# Global exponential stability of discrete-time Hopfield neural network models with unbounded delays

José J. Oliveira

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Centro de Matemática (CMAT) Departamento de Matemática da Universidade do Minho

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# **Hopfield Neural Network Models**

Pioneer Hopfield's work (1984)

$$\kappa'_i(t) = -a_i x_i(t) + \sum_{j=1}^n b_{ij} f_j(x_j(t)), \quad i = 1, ..., n.$$
 (1)

 $n \in \mathbb{N}$  number of neurons,  $t \ge 0$ ;  $\sigma_{ij}(t) \ge 0$ ;  $diag(a_1, \ldots, a_n) > 0$ self-feedback matrix;  $I_i$  external inputs;  $k_{ij}(s) \ge 0$  kernel functions;  $f_j$  activation functions;  $[b_{ij}], [c_{ij}], [d_{ij}]$ , connection matrices;

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# Hopfield Neural Network Models

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 (1)

Generalized Hopfield neural network

$$\begin{aligned} \kappa_{i}'(t) &= -a_{i}x_{i}(t) + \sum_{j=1}^{n} b_{ij}f_{j}(x_{j}(t)) + \sum_{j=1}^{n} c_{ij}f_{j}(x_{j}(t-\sigma_{ij}(t))) \\ &+ \sum_{j=1}^{n} d_{ij}\int_{-\infty}^{0} k_{ij}(s)f_{j}(x_{j}(t+s))ds + I_{i}(t), \end{aligned}$$
(2)

 $n \in \mathbb{N}$  number of neurons,  $t \ge 0$ ;  $\sigma_{ij}(t) \ge 0$ ;  $diag(a_1, \ldots, a_n) > 0$ self-feedback matrix;  $I_i$  external inputs;  $k_{ij}(s) \ge 0$  kernel functions;  $f_j$  activation functions;  $[b_{ij}], [c_{ij}], [d_{ij}]$ , connection matrices;

We take the approximation of (2)

$$egin{aligned} &x_i'(t) = -a_i x_i(t) + \sum_{j=1}^n b_{ij} f_j(x_j([t/h]h)) + \sum_{j=1}^n c_{ij} f_j\left(x_j\left([t/h]h - \left[rac{\sigma_{ij}([t/h]h)}{h}
ight]h
ight) + \sum_{j=1}^n d_{ij} \int_{-\infty}^0 k_{ij}([s/h]h) f_j(x_j([t/h]h + [s/h]h)) ds + I_i([t/h]h), \end{aligned}$$

for  $t \in [mh, (m+1)h[$  and  $m \in \mathbb{N}_0$ , where

- [r] is the integer part of  $r \in \mathbb{R}$ ;
- *h* > 0 is the discretization step size;

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Hopfield discrete-time models

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- [r] is the integer part of  $r \in \mathbb{R}$ ;
- *h* > 0 is the discretization step size;

• We have [t/h] = m, thus

$$egin{split} x_i'(t) &= -a_i x_i(t) + \sum_{j=1}^n b_{ij} f_j(x_j(mh)) + \sum_{j=1}^n c_{ij} f_j\left(x_j\left(mh - \left[rac{\sigma_{ij}(mh)}{h}
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ight) \ &+ \sum_{j=1}^n d_{ij} \int_{-\infty}^0 k_{ij}([s/h]h) f_j(x_j(mh + [s/h]h)) ds + I_i(mh), \end{split}$$

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▶ For 
$$s \in ] - lh, -(l-1)h]$$
 and  $l \in \mathbb{N}$ , we have  $[s/h] = -lh$ 

$$\begin{aligned} x_i'(t) &= -a_i x_i(t) + \sum_{j=1}^n b_{ij} f_j(x_j(mh)) + \sum_{j=1}^n c_{ij} f_j\left(x_j\left(mh - \left[\frac{\sigma_{ij}(mh)}{h}\right]h\right)\right) \\ &+ \sum_{j=1}^n d_{ij} \sum_{l=1}^\infty \left(k_{ij}(-lh) f_j(x_j(mh - lh))h\right) + I_i(mh), \end{aligned}$$

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▶ For 
$$s \in ] - lh, -(l-1)h]$$
 and  $l \in \mathbb{N}$ , we have  $[s/h] = -lh$ 

$$egin{aligned} & x_i'(t) = -a_i x_i(t) + \sum_{j=1}^n b_{ij} f_j(x_j(mh)) + \sum_{j=1}^n c_{ij} f_j\left(x_j\left(mh - \left[rac{\sigma_{ij}(mh)}{h}
ight]h
ight)
ight) \ & + \sum_{j=1}^n d_{ij} \sum_{l=1}^\infty \left(k_{ij}(-lh) f_j(x_j(mh-lh))h
ight) + I_i(mh), \end{aligned}$$

▶ Jumping computations and  $t \rightarrow (m+1)h$   $t \in [mh, (m+1)h[$ 

$$x_{i}((m+1)h) = e^{-a_{i}h} x_{i}(mh) + \theta_{i}(h)I_{i}(mh) + \theta_{i}(h)\sum_{j=1}^{n} \left(b_{ij}f_{j}(x_{j}(mh)) + c_{ij}f_{j}(x_{j}((m-\tau_{ij}(m))h)) + d_{ij}\sum_{l=1}^{\infty} k_{ij}(-lh)f_{j}(x_{j}(mh-lh))h\right),$$

where 
$$heta_i(h) = rac{1-\mathrm{e}^{-a_i h}}{a_i}$$
 and  $au_{ij}(m) = \left\lfloor rac{\sigma_{ij}(mh)}{h} 
ight
ceil$ 

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$$\begin{aligned} x_i'(t) &= -a_i x_i(t) + \sum_{j=1}^n b_{ij} f_j(x_j(mh)) + \sum_{j=1}^n c_{ij} f_j\left(x_j\left(mh - \left[\frac{\sigma_{ij}(mh)}{h}\right]h\right)\right) \\ &+ \sum_{j=1}^n d_{ij} \sum_{l=1}^\infty \left(k_{ij}(-lh) f_j(x_j(mh - lh))h\right) + I_i(mh), \end{aligned}$$

• Identifying  $mh \equiv m$  and  $lh \equiv l$ , we have

$$\begin{aligned} x_i(m+1) &= e^{-a_i h} x_i(m) + \theta_i(h) I_i(m) + \theta_i(h) \sum_{j=1}^n \left( b_{ij} f_j(x_j(m)) + c_{ij} f_j(x_j(m-\tau_{ij}(m))) + d_{ij} \sum_{l=1}^\infty k_{ij}(-l) f_j(x_j(m-l))h \right), \\ &+ c_{ij} f_j(x_j(m-\tau_{ij}(m))) + d_{ij} \sum_{l=1}^\infty k_{ij}(-l) f_j(x_j(m-l))h \right), \end{aligned}$$
where  $\theta_i(h) = \frac{1 - e^{-a_i h}}{a_i}$  and  $\tau_{ij}(m) = \left[ \frac{\sigma_{ij}(mh)}{h} \right]$ 

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## Low-order Hopfield neural network model

$$\begin{aligned} x_{i}(m+1) &= a_{i}x_{i}(m) + \sum_{j=1}^{n} b_{ij}f_{j}(x_{j}(m)) + \sum_{j=1}^{n} c_{ij}f_{j}(x_{j}(m-\tau_{ij}(m))) \\ &+ \sum_{j=1}^{n} d_{ij}\sum_{l=1}^{\infty} \rho_{ijl}f_{j}(x_{j}(m-l)), \ m \in \mathbb{N}_{0} \end{aligned}$$
(3)

$$i=1,\ldots,n$$
, with  $n\in\mathbb{N}$  and

• 
$$a_i \in ]-1,1[;$$

► 
$$b_{ij}, c_{ij}, d_{ij} \in \mathbb{R}$$
;

• 
$$f_j : \mathbb{R} \to \mathbb{R}, \ \tau_{ij} : \mathbb{N}_0 \to \mathbb{N}_0;$$

•  $(\rho_{ijl})_{l \in \mathbb{N}}$  non-negative sequence with  $\sum_{l=1}^{n} \rho_{ijl} < \infty$ .

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# High-order Hopfield neural network model

$$\begin{aligned} x_{i}(m+1) &= a_{i}x_{i}(m) + \sum_{j=1}^{n} b_{ij}f_{j}(x_{j}(m)) \\ &+ \sum_{j=1}^{n} \sum_{k=1}^{n} c_{ijk}g_{j}(x_{j}(m-\tau_{ijk}(m)))g_{k}(x_{k}(m-\tau_{ijk}(m))) \\ &+ \sum_{j=1}^{n} \sum_{k=1}^{n} d_{ijk}\left(\sum_{l=1}^{\infty} \rho_{ijl}g_{j}(x_{j}(m-l))\right)\left(\sum_{l=1}^{\infty} \rho_{ijl}g_{k}(x_{k}(m-l))\right), \end{aligned}$$
(4)

 $m \in \mathbb{N}_0$ ,  $i = 1, \ldots, n$ , with  $n \in \mathbb{N}$  and

▶ 
$$a_i \in ]-1, 1[;$$

► 
$$b_{ijk}, c_{ijk}, d_{ijk} \in \mathbb{R};$$

• 
$$f_j, g_j : \mathbb{R} \to \mathbb{R}, \ \tau_{ijk} : \mathbb{N}_0 \to \mathbb{N}_0;$$

•  $(\rho_{ijl})_{l \in \mathbb{N}}$  non-negative sequence with  $\sum \rho_{ijl} < \infty$ .

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# GOAL

Establish sufficient conditions for the global exponential stability of discrete-time Hopfield models (3) and (4).

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General system of delay difference equations

$$x_i(m+1) = \mathcal{F}_i(m, \overline{x}_m), \quad m \in \mathbb{N}_0, i = 1, \dots, n,$$

where  $n \in \mathbb{N}$  and

•  $X_{\alpha}^{n}$  convenient phase space of sequences in  $\mathbb{R}^{n}$ ;

$$\overline{\mathcal{F}}: \mathbb{N}_0 \times X_{\alpha}^n \to \mathbb{R}^n \text{ with } \\ \overline{\mathcal{F}}(m, \overline{\varphi}) = (\mathcal{F}_1(m, \overline{\varphi}), \dots, \mathcal{F}_n(m, \overline{\varphi}));$$

For  $m \in \mathbb{N}_0$ ,  $\overline{x}_m$  is a sequence in  $\mathbb{R}^n$  which gives the historical information of the solution from  $-\infty$  until m.

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General system and Phase Space Exponential Stability

• For  $\alpha > 0$  we define  $X_{\alpha}^{n}$  as the space of the functions

$$\begin{array}{rcl} \overline{\varphi} : & \mathbb{Z}_0^- & \to & \mathbb{R}^n \\ & j & \mapsto & (\varphi_1(j), \dots, \varphi_n(j)) \end{array}$$

such that

$$\max_{i\in 1,\ldots,n}\left(\sup_{j\in\mathbb{Z}_0^-}|\varphi_i(j)|\,\mathrm{e}^{\alpha j}\right)<\infty.$$

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such that

$$\max_{i\in 1,\ldots,n} \left( \sup_{j\in \mathbb{Z}_0^-} |\varphi_i(j)| \, \mathrm{e}^{\alpha j} \right) < \infty.$$

• Consider  $X_{\alpha}^{n}$  the normed space with the norm

$$\|\overline{\varphi}\|_{\alpha} = \max_{i \in 1,...,n} \left( \sup_{j \in \mathbb{Z}_0^-} |\varphi_i(j)| e^{\alpha j} \right), \quad \overline{\varphi} \in X_{\alpha}^n.$$

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• For  $\alpha > 0$  we define  $X_{\alpha}^{n}$  as the space of the functions

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such that

$$\max_{i\in 1,\ldots,n}\left(\sup_{j\in\mathbb{Z}_0^-}|\varphi_i(j)|\,\mathrm{e}^{\alpha j}\right)<\infty.$$

• Consider  $X_{\alpha}^{n}$  the normed space with the norm

$$\|\overline{\varphi}\|_{lpha} = \max_{i \in 1,...,n} \left( \sup_{j \in \mathbb{Z}_0^-} |\varphi_i(j)| e^{lpha j} \right), \quad \overline{\varphi} \in X_{lpha}^n.$$

► Consider  $\overline{x} : \mathbb{Z} \to \mathbb{R}^n$  with  $\sup_{j \in \mathbb{Z}_0^-} |\overline{x}(j)|_{\infty} e^{\alpha j} < \infty$ , For  $m \in \mathbb{N}_0$ , we define  $\overline{x}_m \in X_{\alpha}^n$  by

$$\overline{x}_m(j) = \overline{x}(m+j), \quad j \in \mathbb{Z}_0^-.$$

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#### Let $\overline{x} : \mathbb{Z} \to \mathbb{R}^n$



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General system and Phase Space Exponential Stability

#### For m = 3 the graph of $\overline{x}_3 \in X_{\alpha}^n$ is



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Consider the delay difference system

$$x_i(m+1) = \mathcal{F}_i(m, \overline{x}_m), \quad m \in \mathbb{N}_0, i = 1, \dots, n,$$
 (5)

where  
$$\overline{\mathcal{F}}:\mathbb{N}_0 imes X^n_lpha o\mathbb{R}^n$$
 with  $\overline{\mathcal{F}}(m,\overline{arphi})=(\mathcal{F}_1(m,\overline{arphi}),\ldots,\mathcal{F}_n(m,\overline{arphi}))$ 

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The initial condition

$$\overline{x}_0 = \overline{\varphi}, \quad \text{for} \quad \overline{\varphi} \in X^n_{\alpha}.$$
 (6)

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$$x_i(m+1) = \mathcal{F}_i(m, \overline{x}_m), \quad m \in \mathbb{N}_0, i = 1, \dots, n,$$
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The initial condition

$$\overline{x}_0 = \overline{\varphi}, \quad \text{for} \quad \overline{\varphi} \in X^n_{\alpha}.$$
 (6)

• We denote by  $\overline{x}(\cdot, 0, \overline{\varphi})$  the unique solution

$$\overline{x}:\mathbb{Z}\to\mathbb{R}^n$$

of (5)-(6).

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# Main stability result

### Theorem 1 If

$$|\mathcal{F}_i(m,\overline{\varphi})| \le e^{-\alpha} \, \|\overline{\varphi}\|_{\alpha},\tag{7}$$

for all  $\overline{\varphi} \in X_{\alpha}^{n}$ ,  $m \in \mathbb{N}_{0}$ , i = 1, ..., n, then the zero solution of (5)  $x_{i}(m+1) = \mathcal{F}_{i}(m, \overline{x}_{m})$ 

is globally exponentially stable,

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# Main stability result

### Theorem 1 If

$$|\mathcal{F}_i(m,\overline{\varphi})| \le e^{-\alpha} \, \|\overline{\varphi}\|_{\alpha},\tag{7}$$

for all  $\overline{\varphi} \in X_{\alpha}^{n}, \ m \in \mathbb{N}_{0}, \ i = 1, \dots, n$ , then the zero solution of (5)

$$x_i(m+1) = \mathcal{F}_i(m, \overline{x}_m)$$

is globally exponentially stable,

That is

$$\|\overline{\mathbf{x}}_{m}\|_{\alpha} \leq e^{-\alpha m} \|\overline{\varphi}\|_{\alpha}, \quad \forall m \in \mathbb{N}_{0}.$$
(8)

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General system and Phase Space Exponential Stability

### ▶ **Proof:** By induction we prove $\|\overline{x}_m\|_{\alpha} \leq e^{-\alpha m} \|\overline{\varphi}\|_{\alpha}$ , $\forall m \in \mathbb{N}_0$

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- ▶ **Proof:** By induction we prove  $\|\overline{x}_m\|_{\alpha} \leq e^{-\alpha m} \|\overline{\varphi}\|_{\alpha}, \forall m \in \mathbb{N}_0$
- For m = 0 is trivial.

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- ▶ **Proof:** By induction we prove  $\|\overline{x}_m\|_{\alpha} \leq e^{-\alpha m} \|\overline{\varphi}\|_{\alpha}$ ,  $\forall m \in \mathbb{N}_0$
- For m = 0 is trivial.
- For  $m \in \mathbb{N}_0$ , assume

$$\|\overline{x}_r\|_{\alpha} \leq e^{-\alpha r} \|\overline{\varphi}\|_{\alpha}, \quad 0 \leq r \leq m.$$

For all i = 1, ..., n, by induction hypotheses and (7)

 $|x_i(m+1)| = |\mathcal{F}_i(m, \overline{x}_m)| \le e^{-\alpha} \|\overline{x}_m\|_{\alpha} \le e^{-\alpha(m+1)} \|\overline{\varphi}\|_{\alpha}$ 

- ▶ **Proof:** By induction we prove  $\|\overline{x}_m\|_{\alpha} \leq e^{-\alpha m} \|\overline{\varphi}\|_{\alpha}$ ,  $\forall m \in \mathbb{N}_0$
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- For  $m \in \mathbb{N}_0$ , assume

$$\|\overline{x}_r\|_{\alpha} \leq e^{-\alpha r} \, \|\overline{\varphi}\|_{\alpha}, \quad 0 \leq r \leq m.$$

For all i = 1, ..., n, by induction hypotheses and (7)

$$|x_i(m+1)| = |\mathcal{F}_i(m, \overline{x}_m)| \le e^{-\alpha} \|\overline{x}_m\|_{\alpha} \le e^{-\alpha(m+1)} \|\overline{\varphi}\|_{\alpha}$$

thus

$$\begin{aligned} \|\overline{x}_{m+1}\|_{\alpha} &= \max_{i} \left\{ \sup_{j \leq -m-1} |x_{i}(m+1+j)| e^{\alpha j}, \max_{-m \leq j \leq 0} |x_{i}(m+1+j)| e^{\alpha j} \right\} \\ &\leq \max_{i} \left\{ \sup_{j \leq -m-1} |\varphi_{i}(j+m+1)| e^{\alpha j}, \max_{-m \leq j \leq 0} e^{-\alpha (m+1+j)+\alpha j} \|\overline{\varphi}\|_{\alpha} \right\} \\ &= \max_{i} \left\{ \sup_{j \leq 0} |\varphi_{i}(j)| e^{\alpha (j-m-1)}, e^{-\alpha (m+1)} \|\overline{\varphi}\|_{\alpha} \right\} \\ &= e^{-\alpha (m+1)} \|\overline{\varphi}\|_{\alpha}. \end{aligned}$$

Low-order Hopfield model with unbounded delays High-order Hopfield model with unbounded delays Numerical example

## Low-order discrete-time Hopfield model

Consider the Low-order Hopfield model (3)

$$egin{aligned} & x_i(m+1) = a_i x_i(m) + \sum_{j=1}^n b_{ij} f_j(x_j(m)) + \sum_{j=1}^n c_{ij} f_j\left(x_j\left(m - au_{ij}(m)
ight)
ight) \ & + \sum_{j=1}^n d_{ij} \sum_{l=1}^\infty 
ho_{ijl} f_j(x_j(m-l)), \ m \in \mathbb{N}_0 \end{aligned}$$

with  $a_i \in ]-1, 1[$ ,  $b_{ij}, c_{ij}, d_{ij} \in \mathbb{R}$ ,  $\rho_{ijl} \ge 0$  and the hypothesis

Low-order Hopfield model with unbounded delays High-order Hopfield model with unbounded delays Numerical example

## Low-order discrete-time Hopfield model

Consider the Low-order Hopfield model (3)

$$egin{aligned} & x_i(m+1) = a_i x_i(m) + \sum_{j=1}^n b_{ij} f_j(x_j(m)) + \sum_{j=1}^n c_{ij} f_j\left(x_j\left(m - au_{ij}(m)
ight)
ight) \ & + \sum_{j=1}^n d_{ij} \sum_{l=1}^\infty 
ho_{ijl} f_j(x_j(m-l)), \ m \in \mathbb{N}_0 \end{aligned}$$

with  $a_i \in ]-1, 1[$ ,  $b_{ij}, c_{ij}, d_{ij} \in \mathbb{R}$ ,  $\rho_{ijl} \ge 0$  and the hypothesis (H1)  $\exists F_j > 0$  such that  $|f_j(u)| \le F_j |u|$ ;

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• (H3) 
$$\exists \xi > 0$$
 such that  $\sum_{l=1}^{\infty} e^{\xi l} \rho_{ijl} < \infty$  and  $\sum_{l=1}^{\infty} \rho_{ijl} = 1$ ;  $\rho_{ijl} \ge 0$ 

Theorem 2 Assume (H1)-(H3). If

 $\mathcal{M} = diag(1 - |a_1|, \dots, 1 - |a_n|) - [F_j(|b_{ij}| + |c_{ij}| + |d_{ij}|)]$ 

is a non-singular M-matrix, then the zero solution of (3) is globally exponentially stable.

That is there are  $C \ge 1$  and  $\alpha > 0$ :

$$\|\overline{x}_m(\cdot, 0, \overline{\varphi})\|_{lpha} \leq C e^{-lpha m} \|\overline{\varphi}\|_{lpha}, \quad \forall \, \overline{\varphi} \in X^n_{lpha}, \, orall m \in \mathbb{N}_0.$$

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Lemma: Assume (H3).
 If γ > 0, then there is η > 0 such that

$$\sum_{l=1}^{\infty} e^{tl} \rho_{ijl} < 1 + \gamma, \quad \forall t \in [0, \eta], \ i, j = 1, \dots, n.$$

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▶ Proof of Theorem 2:  $\mathcal{M}$  is a non-singular M-matrix, thus there is  $\overline{p} = (p_1, \ldots, p_n) \in \mathbb{R}^n_+$  such that  $\mathcal{M}\overline{p}^T > 0$ , i.e.

$$p_i - p_i |a_i| - \sum_{j=1}^n p_j F_j(|b_{ij}| + |c_{ij}| + |d_{ij}|) > 0, \quad i = 1, ..., n.$$

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• Unless a change of variables, assume  $\overline{p} = (1, \dots, 1)$ , that is

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• Thus there is  $\gamma > 0$  such that

$$\mathrm{e}^{-\gamma}-|a_i|-\sum_{j=1}^nF_j(|b_{ij}|+|c_{ij}|\,\mathrm{e}^{\gamma au}+|d_{ij}|(1+\gamma))>0,$$

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▶ By previous Lemma, there is  $\alpha \in ]0, \gamma[$  such that

$$\sum_{l=1}^{\infty} e^{\alpha l} \rho_{ijl} < 1 + \gamma, \quad \forall i, j = 1, \dots, n, \\ \forall i, j \in 1, \dots, n, n, \\ \forall i, j \in n, n, n, n, n$$

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#### • As $0 < \alpha < \gamma$ we obtain

$$e^{-\alpha} > |a_i| + \sum_{j=1}^n F_j(|b_{ij}| + |c_{ij}|e^{\alpha\tau} + |d_{ij}|(1+\gamma)), \ i = 1, \dots, n.$$
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• As  $0 < \alpha < \gamma$  we obtain

$$e^{-\alpha} > |a_i| + \sum_{j=1}^n F_j (|b_{ij}| + |c_{ij}| e^{\alpha \tau} + |d_{ij}|(1+\gamma)), \ i = 1, \dots, n.$$
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Model (3), in the phase space X<sup>n</sup><sub>α</sub>, as the form

$$x_i(m+1) = \mathcal{F}_i(m, \overline{x}_m), \quad i = 1, \dots, n,$$

with

$$\mathcal{F}_{i}(m,\overline{\varphi}) = a_{i}\varphi_{i}(0) + \sum_{j=1}^{n} b_{ij}f_{j}(\varphi_{j}(0)) + \sum_{j=1}^{n} c_{ij}f_{j}(\varphi_{j}(-\tau_{ij}(m)))$$

$$+\sum_{j=1}^{n}d_{ij}\sum_{l=1}^{\infty}\rho_{ijl}f_j(\varphi_j(-l))$$

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• As  $0 < \alpha < \gamma$  we obtain

$$\mathsf{e}^{-\alpha} > |\mathbf{a}_i| + \sum_{j=1}^n F_j \big( |b_{ij}| + |c_{ij}| \, \mathsf{e}^{\alpha \tau} + |d_{ij}|(1+\gamma) \big), \, i = 1, \dots, n.$$

• By hypothesis (H1):  $|f_j(u)| \le F_j|u|$ 

$$|\mathcal{F}_i(m,\overline{arphi})| \leq |a_i arphi_i(0)| + \sum_{j=1}^n |b_{ij}||f_j(arphi_j(0))| + \sum_{j=1}^n |c_{ij}||f_j(arphi_j(- au_{ij}(m)))|$$

$$+\sum_{j=1}^{n}|d_{ij}|\sum_{l=1}^{\infty}\rho_{ijl}|f_{j}(\varphi_{j}(-l))|$$

$$\leq |\mathbf{a}_{i}|\|\overline{\varphi}\|_{\alpha}+\sum_{j=1}^{n}F_{j}\bigg(|b_{ij}|\|\overline{\varphi}\|_{\alpha}+|\mathbf{c}_{ij}|\frac{|\varphi_{j}(-\tau_{ij}(\mathbf{m}))|e^{-\alpha\tau_{ij}(\mathbf{m})}}{e^{-\alpha\tau_{ij}(\mathbf{m})}}$$

$$+ |d_{ij}| \sum_{l=1}^{\infty} \rho_{ijl} \frac{|\varphi_j(-l)| \, \mathrm{e}^{-\alpha l}}{\mathrm{e}^{-\alpha l}} \bigg)$$

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$$\blacktriangleright \ \ \, {\rm Thus,} \ \, \left( {\rm recall} \ \, \sum_{l=1}^\infty {\rm e}^{\alpha l} \, \rho_{ijl} < 1+\gamma \right) \\$$

$$|\mathcal{F}_{i}(m,\overline{\varphi})| \leq |a_{i}|\|\overline{\varphi}\|_{\alpha} + \sum_{j=1}^{n} F_{j}\left(|b_{ij}|\|\overline{\varphi}\|_{\alpha} + \frac{|c_{ij}|}{e^{-\alpha\tau}}\|\overline{\varphi}\|_{\alpha}\right)$$

$$egin{aligned} + |d_{ij}| \sum_{l=1}^{\infty} \mathrm{e}^{lpha l} \, 
ho_{ijl} \| \overline{arphi} \|_{lpha} \ \end{pmatrix} \ &\leq \ \left( |a_i| + \sum_{j=1}^{n} F_j(|b_{ij}| + |c_{ij}| \, \mathrm{e}^{lpha au} + |d_{ij}|(1+\gamma)) 
ight) \| \overline{arphi} \|_{lpha} \end{aligned}$$

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► Thus, 
$$\left(\operatorname{recall} \sum_{l=1}^{\infty} e^{\alpha l} \rho_{ijl} < 1 + \gamma\right)$$

$$|\mathcal{F}_{i}(m,\overline{\varphi})| \leq |a_{i}|\|\overline{\varphi}\|_{\alpha} + \sum_{j=1}^{n} F_{j}\left(|b_{ij}|\|\overline{\varphi}\|_{\alpha} + \frac{|c_{ij}|}{e^{-\alpha\tau}}\|\overline{\varphi}\|_{\alpha}\right)$$

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ight) \| \overline{arphi} \|_{lpha} \end{aligned}$$

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and from (9) we obtain

$$|\mathcal{F}_i(m,\overline{\varphi})| \leq e^{-\alpha} \|\overline{\varphi}\|_{\alpha},$$

and the conclusion follows from Theorem 1.

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## High-order discrete-time Hopfield model

Consider the High-order Hopfield model (4)

$$\begin{aligned} x_{i}(m+1) &= a_{i}x_{i}(m) + \sum_{j=1}^{n} b_{ij}f_{j}(x_{j}(m)) \\ &+ \sum_{j=1}^{n} \sum_{k=1}^{n} c_{ijk}g_{j}(x_{j}(m-\tau_{ijk}(m)))g_{k}(x_{k}(m-\tau_{ijk}(m))) \\ &+ \sum_{j=1}^{n} \sum_{k=1}^{n} d_{ijk}\left(\sum_{l=1}^{\infty} \rho_{ijl}g_{j}(x_{j}(m-l))\right)\left(\sum_{l=1}^{\infty} \rho_{ijl}g_{k}(x_{k}(m-l))\right) \end{aligned}$$

with  $a_i \in ]-1, 1[$ ,  $b_{ij}, c_{ijk}, d_{ijk} \in \mathbb{R}$ ,  $\rho_{ijl} \ge 0$  and the hypothesis (H1), (H2), (H3), and

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# High-order discrete-time Hopfield model

Consider the High-order Hopfield model (4)

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with  $a_i \in ]-1, 1[$ ,  $b_{ij}, c_{ijk}, d_{ijk} \in \mathbb{R}$ ,  $\rho_{ijl} \ge 0$  and the hypothesis (H1), (H2), (H3), and

• (H4)  $\exists G_j, M_j > 0$  such that

$$|g_j(u)| \leq \min \{M_j, G_j|u|\}.$$

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Theorem 3 Assume (H1)-(H4). If

$$diag(1-|a_1|,\ldots,1-|a_n|) - \left[F_j(|b_{ij}|] - \left[G_j\sum_{k=1}^n M_k(|c_{ijk}|+|d_{ijk}|)\right]\right]$$

is a non-singular M-matrix, then the zero solution of (4) is globally exponentially stable.

[1] Z.Dong, X. Wang, and X. Zhang, Appl. Math. Comput. 385 (2020) p.125401

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is a non-singular M-matrix, then the zero solution of (4) is globally exponentially stable.

 In [1] model (4) was studied with finite delays and the additional Lipschitz conditions

$$|f_j(u)-f_j(v)| \leq F_j|u-v|, |g_j(u)-g_j(v)| \leq G_j|u-v|, \quad \forall u, v \in \mathbb{R},$$

and  $f_j(0) = 0$ ,  $g_j(0) = 0$ .

[1] Z.Dong, X. Wang, and X. Zhang, Appl. Math. Comput. 385 (2020) p.125401

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#### Numerical example:

$$\begin{cases} x_1'(t) = -10x_1(t) + 2\tanh(x_2(t-1)) + 15\int_{-\infty}^{0} 4^s \tanh(x_2(t+s))ds \\ x_2'(t) = -10x_2(t) + \tanh(x_1(t-3)) + 2\int_{-\infty}^{0} 2^s \tanh(x_1(t+s))ds \end{cases}$$

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Numerical example:

$$\left\{egin{array}{l} x_1'(t)=-10x_1(t)+2 anh(x_2(t-1))+15\int_{-\infty}^04^s anh(x_2(t+s))ds\ x_2'(t)=-10x_2(t)+ anh(x_1(t-3))+2\int_{-\infty}^02^s anh(x_1(t+s))ds \end{array}
ight.$$

After the discretization process, we obtain

$$\begin{cases} x_1(m+1) &= e^{-10} x_1(m) + \frac{1-e^{-10}}{10} \\ & \cdot \left(2 \tanh(x_2(m-1)) + 5 \sum_{l=1}^{\infty} \frac{3}{4^l} \tanh(x_2(m-l))\right) \\ x_2(m+1) &= e^{-10} x_2(m) + \frac{1-e^{-10}}{10} \\ & \cdot \left(\tanh(x_1(m-3)) + 2 \sum_{l=1}^{\infty} \frac{1}{2^l} \tanh(x_1(m-l))\right) \end{cases}$$
(10)

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We have

$$\mathcal{M} = \left[ egin{array}{ccc} 1-{
m e}^{-10} & -rac{7\left(1-{
m e}^{-10}
ight)}{10} \ -rac{3\left(1-{
m e}^{-10}
ight)}{10} & 1-{
m e}^{-10} \end{array} 
ight]$$

which is a non-singular M-matrix, thus the zero solution of (10) is globally exponentially stable.



Figure: Solution  $(x_1(t), x_2(t))$  of system (10) with initial condition  $\overline{x}_0(j) = \begin{cases} (\cos(j), \sin(j)), & j = -9, \dots, 0 \\ (0, 0), & j \in ] -\infty, -10] \cap \mathbb{Z} \quad \text{is a set in a se$ 

Hopfield neural network models	Low-order Hopfield model with unbounded delays
Stability results	High-order Hopfield model with unbounded delays
Exponential stability of discrete-time Hopfield models	Numerical example

Thank you

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