. If

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- **Proof (idea)**
  We may assume $x^* \equiv 0$, i.e. $b_i(0) + f_i(0) = 0$.
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  Let $x(t, 0, \varphi)$ be a solution of (**??**) with $\varphi \in BC_g$.
- The change of variables

$$z(t) = e^{\varepsilon t} x(t)$$

  transforms (**??**) into

$$\dot{z}_i(t) = F_i(t, z_t), \quad i = 1, \dots, n, \tag{4}$$

where

$$F_i(t,\phi) = \varepsilon\phi_i(0) - a_i(e^{-\varepsilon t}\phi_i(0))e^{\varepsilon t}\left[b_i(e^{-\varepsilon t}\phi_i(0)) + f_i(e^{-\varepsilon(t+\cdot)}\phi)\right]$$

- Let $\phi \in BC_g$ such that $|\phi(s)| < |\phi(0)|$, for $s \in (-\infty, 0)$. Consider $i \in \{1, \dots, n\}$ such that $|\phi_i(0)| = |\phi(0)|$.

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- If $\phi_i(0) > 0$ ($\phi_i(0) < 0$ is analogous)
  From the hypotheses we conclude that

$$\begin{aligned} F_i(t,\phi) &\leq \varepsilon\phi_i(0) - \underline{a}e^{\varepsilon t}\left[b_i(e^{-\varepsilon t}\phi_i(0)) - b_i(0) + \right. \\ &\quad \left. + f_i(e^{-\varepsilon(t+\cdot)}\phi) - f_i(0)\right] \\ &\leq \varepsilon\phi_i(0) - \underline{a}\left[\beta_i\phi_i(0) - l_i \sup_{s\leq 0} e^{(\alpha-\varepsilon)s}|\phi(s)|\right] \\ &\leq \phi_i(0)[\varepsilon - \underline{a}(\beta_i - l_i)] < 0. \end{aligned}$$

Then $F = (F_1, \dots, F_n)$ satisfies **(H)**

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  Then $F = (F_1,\dots,F_n)$ satisfies **(H)**
- From Lemma B, $z(t)$ is defined on $[0,+\infty)$ and

$$|x(t,0,\varphi)| = |e^{-\varepsilon t}z(t,0,e^{\varepsilon\cdot}\varphi)| \leq e^{-\varepsilon t}||\varphi||_\infty.$$

Generalized Cohen-Grossberg Model

Cohen-Grossberg model with unbounded distributed delays (10)

$$\dot{x}_i(t) = -a_i(x_i(t)) \left[b_i(x_i(t)) + \sum_{j=1}^{n} \sum_{p=1}^{P} f_{ij}^{(p)} \left(\int_{-\infty}^{0} g_{ij}^{(p)}(x_j(t+s)) d\eta_{ij}^{(p)}(s) \right) \right] \quad (5$$

- $a_i : \mathbb{R} \to (0, +\infty)$, are continuous functions;

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- $\eta_{ij}^{(p)} : (-\infty, 0] \to \mathbb{R}$ are non-decreasing bounded normalized functions so that

$$\eta_{ij}^{(p)}(0) - \eta_{ij}^{(p)}(-\infty) = 1;$$

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$$\eta_{ij}^{(p)}(0) - \eta_{ij}^{(p)}(-\infty) = 1;$$

- $N := diag(\beta_1, \ldots, \beta_n) - [l_{ij}]$, where $l_{ij} = \sum_{p=1}^{P} \mu_{ij}^{(p)}\sigma_{ij}^{(p)}$.

Example

Cohen-Grossberg neural network with unbounded distributed delays

$$\dot{x}_i(t) = -a_i(x_i(t)) \left[b_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j \left(\int_{-\infty}^{0} k_{ij}(-s) x_j(t+s) ds \right) \right] \quad (6)$$

- $a_{ij} \in \mathbb{R}$, and $a_i : \mathbb{R} \to (0, +\infty)$, $b_i : \mathbb{R} \to \mathbb{R}$ are continuous functions and **(A1)** holds;

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- $f_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz with Lipschitz constant μ_j;

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- $a_{ij} \in \mathbb{R}$, and $a_i : \mathbb{R} \to (0, +\infty)$, $b_i : \mathbb{R} \to \mathbb{R}$ are continuous functions and **(A1)** holds;
- $f_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz with Lipschitz constant μ_j;
- The delay kernel functions $k_{ij} : [0, +\infty) \to \mathbb{R}_0^+$ satisfy

$$\int_0^{+\infty} k_{ij}(t)dt = 1;$$

Example

Cohen-Grossberg neural network with unbounded distributed delays

$$\dot{x}_i(t) = -a_i(x_i(t))\left[b_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j\left(\int_{-\infty}^{0} k_{ij}(-s)x_j(t+s)ds\right)\right] \quad (6)$$

- $a_{ij} \in \mathbb{R}$, and $a_i : \mathbb{R} \to (0, +\infty)$, $b_i : \mathbb{R} \to \mathbb{R}$ are continuous functions and **(A1)** holds;
- $f_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz with Lipschitz constant μ_j;
- The delay kernel functions $k_{ij} : [0, +\infty) \to \mathbb{R}_0^+$ satisfy

$$\int_0^{+\infty} k_{ij}(t)dt = 1;$$

- $N := diag(\beta_1, \ldots, \beta_n) - [l_{ij}]$, where $l_{ij} = |a_{ij}|\mu_j$.

- **Corollary**(GAS)
  If N is a non-singular M-matrix, then there is a unique equilibrium point of (6), which is globally asymptotically stable.

[1] L. Wang, Appl. Math. Comput. 160 (2005) 93-110.

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  System (5) reduces to (6) if $P = 1$, $f_{ij}^{(1)}(u) = a_{ij} f_j(u)$ and

$$\eta_{ij}^{(1)}(s) = \int_{-\infty}^{s} k_{ij}(-\zeta)d\zeta,$$

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  then the result follows from Theorem 3 **(a)**.
- In [1] assumed the additional conditions:
  f_j is bounded; $0 < \underline{a}_i \leq a_i(u) \leq \overline{a}_i$;
  The kernels functions satisfy $\int_0^\infty t k_{ij}(t) dt < \infty$
  $\underline{N} := B\underline{A} - \overline{A}[l_{ij}]$ is a non-singular M-matrix, where
  $\underline{A} = diag(\underline{a}_1, \ldots, \underline{a}_n)$, $\overline{A} = diag(\overline{a}_1, \ldots, \overline{a}_n)$.

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- **Corollary**(GES)
  Assume **(A0)** and that there is $\gamma > 0$ such that

$$\int_0^\infty k_{ij}(t) e^{\gamma t} dt < \infty.$$

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  Proof Analogous to the previous corollary

[2] W. Wu, B.T. Cui, X.Y. Lou, Math. Comput. Modelling, 47 (2008) 868-873.

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  Proof Analogous to the previous corollary
- In [2] assumed the additional conditions:
  $0 < \underline{a}_i \leq a_i(u) \leq \bar{a}_i$;
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Thank you