Global exponential stability criterion for a general system of difference equations with unbounded delays and applications to discrete-time neural network models

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Memory Systems

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- $x(t)$ feeling of the water temperature
- $\tau > 0$ delay time
- $F : \mathbb{R} \to \mathbb{R}$ reaction men on the temperature regulator

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▶

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Delay differential equations

- $x(t)$ feeling of the water temperature
- $\tau > 0$ delay time
- $F : \mathbb{R} \to \mathbb{R}$ man reaction on the temperature regulator

- This situation is modulated by the delay differential equation

$$x'(t) = F(x(t - \tau))$$

Delay differential equations

- For $\tau \in \mathbb{R}^+$ and $n \in \mathbb{N}$, consider

$$\mathcal{C} := C([-\tau, 0]; \mathbb{R}^n) = \left\{ \varphi : [-\tau, 0] \to \mathbb{R}^n \,\middle|\, \varphi \text{ is continuous} \right\}$$

with the norm

$$\|\varphi\| = \sup_{\theta \in [-\tau, 0]} |\varphi(\theta)|_{\mathbb{R}^n}.$$

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$$\|\varphi\| = \sup_{\theta \in [-\tau, 0]} |\varphi(\theta)|_{\mathbb{R}^n}.$$

- The normed space $(\mathcal{C}, \|\cdot\|)$ is a Banach space.

- For a continuous function $F:\mathbb{R}\times\mathcal{C}\to\mathbb{R}^n$, we call a delay differential equation to

$$x'(t)=F(t,x_t),\qquad t\geq 0. \tag{1}$$

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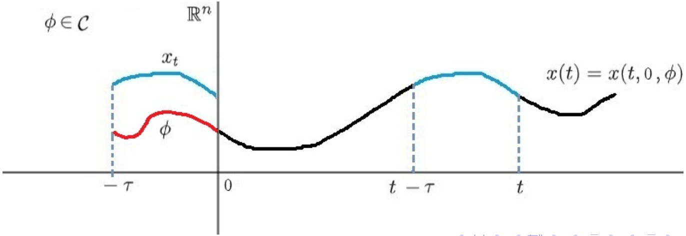
$$x'(t) = F(t, x_t), \qquad t \geq 0. \tag{1}$$

- For $b \in (0, +\infty]$, a continuous function $x : [-\tau, b) \to \mathbb{R}^n$ is a solution of (1) if it is differentiable for $t \geq 0$ and verifies (1) where, for each $t \in [0, b)$, the function $x_t \in \mathcal{C}$ is defined by

$$x_t(s) = x(t + s), \qquad \forall s \in [-\tau, 0].$$



- Thus $\mathcal{C} = C([-\tau, 0]; \mathbb{R}^n)$ is the phase space of (1).

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- For the initial value problem

$$\begin{cases} x'(t) = F(t, x_t) \\ x_0 = \phi \end{cases}, \qquad t \geq 0, \tag{2}$$

where $\phi \in \mathcal{C}$, the standard results about existence, uniqueness, continuation (for the future), and continuous dependence of solutions of (2) is available.

- J. Hale and V. Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, 1993

Infinite delay differential equations

- In case $\tau = +\infty$, the differential equation

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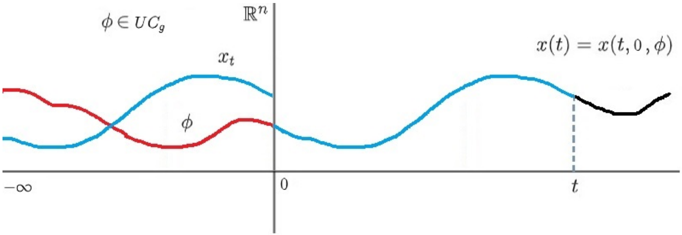
allowed us to consider all history of the solution.

- Here for $t \geq 0$ and a solution $x : [-\infty, b) \to \mathbb{R}^n$, we denote

$$x_t(s) = x(t+s), \qquad \forall s \in (-\infty, 0].$$



- To obtain the basic results of existence, uniqueness and continuations of solutions of infinite delay differential equations another phase space is needed.

[1] J. Hale and J. Kato, *Funkcialaj Ekvacioj*, 21 (1978) 11-41

- To obtain the basic results of existence, uniqueness and continuations of solutions of infinite delay differential equations another phase space is needed.
- Hale and Kato [1] introduced the Banach space

$$UC_g = \left\{ \phi \in C((-\infty,0];\mathbb{R}^n) : \sup_{s\leq 0} \frac{|\phi(s)|_{\mathbb{R}^n}}{g(s)} < \infty, \frac{\phi(s)}{g(s)} \text{ unif. cont.} \right\}$$

$$\|\phi\|_g = \sup_{s\leq 0} \frac{|\phi(s)|_{\mathbb{R}^n}}{g(s)}, \text{ with } |x|_{\mathbb{R}^n} \text{ a norm in } \mathbb{R}^n$$

where:

(g1) $g : (-\infty,0] \to [1,\infty)$ non-increasing, continuous, with $g(0)=1$;

(g2) $\lim_{u\to 0^-} \frac{g(s+u)}{g(s)} = 1$ uniformly on $(-\infty,0]$;

(g3) $g(s) \to \infty$ as $s \to -\infty$.

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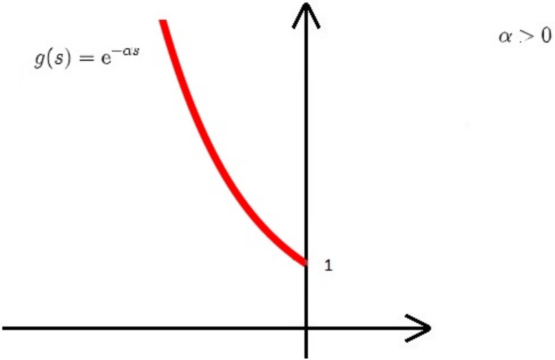
For $\alpha > 0$, the exponential function $g(s) = e^{-\alpha s}$ verifies **(g1)-(g3)**.



In this case, the norm in UC_g is

$$\|\phi\|_g = \sup_{s\leq 0} \frac{|\phi(s)|_{\mathbb{R}^n}}{g(s)} = \sup_{s\leq 0} |\phi(s)|_{\mathbb{R}^n} e^{\alpha s} \in [0, +\infty)$$

- In the phase space UC_g consider the initial value problem

$$\begin{cases} x'(t) = F(t, x_t) \\ x_0 = \phi \end{cases}, \qquad t \geq 0, \tag{3}$$

where $\phi \in UC_g$, and $F : \mathbb{R} \times UC_g \to \mathbb{R}^n$ is a continuous functions.

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- The standard results about existence, uniqueness, continuation (for the future), and continuous dependence of solutions of (3) is available.

- Y. Hino, S. Murakami, and T. Naito, *Functional Differential Equations with Infinite Delay*, Springer-Verlag, 1991

Hopfield Neural Network Models

- Pioneer Hopfield's work (1984)

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)), \qquad i = 1, \dots, n. \tag{4}$$

$n \in \mathbb{N}$ number of neurons, $t \geq 0$; $\sigma_{ij}(t) \geq 0$; $diag(a_1, \dots, a_n) > 0$ self-feedback matrix; I_i external inputs; $k_{ij}(s) \geq 0$ kernel functions; f_j activation functions; $[b_{ij}], [c_{ij}], [d_{ij}]$, connection matrices;

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- Generalized Hopfield neural network

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} f_j(x_j(t - \sigma_{ij}(t))) + \sum_{j=1}^{n} d_{ij} \int_{-\infty}^{0} k_{ij}(s) f_j(x_j(t+s)) ds + I_i(t), \tag{5}$$

$n \in \mathbb{N}$ number of neurons, $t \geq 0$; $\sigma_{ij}(t) \geq 0$; $diag(a_1, \ldots, a_n) > 0$ self-feedback matrix; I_i external inputs; $k_{ij}(s) \geq 0$ kernel functions; f_j activation functions; $[b_{ij}], [c_{ij}], [d_{ij}]$, connection matrices;

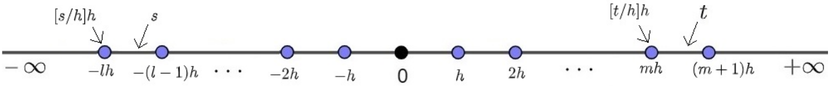
- We take the approximation of (5)

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j([t/h]h)) + \sum_{j=1}^{n} c_{ij} f_j\left(x_j\left([t/h]h - \left[\frac{\sigma_{ij}([t/h]h)}{h}\right] h\right)\right)$$
$$+ \sum_{j=1}^{n} d_{ij} \int_{-\infty}^{0} k_{ij}([s/h]h) f_j(x_j([t/h]h + [s/h]h)) ds + I_i([t/h]h),$$

for $t \in [mh, (m+1)h[$ and $m \in \mathbb{N}_0$, where

  - $[r]$ is the integer part of $r \in \mathbb{R}$;
  - $h > 0$ is the discretization step size;

- Time line



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for $t \in [mh, (m+1)h[$ and $m \in \mathbb{N}_0$, where
  - $[r]$ is the integer part of $r \in \mathbb{R}$;
  - $h > 0$ is the discretization step size;
- We have $[t/h] = m$, thus

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(mh)) + \sum_{j=1}^{n} c_{ij} f_j\left(x_j\left(mh - \left[\frac{\sigma_{ij}(mh)}{h}\right] h\right)\right)$$
$$+ \sum_{j=1}^{n} d_{ij} \int_{-\infty}^{0} k_{ij}([s/h]h) f_j(x_j(mh + [s/h]h))ds + I_i(mh),$$

- For $s \in]-lh, -(l-1)h]$ and $l \in \mathbb{N}$, we have $[s/h] = -l$

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$$+ \sum_{j=1}^{n} d_{ij} \sum_{l=1}^{\infty} (k_{ij}(-lh) f_j(x_j(mh - lh))h) + I_i(mh),$$

- Jumping computations and $t \to (m+1)h$ $t \in [mh, (m+1)h[$

$$x_i((m+1)h) = e^{-a_i h} x_i(mh) + \theta_i(h) I_i(mh) + \theta_i(h) \sum_{j=1}^{n} \Bigg(b_{ij} f_j(x_j(mh))$$

$$+ c_{ij} f_j \left(x_j \left((m - \tau_{ij}(m))h \right) \right) + d_{ij} \sum_{l=1}^{\infty} k_{ij}(-lh) f_j(x_j(mh - lh))h \Bigg),$$

where $\theta_i(h) = \frac{1 - e^{-a_i h}}{a_i}$ and $\tau_{ij}(m) = \left[\frac{\sigma_{ij}(mh)}{h} \right]$

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$$+ \sum_{j=1}^{n} d_{ij} \sum_{l=1}^{\infty} (k_{ij}(-lh) f_j(x_j(mh - lh))h) + I_i(mh),$$

- Identifying $mh \equiv m$ and $lh \equiv l$, we have

$$x_i(m+1) = \mathrm{e}^{-a_i h} x_i(m) + \theta_i(h) I_i(m) + \theta_i(h) \sum_{j=1}^{n} \Bigg(b_{ij} f_j(x_j(m))$$

$$+ c_{ij} f_j\left(x_j\left(m - \tau_{ij}(m)\right)\right) + d_{ij} \sum_{l=1}^{\infty} k_{ij}(-l) f_j(x_j(m-l))h\Bigg),$$

where $\theta_i(h) = \frac{1-\mathrm{e}^{-a_i h}}{a_i}$ and $\tau_{ij}(m) = \left[\frac{\sigma_{ij}(mh)}{h}\right]$

Low-order Hopfield neural network model

$$x_i(m+1) = a_i x_i(m) + \sum_{j=1}^{n} b_{ij} f_j(x_j(m)) + \sum_{j=1}^{n} c_{ij} f_j\left(x_j\left(m-\tau_{ij}(m)\right)\right)$$
$$+ \sum_{j=1}^{n} d_{ij} \sum_{l=1}^{\infty} \rho_{ijl} f_j(x_j(m-l)), \; m \in \mathbb{N}_0 \qquad (6)$$

$i = 1, \dots, n$, with $n \in \mathbb{N}$ and

- $a_i \in]-1,1[$;
- $b_{ij}, c_{ij}, d_{ij} \in \mathbb{R}$;
- $f_j : \mathbb{R} \to \mathbb{R}$, $\tau_{ij} : \mathbb{N}_0 \to \mathbb{N}_0$;
- $(\rho_{ijl})_{l\in\mathbb{N}}$ non-negative sequence with $\sum_{l=1}^{\infty} \rho_{ijl} < \infty$.

High-order Hopfield neural network model

$$
\begin{aligned}
x_i(m+1) \quad = \quad & a_i x_i(m) + \sum_{j=1}^{n} b_{ij} f_j(x_j(m)) \\
& + \sum_{j=1}^{n}\sum_{k=1}^{n} c_{ijk} g_j\left(x_j\left(m-\tau_{ijk}(m)\right)\right) g_k\left(x_k\left(m-\tau_{ijk}(m)\right)\right) \\
& + \sum_{j=1}^{n}\sum_{k=1}^{n} d_{ijk}\left(\sum_{l=1}^{\infty}\rho_{ijl} g_j(x_j(m-l))\right)\left(\sum_{l=1}^{\infty}\rho_{ijl} g_k(x_k(m-l))\right),
\end{aligned}
\tag{7}
$$

$m \in \mathbb{N}_0$, $i = 1, \dots, n$, with $n \in \mathbb{N}$ and

- $a_i \in]-1,1[$;
- $b_{ijk}, c_{ijk}, d_{ijk} \in \mathbb{R}$;
- $f_j, g_j : \mathbb{R} \to \mathbb{R}$, $\tau_{ijk} : \mathbb{N}_0 \to \mathbb{N}_0$;
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GOAL

Establish sufficient conditions for the global exponential stability of discrete-time Hopfield models (6) and (7).

General system of delay difference equations

$$x_i(m+1) = \mathcal{F}_i(m, \overline{x}_m), \quad m \in \mathbb{N}_0,\ i = 1, \dots, n,$$

where $n \in \mathbb{N}$ and

- X^n_α convenient phase space of sequences in $\mathbb{R}^n$;
- $\overline{\mathcal{F}} : \mathbb{N}_0 \times X^n_\alpha \to \mathbb{R}^n$ with $\overline{\mathcal{F}}(m, \overline{\varphi}) = (\mathcal{F}_1(m, \overline{\varphi}), \dots, \mathcal{F}_n(m, \overline{\varphi}))$;
- For $m \in \mathbb{N}_0$, $\overline{x}_m$ is a sequence in $\mathbb{R}^n$ which gives the historical information of the solution from $-\infty$ until m.

- Given $\alpha > 0$, we define X_α^n as the space of sequences

$$\begin{array}{rccl} \overline{\varphi}: & \mathbb{Z}_0^- & \to & \mathbb{R}^n \\ & j & \mapsto & (\varphi_1(j), \ldots, \varphi_n(j)) \end{array}$$

such that

$$\max_{i\in 1,\ldots,n} \left(\sup_{j\in\mathbb{Z}_0^-} |\varphi_i(j)|\, e^{\alpha j} \right) < \infty.$$

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  such that

$$\max_{i \in 1,\ldots,n} \left(\sup_{j \in \mathbb{Z}_0^-} |\varphi_i(j)| \, e^{\alpha j} \right) < \infty.$$

- Consider X_α^n the normed space with the norm

$$\|\overline{\varphi}\|_\alpha = \max_{i \in 1,\ldots,n} \left(\sup_{j \in \mathbb{Z}_0^-} |\varphi_i(j)| \, e^{\alpha j} \right), \quad \overline{\varphi} \in X_\alpha^n.$$

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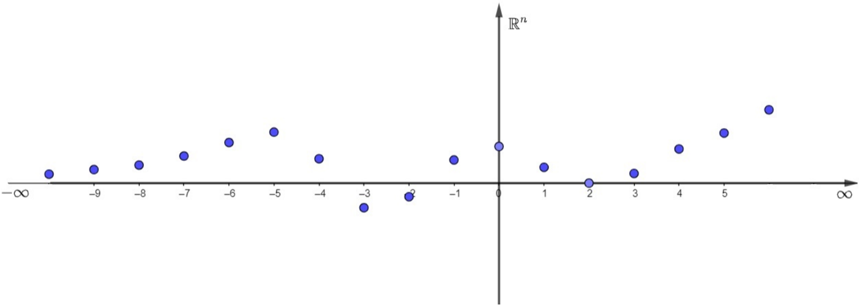
- Consider $\overline{x} : \mathbb{Z} \to \mathbb{R}^n$ with $\sup_{j \in \mathbb{Z}_0^-} |\overline{x}(j)|_\infty \, e^{\alpha j} < \infty$,

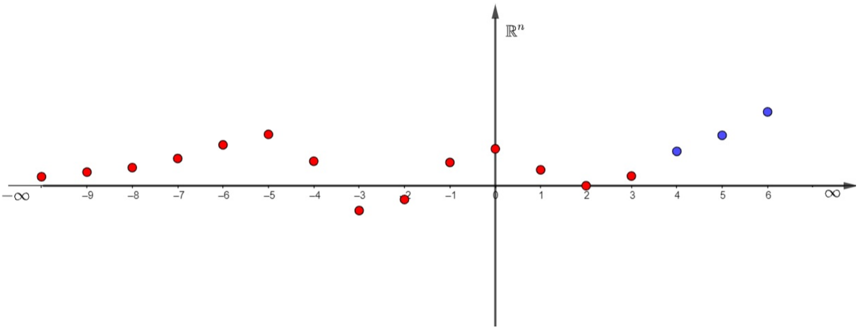
For $m \in \mathbb{N}_0$, we define $\overline{x}_m \in X_\alpha^n$ by

$$\overline{x}_m(j) = \overline{x}(m + j), \qquad j \in \mathbb{Z}_0^-.$$

Let $\overline{x} : \mathbb{Z} \to \mathbb{R}^n$



For $m = 3$ the graph of $\overline{x}_3 \in X_\alpha^n$ is



$$\begin{array}{rccc} \overline{x}_3 : & \mathbb{Z}_0^- & \to & \mathbb{R}^n \\ & j & \mapsto & \overline{x}(3+j) \end{array}$$

- Consider the delay difference system

$$x_i(m+1) = \mathcal{F}_i(m, \overline{x}_m), \qquad m \in \mathbb{N}_0,\ i = 1, \dots, n, \tag{8}$$

where
$\overline{\mathcal{F}} : \mathbb{N}_0 \times X_\alpha^n \to \mathbb{R}^n$ with $\overline{\mathcal{F}}(m, \overline{\varphi}) = (\mathcal{F}_1(m, \overline{\varphi}), \dots, \mathcal{F}_n(m, \overline{\varphi}))$

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- The initial condition

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- We denote by $\overline{x}(\cdot, 0, \overline{\varphi})$ the unique solution

$$\overline{x} : \mathbb{Z} \to \mathbb{R}^n$$

of (8)-(9).

Main stability result

- **Theorem 1** If

$$|\mathcal{F}_i(m,\overline{\varphi})| \leq \mathrm{e}^{-\alpha} \, \|\overline{\varphi}\|_\alpha, \tag{10}$$

for all $\overline{\varphi} \in X_\alpha^n$, $m \in \mathbb{N}_0$, $i = 1, \ldots, n$, then the zero solution of (8)

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is globally exponentially stable,

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is globally exponentially stable,

- That is

$$\|\overline{x}_m\|_\alpha \leq \mathrm{e}^{-\alpha m} \|\overline{\varphi}\|_\alpha, \qquad \forall m \in \mathbb{N}_0. \tag{11}$$

- **Proof:** By induction we prove $\|\overline{x}_m\|_\alpha \leq e^{-\alpha m}\,\|\overline{\varphi}\|_\alpha$, $\forall m \in \mathbb{N}_0$

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- For $m = 0$ is trivial.
- For $m \in \mathbb{N}_0$, assume

$$\|\overline{x}_r\|_\alpha \leq \mathrm{e}^{-\alpha r}\|\overline{\varphi}\|_\alpha, \qquad 0 \leq r \leq m.$$

For all $i = 1, \dots, n$, by induction hypotheses and (10)

$$|x_i(m+1)| = |\mathcal{F}_i(m, \overline{x}_m)| \leq \mathrm{e}^{-\alpha}\|\overline{x}_m\|_\alpha \leq \mathrm{e}^{-\alpha(m+1)}\|\overline{\varphi}\|_\alpha$$

- **Proof:** By induction we prove $\|\overline{x}_m\|_\alpha \leq e^{-\alpha m}\|\overline{\varphi}\|_\alpha$, $\forall m \in \mathbb{N}_0$
- For $m = 0$ is trivial.
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For all $i = 1, \dots, n$, by induction hypotheses and (10)

$$|x_i(m+1)| = |\mathcal{F}_i(m, \overline{x}_m)| \leq e^{-\alpha}\|\overline{x}_m\|_\alpha \leq e^{-\alpha(m+1)}\|\overline{\varphi}\|_\alpha$$

- thus

$$\begin{aligned}
\|\overline{x}_{m+1}\|_\alpha &= \max_i \left\{ \sup_{j \leq -m-1} |x_i(m+1+j)| e^{\alpha j}, \max_{-m \leq j \leq 0} |x_i(m+1+j)| e^{\alpha j} \right\} \\
&\leq \max_i \left\{ \sup_{j \leq -m-1} |\varphi_i(j+m+1)| e^{\alpha j}, \max_{-m \leq j \leq 0} e^{-\alpha(m+1+j)+\alpha j}\|\overline{\varphi}\|_\alpha \right\} \\
&= \max_i \left\{ \sup_{j \leq 0} |\varphi_i(j)| e^{\alpha(j-m-1)}, e^{-\alpha(m+1)}\|\overline{\varphi}\|_\alpha \right\} \\
&= e^{-\alpha(m+1)}\|\overline{\varphi}\|_\alpha.
\end{aligned}$$

Low-order discrete-time Hopfield model

- Consider the Low-order Hopfield model (6)

$$x_i(m+1) = a_i x_i(m) + \sum_{j=1}^{n} b_{ij} f_j(x_j(m)) + \sum_{j=1}^{n} c_{ij} f_j\left(x_j\left(m - \tau_{ij}(m)\right)\right) + \sum_{j=1}^{n} d_{ij} \sum_{l=1}^{\infty} \rho_{ijl} f_j(x_j(m-l)), \;\; m \in \mathbb{N}_0$$

with $a_i \in]-1,1[$, $b_{ij}, c_{ij}, d_{ij} \in \mathbb{R}$, $\rho_{ijl} \geq 0$ and the hypothesis

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- (H3) $\exists \xi > 0$ such that $\sum_{l=1}^{\infty} e^{\xi l} \rho_{ijl} < \infty$ and $\sum_{l=1}^{\infty} \rho_{ijl} = 1$; $\rho_{ijl} \geq 0$

- **Theorem 2** Assume (H1)-(H3). If

$$\mathcal{M} = diag(1-|a_1|,\dots,1-|a_n|) - \left[F_j(|b_{ij}|+|c_{ij}|+|d_{ij}|)\right]$$

is a non-singular M-matrix, then the zero solution of (6) is globally exponentially stable.
That is there are $C \geq 1$ and $\alpha > 0$:

$$\|\overline{x}_m(\cdot,0,\overline{\varphi})\|_\alpha \leq C\, \mathrm{e}^{-\alpha m}\, \|\overline{\varphi}\|_\alpha, \qquad \forall\, \overline{\varphi} \in X^n_\alpha,\ \forall m \in \mathbb{N}_0.$$

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- **Lemma:** Assume (H3).
If $\gamma > 0$, then there is $\eta > 0$ such that

$$\sum_{l=1}^{\infty}\mathrm{e}^{tl}\,\rho_{ijl} < 1+\gamma, \qquad \forall t\in[0,\eta],\ i,j=1,\ldots,n.$$

- **Proof of Theorem 2:** $\mathcal{M}$ is a non-singular M-matrix, thus there is $\overline{p} = (p_1, \dots, p_n) \in \mathbb{R}^n_+$ such that $\mathcal{M}\overline{p}^T > 0$, i.e.

$$p_i - p_i|a_i| - \sum_{j=1}^{n} p_j F_j(|b_{ij}| + |c_{ij}| + |d_{ij}|) > 0, \qquad i = 1, \dots, n.$$

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- Unless a change of variables, assume $\overline{p}=(1,\dots,1)$, that is

$$1 - |a_i| - \sum_{j=1}^{n} F_j(|b_{ij}|+|c_{ij}|+|d_{ij}|) > 0, \qquad i=1,\dots,n.$$

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- Thus there is $\gamma>0$ such that

$$e^{-\gamma}-|a_i|-\sum_{j=1}^n F_j(|b_{ij}|+|c_{ij}|\,e^{\gamma\tau}+|d_{ij}|(1+\gamma))>0,$$

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$$\mathrm{e}^{-\gamma}-|a_i|-\sum_{j=1}^{n}F_j(|b_{ij}|+|c_{ij}|\,\mathrm{e}^{\gamma\tau}+|d_{ij}|(1+\gamma))>0,$$

- By previous Lemma, there is $\alpha\in]0,\gamma[$ such that

$$\sum_{l=1}^{\infty}\mathrm{e}^{\alpha l}\rho_{ijl}<1+\gamma,\qquad\forall i,j=1,\dots,n,$$

- As $0 < \alpha < \gamma$ we obtain

$$e^{-\alpha} > |a_i| + \sum_{j=1}^{n} F_j\big(|b_{ij}| + |c_{ij}|\, e^{\alpha\tau} + |d_{ij}|(1+\gamma)\big),\ i = 1, \dots, n. \quad (12)$$

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- Model (6), in the phase space X_α^n, as the form

$$x_i(m+1) = \mathcal{F}_i(m, \overline{x}_m), \qquad i = 1, \dots, n,$$

with

$$\mathcal{F}_i(m, \overline{\varphi}) = a_i\varphi_i(0) + \sum_{j=1}^{n} b_{ij} f_j(\varphi_j(0)) + \sum_{j=1}^{n} c_{ij} f_j(\varphi_j(-\tau_{ij}(m)))$$

$$+ \sum_{j=1}^{n} d_{ij} \sum_{l=1}^{\infty} \rho_{ijl} f_j(\varphi_j(-l))$$

- As $0<\alpha<\gamma$ we obtain

$$e^{-\alpha} > |a_i| + \sum_{j=1}^{n} F_j\big(|b_{ij}| + |c_{ij}|\, e^{\alpha\tau} + |d_{ij}|(1+\gamma)\big), \; i=1,\dots,n.$$

- By hypothesis (H1): $|f_j(u)| \le F_j|u|$

$$\begin{aligned}
|\mathcal{F}_i(m,\overline{\varphi})| \quad &\le \quad |a_i\varphi_i(0)| + \sum_{j=1}^{n}|b_{ij}||f_j(\varphi_j(0))| + \sum_{j=1}^{n}|c_{ij}||f_j(\varphi_j(-\tau_{ij}(m)))| \\
&\quad + \sum_{j=1}^{n}|d_{ij}|\sum_{l=1}^{\infty}\rho_{ijl}|f_j(\varphi_j(-l))| \\
&\le \quad |a_i|\|\overline{\varphi}\|_\alpha + \sum_{j=1}^{n}F_j\Bigg(|b_{ij}|\|\overline{\varphi}\|_\alpha + |c_{ij}|\frac{|\varphi_j(-\tau_{ij}(m))|\, e^{-\alpha\tau_{ij}(m)}}{e^{-\alpha\tau_{ij}(m)}} \\
&\quad + |d_{ij}|\sum_{l=1}^{\infty}\rho_{ijl}\frac{|\varphi_j(-l)|\, e^{-\alpha l}}{e^{-\alpha l}}\Bigg)
\end{aligned}$$

- Thus, $\left(\text{recall } \sum_{l=1}^{\infty} e^{\alpha l}\rho_{ijl} < 1+\gamma\right)$

$$\begin{aligned}|\mathcal{F}_i(m,\overline{\varphi})| &\leq |a_i|\|\overline{\varphi}\|_\alpha + \sum_{j=1}^{n} F_j\Bigg(|b_{ij}|\|\overline{\varphi}\|_\alpha + \frac{|c_{ij}|}{e^{-\alpha\tau}}\|\overline{\varphi}\|_\alpha \\ &\qquad + |d_{ij}|\sum_{l=1}^{\infty} e^{\alpha l}\rho_{ijl}\|\overline{\varphi}\|_\alpha\Bigg) \\ &\leq \left(|a_i| + \sum_{j=1}^{n} F_j(|b_{ij}| + |c_{ij}|\, e^{\alpha\tau} + |d_{ij}|(1+\gamma))\right)\|\overline{\varphi}\|_\alpha\end{aligned}$$

- Thus, $\left(\text{recall } \sum_{l=1}^{\infty} e^{\alpha l} \rho_{ijl} < 1+\gamma\right)$

$$\begin{aligned}|\mathcal{F}_i(m,\overline{\varphi})| &\leq |a_i|\|\overline{\varphi}\|_\alpha + \sum_{j=1}^{n} F_j\Bigg(|b_{ij}|\|\overline{\varphi}\|_\alpha + \frac{|c_{ij}|}{e^{-\alpha\tau}}\|\overline{\varphi}\|_\alpha \\ &\qquad + |d_{ij}|\sum_{l=1}^{\infty} e^{\alpha l}\rho_{ijl}\|\overline{\varphi}\|_\alpha\Bigg) \\ &\leq \left(|a_i| + \sum_{j=1}^{n} F_j(|b_{ij}| + |c_{ij}|\, e^{\alpha\tau} + |d_{ij}|(1+\gamma))\right)\|\overline{\varphi}\|_\alpha\end{aligned}$$

- and from (12) we obtain

$$|\mathcal{F}_i(m,\overline{\varphi})| \leq e^{-\alpha}\|\overline{\varphi}\|_\alpha,$$

and the conclusion follows from Theorem 1.

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- Consider the High-order Hopfield model (7)

$$\begin{aligned} x_i(m+1) &= a_i x_i(m) + \sum_{j=1}^{n} b_{ij} f_j(x_j(m)) \\ &+ \sum_{j=1}^{n}\sum_{k=1}^{n} c_{ijk} g_j\left(x_j\left(m-\tau_{ijk}(m)\right)\right) g_k\left(x_k\left(m-\tau_{ijk}(m)\right)\right) \\ &+ \sum_{j=1}^{n}\sum_{k=1}^{n} d_{ijk}\left(\sum_{l=1}^{\infty}\rho_{ijl} g_j(x_j(m-l))\right)\left(\sum_{l=1}^{\infty}\rho_{ijl} g_k(x_k(m-l))\right), \end{aligned}$$

with $a_i \in]-1,1[$, $b_{ij}, c_{ijk}, d_{ijk} \in \mathbb{R}$, $\rho_{ijl} \geq 0$ and the hypothesis (H1), (H2), (H3), and

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- Consider the High-order Hopfield model (7)

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  with $a_i \in]-1,1[$, $b_{ij}, c_{ijk}, d_{ijk} \in \mathbb{R}$, $\rho_{ijl} \geq 0$ and the hypothesis (H1), (H2), (H3), and
- (H4) $\exists G_j, M_j > 0$ such that

$$|g_j(u)| \leq \min\{M_j, G_j|u|\}.$$

- **Theorem 3** Assume (H1)-(H4). If

$$diag(1-|a_1|,\dots,1-|a_n|)-[F_j(|b_{ij}|]-\left[G_j\sum_{k=1}^{n}M_k(|c_{ijk}|+|d_{ijk}|)\right]$$

is a non-singular M-matrix, then the zero solution of

$$\begin{aligned} x_i(m+1) &= a_i x_i(m)+\sum_{j=1}^{n} b_{ij} f_j(x_j(m)) \\ &+\sum_{j=1}^{n}\sum_{k=1}^{n} c_{ijk} g_j\left(x_j\left(m-\tau_{ijk}(m)\right)\right) g_k\left(x_k\left(m-\tau_{ijk}(m)\right)\right) \\ &+\sum_{j=1}^{n}\sum_{k=1}^{n} d_{ijk}\left(\sum_{l=1}^{\infty}\rho_{ijl} g_j(x_j(m-l))\right)\left(\sum_{l=1}^{\infty}\rho_{ijl} g_k(x_k(m-l))\right), \end{aligned}$$

is globally exponentially stable.

- In [2], the global exponential stability of the zero solution of

$$\begin{aligned} x_i(m+1) &= a_i x_i(m) + \sum_{j=1}^{n} b_{ij} f_j(x_j(m)) \\ &\quad + \sum_{j=1}^{n}\sum_{k=1}^{n} c_{ijk} g_j\left(x_j\left(m-\tau_{ijk}(m)\right)\right) g_k\left(x_k\left(m-\tau_{ijk}(m)\right)\right), \end{aligned}$$

  was obtained assuming that:

  - $diag(1-|a_1|,\dots,1-|a_n|) - \left[F_j|b_{ij}|\right] - \left[G_j \sum_{k=1}^{n} M_k |c_{ijk}|\right]$

    is a non-singular M-matrix;
  - τ_{ijk} are bounded;
  - $|g_j(u)| \leq M_j$
  - $|f_j(u)-f_j(v)| \leq F_j|u-v|,\ |g_j(u)-g_j(v)| \leq G_j|u-v|,$
    $\forall u, v \in \mathbb{R},$
    and $f_j(0)=0,\ g_j(0)=0.$

[2] Z.Dong, X. Wang, and X. Zhang, *Appl. Math. Comput.* 385 (2020) p.125401

- Numerical example

$$\begin{cases} x_1'(t) = -10x_1(t) + 2\tanh(x_2(t-1)) + 15\displaystyle\int_{-\infty}^{0} 4^s \tanh(x_2(t+s))ds \\ x_2'(t) = -10x_2(t) + \tanh(x_1(t-3)) + 2\displaystyle\int_{-\infty}^{0} 2^s \tanh(x_1(t+s))ds \end{cases}$$

- Numerical example

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- After the discretization process, we obtain

$$\left\{ \begin{array}{rcl} x_1(m+1) & = & e^{-10}\, x_1(m) + \dfrac{1-e^{-10}}{10} \\ & & \cdot \left(2\tanh(x_2(m-1)) + 5\displaystyle\sum_{l=1}^{\infty} \frac{3}{4^l} \tanh(x_2(m-l)) \right) \\ x_2(m+1) & = & e^{-10}\, x_2(m) + \dfrac{1-e^{-10}}{10} \\ & & \cdot \left(\tanh(x_1(m-3)) + 2\displaystyle\sum_{l=1}^{\infty} \frac{1}{2^l} \tanh(x_1(m-l)) \right) \end{array} \right. . \quad (13)$$

We have

$$\mathcal{M} = \begin{bmatrix} 1 - e^{-10} & -\frac{7(1-e^{-10})}{10} \\ -\frac{3(1-e^{-10})}{10} & 1 - e^{-10} \end{bmatrix}$$

which is a non-singular M-matrix, thus the zero solution of (13) is globally exponentially stable.

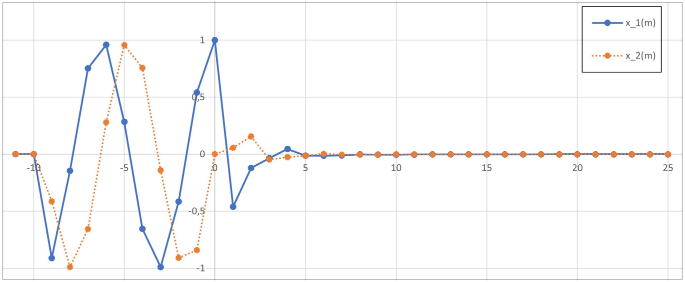

Figure: Solution $(x_1(t), x_2(t))$ of system (13) with initial condition

$$\overline{x}_0(j) = \begin{cases} (\cos(j), \sin(j)), & j = -9, \dots, 0 \\ (0,0), & j \in]-\infty, -10] \cap \mathbb{Z} \end{cases}$$

Thank you

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