Global exponential stability of discrete-time Hopfield neural network models with unbounded delays

José J. Oliveira

September 23, 2022

Centro de Matemática (CMAT)
Departamento de Matemática da Universidade do Minho

Hopfield Neural Network Models

- Pioneer Hopfield's work (1984)

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)), \qquad i = 1, \dots, n. \tag{1}$$

$n \in \mathbb{N}$ number of neurons, $t \geq 0$; $\sigma_{ij}(t) \geq 0$; $diag(a_1, \dots, a_n) > 0$ self-feedback matrix; I_i external inputs; $k_{ij}(s) \geq 0$ kernel functions; f_j activation functions; $[b_{ij}], [c_{ij}], [d_{ij}]$, connection matrices;

Hopfield Neural Network Models

- Pioneer Hopfield's work (1984)

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)), \qquad i = 1, \dots, n. \tag{1}$$

- Generalized Hopfield neural network

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} f_j(x_j(t - \sigma_{ij}(t))) + \sum_{j=1}^{n} d_{ij} \int_{-\infty}^{0} k_{ij}(s) f_j(x_j(t+s)) ds + I_i(t), \tag{2}$$

$n \in \mathbb{N}$ number of neurons, $t \geq 0$; $\sigma_{ij}(t) \geq 0$; $diag(a_1, \dots, a_n) > 0$ self-feedback matrix; I_i external inputs; $k_{ij}(s) \geq 0$ kernel functions; f_j activation functions; $[b_{ij}], [c_{ij}], [d_{ij}]$, connection matrices;

- We take the approximation of (2)

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j([t/h]h)) + \sum_{j=1}^{n} c_{ij} f_j \left(x_j \left([t/h]h - \left[\frac{\sigma_{ij}([t/h]h)}{h} \right] h \right) \right.$$
$$\left. + \sum_{j=1}^{n} d_{ij} \int_{-\infty}^{0} k_{ij}([s/h]h) f_j(x_j([t/h]h + [s/h]h)) ds + I_i([t/h]h), \right.$$

for $t \in [mh, (m+1)h[$ and $m \in \mathbb{N}_0$, where

- $[r]$ is the integer part of $r \in \mathbb{R}$;
- $h > 0$ is the discretization step size;

- We take the approximation of (2)

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j([t/h]h)) + \sum_{j=1}^{n} c_{ij} f_j \left(x_j \left([t/h]h - \left[\frac{\sigma_{ij}([t/h]h)}{h} \right] h \right) \right.$$

$$\left. + \sum_{j=1}^{n} d_{ij} \int_{-\infty}^{0} k_{ij}([s/h]h) f_j(x_j([t/h]h + [s/h]h)) ds + I_i([t/h]h), \right.$$

for $t \in [mh, (m+1)h[$ and $m \in \mathbb{N}_0$, where
  - $[r]$ is the integer part of $r \in \mathbb{R}$;
  - $h > 0$ is the discretization step size;
- We have $[t/h] = m$, thus

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(mh)) + \sum_{j=1}^{n} c_{ij} f_j \left(x_j \left(mh - \left[\frac{\sigma_{ij}(mh)}{h} \right] h \right) \right)$$

$$+ \sum_{j=1}^{n} d_{ij} \int_{-\infty}^{0} k_{ij}([s/h]h) f_j(x_j(mh + [s/h]h)) ds + I_i(mh),$$

- For $s \in]-lh, -(l-1)h]$ and $l \in \mathbb{N}$, we have $[s/h] = -l$

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(mh)) + \sum_{j=1}^{n} c_{ij} f_j \left(x_j \left(mh - \left[\frac{\sigma_{ij}(mh)}{h} \right] h \right) \right)$$

$$+ \sum_{j=1}^{n} d_{ij} \sum_{l=1}^{\infty} (k_{ij}(-lh) f_j(x_j(mh - lh)) h) + I_i(mh),$$

- For $s \in]-lh, -(l-1)h]$ and $l \in \mathbb{N}$, we have $[s/h] = -l$

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(mh)) + \sum_{j=1}^{n} c_{ij} f_j\left(x_j\left(mh - \left[\frac{\sigma_{ij}(mh)}{h}\right]h\right)\right)$$
$$+ \sum_{j=1}^{n} d_{ij} \sum_{l=1}^{\infty} (k_{ij}(-lh) f_j(x_j(mh-lh))h) + I_i(mh),$$

- Jumping computations and $t \to (m+1)h$ $t \in [mh, (m+1)h[$

$$x_i((m+1)h) = \mathrm{e}^{-a_i h} x_i(mh) + \theta_i(h) I_i(mh) + \theta_i(h) \sum_{j=1}^{n} \Big(b_{ij} f_j(x_j(mh))$$
$$+ c_{ij} f_j\left(x_j\left((m - \tau_{ij}(m))h\right)\right) + d_{ij} \sum_{l=1}^{\infty} k_{ij}(-lh) f_j(x_j(mh-lh))h \Big),$$

where $\theta_i(h) = \frac{1-\mathrm{e}^{-a_i h}}{a_i}$ and $\tau_{ij}(m) = \left[\frac{\sigma_{ij}(mh)}{h}\right]$

- For $s \in]-lh, -(l-1)h]$ and $l \in \mathbb{N}$, we have $[s/h] = -l$

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(mh)) + \sum_{j=1}^{n} c_{ij} f_j\left(x_j\left(mh - \left[\frac{\sigma_{ij}(mh)}{h}\right] h\right)\right)$$
$$+ \sum_{j=1}^{n} d_{ij} \sum_{l=1}^{\infty} (k_{ij}(-lh) f_j(x_j(mh - lh))h) + I_i(mh),$$

- Identifying $mh \equiv m$ and $lh \equiv l$, we have

$$x_i(m+1) = \mathrm{e}^{-a_i h} x_i(m) + \theta_i(h) I_i(m) + \theta_i(h) \sum_{j=1}^{n} \Big(b_{ij} f_j(x_j(m))$$
$$+ c_{ij} f_j\left(x_j\left(m - \tau_{ij}(m)\right)\right) + d_{ij} \sum_{l=1}^{\infty} k_{ij}(-l) f_j(x_j(m-l))h \Big),$$

where $\theta_i(h) = \frac{1 - \mathrm{e}^{-a_i h}}{a_i}$ and $\tau_{ij}(m) = \left[\frac{\sigma_{ij}(mh)}{h}\right]$

Low-order Hopfield neural network model

$$x_i(m+1) = a_i x_i(m) + \sum_{j=1}^{n} b_{ij} f_j(x_j(m)) + \sum_{j=1}^{n} c_{ij} f_j\left(x_j\left(m-\tau_{ij}(m)\right)\right) + \sum_{j=1}^{n} d_{ij} \sum_{l=1}^{\infty} \rho_{ijl} f_j(x_j(m-l)), \quad m \in \mathbb{N}_0 \tag{3}$$

$i = 1, \ldots, n$, with $n \in \mathbb{N}$ and

- $a_i \in]-1, 1[$;
- $b_{ij}, c_{ij}, d_{ij} \in \mathbb{R}$;
- $f_j : \mathbb{R} \to \mathbb{R}$, $\tau_{ij} : \mathbb{N}_0 \to \mathbb{N}_0$;
- $(\rho_{ijl})_{l \in \mathbb{N}}$ non-negative sequence with $\sum_{l=1}^{\infty} \rho_{ijl} < \infty$.

High-order Hopfield neural network model

$$
\begin{aligned}
x_i(m+1) \quad = \quad & a_i x_i(m) + \sum_{j=1}^{n} b_{ij} f_j(x_j(m)) \\
& + \sum_{j=1}^{n}\sum_{k=1}^{n} c_{ijk} g_j\left(x_j\left(m-\tau_{ijk}(m)\right)\right) g_k\left(x_k\left(m-\tau_{ijk}(m)\right)\right) \\
& + \sum_{j=1}^{n}\sum_{k=1}^{n} d_{ijk}\left(\sum_{l=1}^{\infty}\rho_{ijl} g_j(x_j(m-l))\right)\left(\sum_{l=1}^{\infty}\rho_{ijl} g_k(x_k(m-l))\right),
\end{aligned} \tag{4}
$$

$m \in \mathbb{N}_0$, $i = 1, \dots, n$, with $n \in \mathbb{N}$ and

- $a_i \in]-1, 1[$;
- $b_{ijk}, c_{ijk}, d_{ijk} \in \mathbb{R}$;
- $f_j, g_j : \mathbb{R} \to \mathbb{R}$, $\tau_{ijk} : \mathbb{N}_0 \to \mathbb{N}_0$;
- $(\rho_{ijl})_{l\in\mathbb{N}}$ non-negative sequence with $\displaystyle\sum_{l=1}^{\infty} \rho_{ijl} < \infty$.

GOAL

Establish sufficient conditions for the global exponential stability of discrete-time Hopfield models (3) and (4).

General system of delay difference equations

$$x_i(m+1) = \mathcal{F}_i(m, \overline{x}_m), \qquad m \in \mathbb{N}_0,\ i = 1, \dots, n,$$

where $n \in \mathbb{N}$ and

- X^n_α convenient phase space of sequences in $\mathbb{R}^n$;
- $\overline{\mathcal{F}} : \mathbb{N}_0 \times X^n_\alpha \to \mathbb{R}^n$ with $\overline{\mathcal{F}}(m, \overline{\varphi}) = (\mathcal{F}_1(m, \overline{\varphi}), \dots, \mathcal{F}_n(m, \overline{\varphi}))$;
- For $m \in \mathbb{N}_0$, $\overline{x}_m$ is a sequence in $\mathbb{R}^n$ which gives the historical information of the solution from $-\infty$ until m.

- For $\alpha > 0$ we define X_α^n as the space of the functions

$$\begin{array}{rccl} \overline{\varphi}: & \mathbb{Z}_0^- & \to & \mathbb{R}^n \\ & j & \mapsto & (\varphi_1(j), \ldots, \varphi_n(j)) \end{array}$$

  such that

$$\max_{i \in 1,\ldots,n} \left(\sup_{j \in \mathbb{Z}_0^-} |\varphi_i(j)| \, e^{\alpha j} \right) < \infty.$$

- For $\alpha > 0$ we define X_α^n as the space of the functions

$$\begin{array}{rccl} \overline{\varphi}: & \mathbb{Z}_0^- & \to & \mathbb{R}^n \\ & j & \mapsto & (\varphi_1(j), \ldots, \varphi_n(j)) \end{array}$$

  such that

$$\max_{i \in 1,\ldots,n} \left(\sup_{j \in \mathbb{Z}_0^-} |\varphi_i(j)| \, e^{\alpha j} \right) < \infty.$$

- Consider X_α^n the normed space with the norm

$$\|\overline{\varphi}\|_\alpha = \max_{i \in 1,\ldots,n} \left(\sup_{j \in \mathbb{Z}_0^-} |\varphi_i(j)| \, e^{\alpha j} \right), \quad \overline{\varphi} \in X_\alpha^n.$$

- For $\alpha > 0$ we define X^n_α as the space of the functions

$$\begin{array}{rccc} \overline{\varphi}: & \mathbb{Z}_0^- & \to & \mathbb{R}^n \\ & j & \mapsto & (\varphi_1(j), \dots, \varphi_n(j)) \end{array}$$

such that

$$\max_{i \in 1,\dots,n} \left(\sup_{j \in \mathbb{Z}_0^-} |\varphi_i(j)| \, e^{\alpha j} \right) < \infty.$$

- Consider X^n_α the normed space with the norm

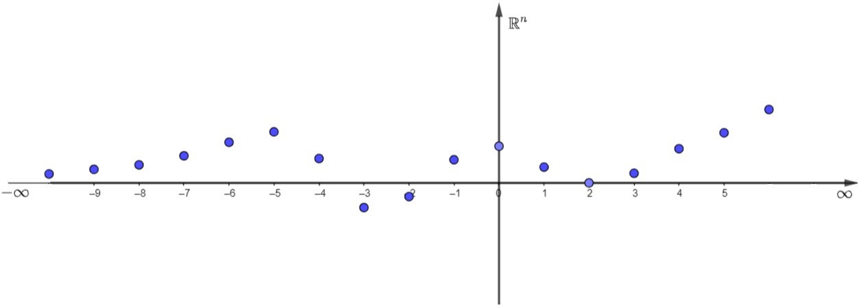
$$\|\overline{\varphi}\|_\alpha = \max_{i \in 1,\dots,n} \left(\sup_{j \in \mathbb{Z}_0^-} |\varphi_i(j)| \, e^{\alpha j} \right), \qquad \overline{\varphi} \in X^n_\alpha.$$

- Consider $\overline{x} : \mathbb{Z} \to \mathbb{R}^n$ with $\sup_{j \in \mathbb{Z}_0^-} |\overline{x}(j)|_\infty \, e^{\alpha j} < \infty$,

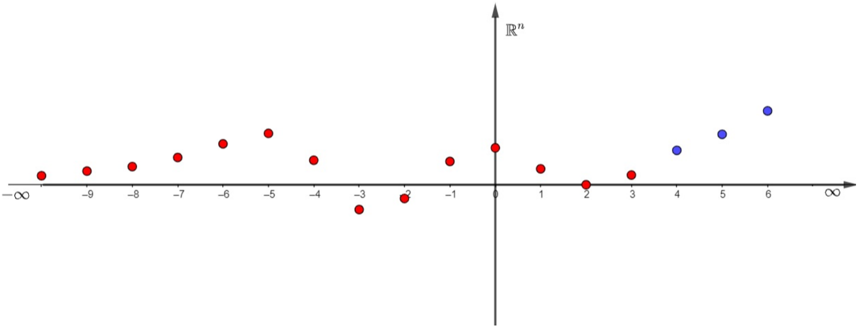
  For $m \in \mathbb{N}_0$, we define $\overline{x}_m \in X^n_\alpha$ by

$$\overline{x}_m(j) = \overline{x}(m+j), \qquad j \in \mathbb{Z}_0^-.$$

Let $\overline{x} : \mathbb{Z} \to \mathbb{R}^n$



For $m = 3$ the graph of $\overline{x}_3 \in X_\alpha^n$ is



$$\begin{array}{cccc} \overline{x}_3 : & \mathbb{Z}_0^- & \rightarrow & \mathbb{R}^n \\ & j & \mapsto & \overline{x}(3+j) \end{array}$$

- Consider the delay difference system

$$x_i(m+1) = \mathcal{F}_i(m, \overline{x}_m), \qquad m \in \mathbb{N}_0,\ i = 1, \dots, n, \tag{5}$$

where
$\overline{\mathcal{F}} : \mathbb{N}_0 \times X_\alpha^n \to \mathbb{R}^n$ with $\overline{\mathcal{F}}(m, \overline{\varphi}) = (\mathcal{F}_1(m, \overline{\varphi}), \dots, \mathcal{F}_n(m, \overline{\varphi}))$

- Consider the delay difference system

$$x_i(m+1) = \mathcal{F}_i(m, \overline{x}_m), \qquad m \in \mathbb{N}_0,\ i = 1, \dots, n, \tag{5}$$

where
$\overline{\mathcal{F}} : \mathbb{N}_0 \times X_\alpha^n \to \mathbb{R}^n$ with $\overline{\mathcal{F}}(m, \overline{\varphi}) = (\mathcal{F}_1(m, \overline{\varphi}), \dots, \mathcal{F}_n(m, \overline{\varphi}))$

- The initial condition

$$\overline{x}_0 = \overline{\varphi}, \qquad \text{for} \qquad \overline{\varphi} \in X_\alpha^n. \tag{6}$$

- Consider the delay difference system

$$x_i(m+1) = \mathcal{F}_i(m, \overline{x}_m), \qquad m \in \mathbb{N}_0,\ i = 1, \dots, n, \tag{5}$$

where
$\overline{\mathcal{F}} : \mathbb{N}_0 \times X_\alpha^n \to \mathbb{R}^n$ with $\overline{\mathcal{F}}(m, \overline{\varphi}) = (\mathcal{F}_1(m, \overline{\varphi}), \dots, \mathcal{F}_n(m, \overline{\varphi}))$

- The initial condition

$$\overline{x}_0 = \overline{\varphi}, \qquad \text{for} \qquad \overline{\varphi} \in X_\alpha^n. \tag{6}$$

- We denote by $\overline{x}(\cdot, 0, \overline{\varphi})$ the unique solution

$$\overline{x} : \mathbb{Z} \to \mathbb{R}^n$$

of (5)-(6).

Main stability result

- **Theorem 1** If

$$|\mathcal{F}_i(m,\overline{\varphi})| \leq \mathrm{e}^{-\alpha}\, \|\overline{\varphi}\|_\alpha, \tag{7}$$

for all $\overline{\varphi} \in X_\alpha^n$, $m \in \mathbb{N}_0$, $i = 1, \dots, n$, then the zero solution of (5)

$$x_i(m+1) = \mathcal{F}_i(m, \overline{x}_m)$$

is globally exponentially stable,

Main stability result

- **Theorem 1** If

$$|\mathcal{F}_i(m,\overline{\varphi})| \leq e^{-\alpha}\|\overline{\varphi}\|_\alpha, \tag{7}$$

for all $\overline{\varphi} \in X_\alpha^n$, $m \in \mathbb{N}_0$, $i = 1, \dots, n$, then the zero solution of (5)

$$x_i(m+1) = \mathcal{F}_i(m, \overline{x}_m)$$

is globally exponentially stable,

- That is

$$\|\overline{x}_m\|_\alpha \leq e^{-\alpha m}\|\overline{\varphi}\|_\alpha, \qquad \forall m \in \mathbb{N}_0. \tag{8}$$

- **Proof:** By induction we prove $\|\overline{x}_m\|_\alpha \leq \mathrm{e}^{-\alpha m}\|\overline{\varphi}\|_\alpha$, $\forall m \in \mathbb{N}_0$

- **Proof:** By induction we prove $\|\overline{x}_m\|_\alpha \leq e^{-\alpha m}\|\overline{\varphi}\|_\alpha$, $\forall m \in \mathbb{N}_0$
- For $m = 0$ is trivial.

- **Proof:** By induction we prove $\|\overline{x}_m\|_\alpha \leq e^{-\alpha m} \|\overline{\varphi}\|_\alpha$, $\forall m \in \mathbb{N}_0$
- For $m = 0$ is trivial.
- For $m \in \mathbb{N}_0$, assume

$$\|\overline{x}_r\|_\alpha \leq e^{-\alpha r} \|\overline{\varphi}\|_\alpha, \qquad 0 \leq r \leq m.$$

For all $i = 1, \ldots, n$, by induction hypotheses and (7)

$$|x_i(m+1)| = |\mathcal{F}_i(m, \overline{x}_m)| \leq e^{-\alpha} \|\overline{x}_m\|_\alpha \leq e^{-\alpha(m+1)} \|\overline{\varphi}\|_\alpha$$

- **Proof:** By induction we prove $\|\overline{x}_m\|_\alpha \leq e^{-\alpha m}\|\overline{\varphi}\|_\alpha$, $\forall m \in \mathbb{N}_0$
- For $m = 0$ is trivial.
- For $m \in \mathbb{N}_0$, assume

$$\|\overline{x}_r\|_\alpha \leq e^{-\alpha r}\|\overline{\varphi}\|_\alpha, \qquad 0 \leq r \leq m.$$

For all $i = 1, \ldots, n$, by induction hypotheses and (7)

$$|x_i(m+1)| = |\mathcal{F}_i(m, \overline{x}_m)| \leq e^{-\alpha}\|\overline{x}_m\|_\alpha \leq e^{-\alpha(m+1)}\|\overline{\varphi}\|_\alpha$$

- thus

$$\begin{aligned}
\|\overline{x}_{m+1}\|_\alpha &= \max_i \left\{ \sup_{j \leq -m-1} |x_i(m+1+j)| e^{\alpha j}, \max_{-m \leq j \leq 0} |x_i(m+1+j)| e^{\alpha j} \right\} \\
&\leq \max_i \left\{ \sup_{j \leq -m-1} |\varphi_i(j+m+1)| e^{\alpha j}, \max_{-m \leq j \leq 0} e^{-\alpha(m+1+j)+\alpha j}\|\overline{\varphi}\|_\alpha \right\} \\
&= \max_i \left\{ \sup_{j \leq 0} |\varphi_i(j)| e^{\alpha(j-m-1)}, e^{-\alpha(m+1)}\|\overline{\varphi}\|_\alpha \right\} \\
&= e^{-\alpha(m+1)}\|\overline{\varphi}\|_\alpha.
\end{aligned}$$

Low-order discrete-time Hopfield model

- Consider the Low-order Hopfield model (3)

$$x_i(m+1) = a_i x_i(m) + \sum_{j=1}^{n} b_{ij} f_j(x_j(m)) + \sum_{j=1}^{n} c_{ij} f_j\left(x_j\left(m - \tau_{ij}(m)\right)\right) + \sum_{j=1}^{n} d_{ij} \sum_{l=1}^{\infty} \rho_{ijl} f_j(x_j(m-l)), \;\; m \in \mathbb{N}_0$$

with $a_i \in]-1,1[$, $b_{ij}, c_{ij}, d_{ij} \in \mathbb{R}$, $\rho_{ijl} \geq 0$ and the hypothesis

Low-order discrete-time Hopfield model

- Consider the Low-order Hopfield model (3)

$$x_i(m+1) = a_i x_i(m) + \sum_{j=1}^{n} b_{ij} f_j(x_j(m)) + \sum_{j=1}^{n} c_{ij} f_j\left(x_j\left(m - \tau_{ij}(m)\right)\right) + \sum_{j=1}^{n} d_{ij} \sum_{l=1}^{\infty} \rho_{ijl} f_j(x_j(m-l)),\ \ m \in \mathbb{N}_0$$

with $a_i \in]-1,1[$, $b_{ij}, c_{ij}, d_{ij} \in \mathbb{R}$, $\rho_{ijl} \geq 0$ and the hypothesis

- (H1) $\exists F_j > 0$ such that $|f_j(u)| \leq F_j |u|$;

Low-order discrete-time Hopfield model

- Consider the Low-order Hopfield model (3)

$$x_i(m+1) = a_i x_i(m) + \sum_{j=1}^{n} b_{ij} f_j(x_j(m)) + \sum_{j=1}^{n} c_{ij} f_j\left(x_j\left(m - \tau_{ij}(m)\right)\right) + \sum_{j=1}^{n} d_{ij} \sum_{l=1}^{\infty} \rho_{ijl} f_j(x_j(m-l)),\ \ m \in \mathbb{N}_0$$

with $a_i \in]-1,1[$, $b_{ij}, c_{ij}, d_{ij} \in \mathbb{R}$, $\rho_{ijl} \geq 0$ and the hypothesis

- (H1) $\exists F_j > 0$ such that $|f_j(u)| \leq F_j |u|$;
- (H2) $\exists \tau > 0$ such that $\tau_{ij}(m) < \tau$;

Low-order discrete-time Hopfield model

- Consider the Low-order Hopfield model (3)

$$x_i(m+1)=a_ix_i(m)+\sum_{j=1}^{n}b_{ij}f_j(x_j(m))+\sum_{j=1}^{n}c_{ij}f_j\left(x_j\left(m-\tau_{ij}(m)\right)\right)$$
$$+\sum_{j=1}^{n}d_{ij}\sum_{l=1}^{\infty}\rho_{ijl}f_j(x_j(m-l)),\ \ m\in\mathbb{N}_0$$

with $a_i\in]-1,1[$, $b_{ij},c_{ij},d_{ij}\in\mathbb{R}$, $\rho_{ijl}\geq 0$ and the hypothesis

- (H1) $\exists F_j>0$ such that $|f_j(u)|\leq F_j|u|$;
- (H2) $\exists\tau>0$ such that $\tau_{ij}(m)<\tau$;
- (H3) $\exists\xi>0$ such that $\sum_{l=1}^{\infty}e^{\xi l}\rho_{ijl}<\infty$ and $\sum_{l=1}^{\infty}\rho_{ijl}=1$; $\rho_{ijl}\geq 0$

- **Theorem 2** Assume (H1)-(H3). If

$$\mathcal{M} = diag(1 - |a_1|, \dots, 1 - |a_n|) - \left[F_j(|b_{ij}| + |c_{ij}| + |d_{ij}|)\right]$$

is a non-singular M-matrix, then the zero solution of (3) is globally exponentially stable.
That is there are $C \geq 1$ and $\alpha > 0$:

$$\|\overline{x}_m(\cdot, 0, \overline{\varphi})\|_\alpha \leq C\,\mathrm{e}^{-\alpha m}\,\|\overline{\varphi}\|_\alpha, \qquad \forall\,\overline{\varphi} \in X_\alpha^n,\ \forall m \in \mathbb{N}_0.$$

- **Theorem 2** Assume (H1)-(H3). If

$$\mathcal{M} = diag(1-|a_1|,\dots,1-|a_n|) - \left[F_j(|b_{ij}|+|c_{ij}|+|d_{ij}|)\right]$$

is a non-singular M-matrix, then the zero solution of (3) is globally exponentially stable.
That is there are $C \geq 1$ and $\alpha > 0$:

$$\|\overline{x}_m(\cdot,0,\overline{\varphi})\|_\alpha \leq C\,\mathrm{e}^{-\alpha m}\,\|\overline{\varphi}\|_\alpha, \qquad \forall\,\overline{\varphi}\in X^n_\alpha,\ \forall m\in\mathbb{N}_0.$$

- **Lemma:** Assume (H3).
If $\gamma > 0$, then there is $\eta > 0$ such that

$$\sum_{l=1}^{\infty} \mathrm{e}^{tl}\,\rho_{ijl} < 1+\gamma, \qquad \forall t\in[0,\eta],\ i,j=1,\dots,n.$$

- **Proof of Theorem 2:** $\mathcal{M}$ is a non-singular M-matrix, thus there is $\overline{p}=(p_1,\dots,p_n)\in\mathbb{R}^n_+$ such that $\mathcal{M}\overline{p}^T>0$, i.e.

$$p_i - p_i|a_i| - \sum_{j=1}^{n} p_j F_j(|b_{ij}|+|c_{ij}|+|d_{ij}|) > 0, \qquad i=1,\dots,n.$$

- **Proof of Theorem 2:** $\mathcal{M}$ is a non-singular M-matrix, thus there is $\overline{p} = (p_1, \dots, p_n) \in \mathbb{R}^n_+$ such that $\mathcal{M}\overline{p}^T > 0$, i.e.

$$p_i - p_i|a_i| - \sum_{j=1}^{n} p_j F_j(|b_{ij}| + |c_{ij}| + |d_{ij}|) > 0, \qquad i = 1, \dots, n.$$

- Unless a change of variables, assume $\overline{p} = (1, \dots, 1)$, that is

$$1 - |a_i| - \sum_{j=1}^{n} F_j(|b_{ij}| + |c_{ij}| + |d_{ij}|) > 0, \qquad i = 1, \dots, n.$$

- **Proof of Theorem 2:** $\mathcal{M}$ is a non-singular M-matrix, thus there is $\overline{p} = (p_1, \dots, p_n) \in \mathbb{R}^n_+$ such that $\mathcal{M}\overline{p}^T > 0$, i.e.

$$p_i - p_i|a_i| - \sum_{j=1}^{n} p_j F_j(|b_{ij}| + |c_{ij}| + |d_{ij}|) > 0, \qquad i = 1, \dots, n.$$

- Unless a change of variables, assume $\overline{p} = (1, \dots, 1)$, that is

$$1 - |a_i| - \sum_{j=1}^{n} F_j(|b_{ij}| + |c_{ij}| + |d_{ij}|) > 0, \qquad i = 1, \dots, n.$$

- Thus there is $\gamma > 0$ such that

$$\mathrm{e}^{-\gamma} - |a_i| - \sum_{j=1}^{n} F_j(|b_{ij}| + |c_{ij}|\,\mathrm{e}^{\gamma\tau} + |d_{ij}|(1+\gamma)) > 0,$$

- **Proof of Theorem 2:** $\mathcal{M}$ is a non-singular M-matrix, thus there is $\overline{p}=(p_1,\dots,p_n)\in\mathbb{R}^n_+$ such that $\mathcal{M}\overline{p}^T>0$, i.e.

$$p_i - p_i|a_i| - \sum_{j=1}^{n} p_j F_j(|b_{ij}|+|c_{ij}|+|d_{ij}|) > 0, \qquad i=1,\dots,n.$$

- Unless a change of variables, assume $\overline{p}=(1,\dots,1)$, that is

$$1-|a_i| - \sum_{j=1}^{n} F_j(|b_{ij}|+|c_{ij}|+|d_{ij}|) > 0, \qquad i=1,\dots,n.$$

- Thus there is $\gamma>0$ such that

$$\mathrm{e}^{-\gamma} - |a_i| - \sum_{j=1}^{n} F_j(|b_{ij}|+|c_{ij}|\,\mathrm{e}^{\gamma\tau} + |d_{ij}|(1+\gamma)) > 0,$$

- By previous Lemma, there is $\alpha\in]0,\gamma[$ such that

$$\sum_{l=1}^{\infty} \mathrm{e}^{\alpha l}\rho_{ijl} < 1+\gamma, \qquad \forall i,j=1,\dots,n,$$

- As $0 < \alpha < \gamma$ we obtain

$$\mathrm{e}^{-\alpha} > |a_i| + \sum_{j=1}^{n} F_j\Big(|b_{ij}| + |c_{ij}|\,\mathrm{e}^{\alpha\tau} + |d_{ij}|(1+\gamma)\Big),\ i = 1, \ldots, n. \quad (9)$$

- As $0 < \alpha < \gamma$ we obtain

$$e^{-\alpha} > |a_i| + \sum_{j=1}^{n} F_j\big(|b_{ij}| + |c_{ij}|\, e^{\alpha\tau} + |d_{ij}|(1+\gamma)\big), \; i = 1, \dots, n. \quad (9)$$

- Model (3), in the phase space X_α^n, as the form

$$x_i(m+1) = \mathcal{F}_i(m, \overline{x}_m), \qquad i = 1, \dots, n,$$

with

$$\mathcal{F}_i(m, \overline{\varphi}) = a_i\varphi_i(0) + \sum_{j=1}^{n} b_{ij} f_j(\varphi_j(0)) + \sum_{j=1}^{n} c_{ij} f_j(\varphi_j(-\tau_{ij}(m)))$$

$$+ \sum_{j=1}^{n} d_{ij} \sum_{l=1}^{\infty} \rho_{ijl} f_j(\varphi_j(-l))$$

- As $0 < \alpha < \gamma$ we obtain

$$e^{-\alpha} > |a_i| + \sum_{j=1}^{n} F_j\left(|b_{ij}| + |c_{ij}|\, e^{\alpha\tau} + |d_{ij}|(1+\gamma)\right), \; i = 1, \dots, n.$$

- By hypothesis (H1): $|f_j(u)| \leq F_j|u|$

$$\begin{aligned}
|\mathcal{F}_i(m,\overline{\varphi})| &\leq |a_i\varphi_i(0)| + \sum_{j=1}^{n} |b_{ij}||f_j(\varphi_j(0))| + \sum_{j=1}^{n} |c_{ij}||f_j(\varphi_j(-\tau_{ij}(m)))| \\
&\quad + \sum_{j=1}^{n} |d_{ij}| \sum_{l=1}^{\infty} \rho_{ijl}|f_j(\varphi_j(-l))| \\
&\leq |a_i|\|\overline{\varphi}\|_\alpha + \sum_{j=1}^{n} F_j\Bigg(|b_{ij}|\|\overline{\varphi}\|_\alpha + |c_{ij}| \frac{|\varphi_j(-\tau_{ij}(m))|\, e^{-\alpha\tau_{ij}(m)}}{e^{-\alpha\tau_{ij}(m)}} \\
&\quad + |d_{ij}| \sum_{l=1}^{\infty} \rho_{ijl} \frac{|\varphi_j(-l)|\, e^{-\alpha l}}{e^{-\alpha l}}\Bigg)
\end{aligned}$$

- Thus, $\left(\text{recall } \sum_{l=1}^{\infty} e^{\alpha l} \rho_{ijl} < 1+\gamma\right)$

$$\begin{aligned}
|\mathcal{F}_i(m,\overline{\varphi})| \quad &\leq \quad |a_i|\|\overline{\varphi}\|_\alpha + \sum_{j=1}^{n} F_j \Bigg(|b_{ij}|\|\overline{\varphi}\|_\alpha + \frac{|c_{ij}|}{e^{-\alpha\tau}}\|\overline{\varphi}\|_\alpha \\
&\qquad + |d_{ij}| \sum_{l=1}^{\infty} e^{\alpha l} \rho_{ijl} \|\overline{\varphi}\|_\alpha \Bigg) \\
&\leq \quad \left(|a_i| + \sum_{j=1}^{n} F_j(|b_{ij}| + |c_{ij}|\, e^{\alpha\tau} + |d_{ij}|(1+\gamma)) \right) \|\overline{\varphi}\|_\alpha
\end{aligned}$$

- Thus, $\left(\text{recall } \sum_{l=1}^{\infty} e^{\alpha l} \rho_{ijl} < 1+\gamma\right)$

$$\begin{aligned}|\mathcal{F}_i(m,\overline{\varphi})| &\leq |a_i|\|\overline{\varphi}\|_\alpha + \sum_{j=1}^{n} F_j\Bigg(|b_{ij}|\|\overline{\varphi}\|_\alpha + \frac{|c_{ij}|}{e^{-\alpha\tau}}\|\overline{\varphi}\|_\alpha \\ &\quad + |d_{ij}|\sum_{l=1}^{\infty} e^{\alpha l} \rho_{ijl}\|\overline{\varphi}\|_\alpha\Bigg) \\ &\leq \left(|a_i| + \sum_{j=1}^{n} F_j(|b_{ij}| + |c_{ij}|\, e^{\alpha\tau} + |d_{ij}|(1+\gamma))\right)\|\overline{\varphi}\|_\alpha\end{aligned}$$

- and from (9) we obtain

$$|\mathcal{F}_i(m,\overline{\varphi})| \leq e^{-\alpha}\,\|\overline{\varphi}\|_\alpha,$$

and the conclusion follows from Theorem 1.

High-order discrete-time Hopfield model

- Consider the High-order Hopfield model (4)

$$
\begin{aligned}
x_i(m+1) \quad = \quad & a_i x_i(m) + \sum_{j=1}^{n} b_{ij} f_j(x_j(m)) \\
& + \sum_{j=1}^{n}\sum_{k=1}^{n} c_{ijk} g_j\left(x_j\left(m-\tau_{ijk}(m)\right)\right) g_k\left(x_k\left(m-\tau_{ijk}(m)\right)\right) \\
& + \sum_{j=1}^{n}\sum_{k=1}^{n} d_{ijk}\left(\sum_{l=1}^{\infty} \rho_{ijl} g_j(x_j(m-l))\right)\left(\sum_{l=1}^{\infty} \rho_{ijl} g_k(x_k(m-l))\right),
\end{aligned}
$$

with $a_i \in]-1,1[$, $b_{ij}, c_{ijk}, d_{ijk} \in \mathbb{R}$, $\rho_{ijl} \geq 0$ and the hypothesis (H1), (H2), (H3), and

High-order discrete-time Hopfield model

- Consider the High-order Hopfield model (4)

$$
\begin{aligned}
x_i(m+1) \quad = \quad & a_i x_i(m) + \sum_{j=1}^{n} b_{ij} f_j(x_j(m)) \\
& + \sum_{j=1}^{n}\sum_{k=1}^{n} c_{ijk} g_j\left(x_j\left(m-\tau_{ijk}(m)\right)\right) g_k\left(x_k\left(m-\tau_{ijk}(m)\right)\right) \\
& + \sum_{j=1}^{n}\sum_{k=1}^{n} d_{ijk}\left(\sum_{l=1}^{\infty} \rho_{ijl} g_j(x_j(m-l))\right)\left(\sum_{l=1}^{\infty} \rho_{ijl} g_k(x_k(m-l))\right),
\end{aligned}
$$

  with $a_i \in]-1,1[$, $b_{ij}, c_{ijk}, d_{ijk} \in \mathbb{R}$, $\rho_{ijl} \geq 0$ and the hypothesis (H1), (H2), (H3), and
- (H4) $\exists G_j, M_j > 0$ such that

$$
|g_j(u)| \leq \min\left\{M_j, G_j|u|\right\}.
$$

- **Theorem 3** Assume (H1)-(H4). If

$$diag(1-|a_1|,\dots,1-|a_n|)-\left[F_j(|b_{ij}|\right]-\left[G_j\sum_{k=1}^{n}M_k(|c_{ijk}|+|d_{ijk}|)\right]$$

is a non-singular M-matrix, then the zero solution of (4) is globally exponentially stable.

[1] Z.Dong, X. Wang, and X. Zhang, *Appl. Math. Comput.* 385 (2020) p.125401

- **Theorem 3** Assume (H1)-(H4). If

$$diag(1-|a_1|,\dots,1-|a_n|)-[F_j(|b_{ij}|]-\left[G_j\sum_{k=1}^{n}M_k(|c_{ijk}|+|d_{ijk}|)\right]$$

  is a non-singular M-matrix, then the zero solution of (4) is globally exponentially stable.
- In [1] model (4) was studied with finite delays and the additional Lipschitz conditions

$$|f_j(u)-f_j(v)|\leq F_j|u-v|,\ |g_j(u)-g_j(v)|\leq G_j|u-v|,\quad \forall u,v\in\mathbb{R},$$

  and $f_j(0)=0$, $g_j(0)=0$.

[1] Z.Dong, X. Wang, and X. Zhang, *Appl. Math. Comput.* 385 (2020) p.125401

- *Numerical example*:

$$\begin{cases} x_1'(t) = -10x_1(t) + 2\tanh(x_2(t-1)) + 15\displaystyle\int_{-\infty}^{0} 4^s \tanh(x_2(t+s))ds \\ x_2'(t) = -10x_2(t) + \tanh(x_1(t-3)) + 2\displaystyle\int_{-\infty}^{0} 2^s \tanh(x_1(t+s))ds \end{cases}$$

- *Numerical example*:

$$\begin{cases} x_1'(t) = -10x_1(t) + 2\tanh(x_2(t-1)) + 15\int_{-\infty}^{0} 4^s \tanh(x_2(t+s))ds \\ x_2'(t) = -10x_2(t) + \tanh(x_1(t-3)) + 2\int_{-\infty}^{0} 2^s \tanh(x_1(t+s))ds \end{cases}$$

- After the discretization process, we obtain

$$\left\{ \begin{array}{rcl} x_1(m+1) & = & e^{-10} x_1(m) + \dfrac{1-e^{-10}}{10} \\ & & \cdot \left(2\tanh(x_2(m-1)) + 5\displaystyle\sum_{l=1}^{\infty} \frac{3}{4^l} \tanh(x_2(m-l)) \right) \\ x_2(m+1) & = & e^{-10} x_2(m) + \dfrac{1-e^{-10}}{10} \\ & & \cdot \left(\tanh(x_1(m-3)) + 2\displaystyle\sum_{l=1}^{\infty} \frac{1}{2^l} \tanh(x_1(m-l)) \right) \end{array} \right. . \quad (10)$$

We have

$$\mathcal{M} = \begin{bmatrix} 1 - e^{-10} & -\frac{7\left(1-e^{-10}\right)}{10} \\ -\frac{3\left(1-e^{-10}\right)}{10} & 1 - e^{-10} \end{bmatrix}$$

which is a non-singular M-matrix, thus the zero solution of (10) is globally exponentially stable.

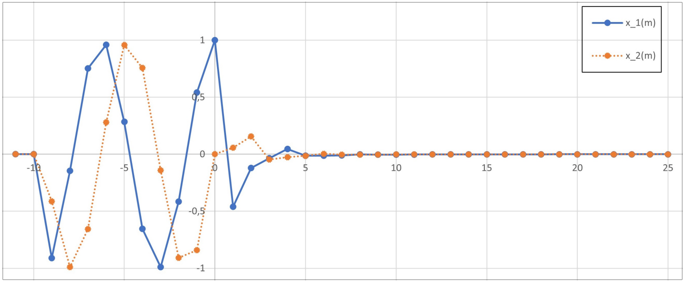


Figure: Solution $(x_1(t), x_2(t))$ of system (10) with initial condition

$$\overline{x}_0(j) = \begin{cases} (\cos(j), \sin(j)), & j = -9, \ldots, 0 \\ (0,0), & j \in]-\infty, -10] \cap \mathbb{Z} \end{cases}$$

Thank you

This work is published in *Journal of Difference Equations and Applications*, 28(2022) 725-751.