Stability of discrete-time Hopfield neural network with delay

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Joint work with
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$$x_i'(t) = -a_i(t)x_i(t) + \sum_{j=1}^{N} k_{ij}(t, x_j(t-\alpha_{ij}(t))), \quad t \geqslant 0,\, i = 1, \ldots, N$$

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neuron state vector at time t

$(x_1(t), \dots, x_N(t)) \in \mathbb{R}^N$

neuron charging time

$a_i : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ continuous

neuron activation functions

$k_{ij} : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ continuous, Lip on the second variable

time delay

$\alpha_{ij} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ continuous, bounded

$$x_i'(t) = -a_i(t)x_i(t) + \sum_{j=1}^{N} k_{ij}(t, x_j(t - \alpha_{ij}(t)))$$

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Discretization method in:
S. Mohamad, K. Gopalsamy, Exponential stability of continuous-time and discrete-time cellular neural networks with delays, Appl. Math. Comput. 135 (1) (2003) 17–38

$$x_i'(t) = -a_i(t)x_i(t) + \sum_{j=1}^{N} k_{ij}(t, x_j(t-\alpha_{ij}(t)))$$

$$x_i'(t) = -a_i([t/h]h)x_i(t) + \sum_{j=1}^{N} k_{ij}\left([t/h]h, x_j\left([t/h]h - \left[\frac{\alpha_{ij}([t/h]h)}{h}\right]h\right)\right)$$

$$\forall m \in \mathbb{N}_0,\ t \in [mh, (m+1)h[\quad \Rightarrow \quad [t/h] = m; \qquad \tau_{ij}(m) := \left[\frac{\alpha_{ij}(mh)}{h}\right]$$

$$e^{a_i(mh)t}x_i'(t) + a_i(mh)e^{a_i(mh)t}x_i(t) = e^{a_i(mh)t}\sum_{j=1}^{N} k_{ij}\left(mh, x_j\left((m-\tau_{ij}(m))h\right)\right)$$

$$\int_{mh}^{t} (e^{a_i(mh)s}x_i(s))'ds = \frac{e^{a_i(mh)t} - e^{a_i(mh)mh}}{a_i(mh)}\sum_{j=1}^{N} k_{ij}\left(mh, x_j\left((m-\tau_{ij}(m))h\right)\right)$$

$$x_i(m+1) = e^{-a_i(m)h}x_i(m) + \frac{1-e^{-a_i(m)h}}{a_i(m)}\sum_{j=1}^{N} k_{ij}\left(m, x_j(m-\tau_{ij}(m))\right)$$

$$x_i(m+1) = c_i(m)x_i(m) + \sum_{j=1}^{N} h_{ij}\big(m, x_j(m - \tau_{ij}(m))\big), \quad i = 1, \dots, N$$

- $c_i : \mathbb{N}_0 \to]0,1[$, $\tau_{ij} : \mathbb{N}_0 \to \mathbb{N}_0$ bounded with $\tau := \max\limits_{i,j,m}\{\tau_{ij}(m)\}$
- $h_{ij} : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}$ are Lip on the second variable, i.e.,

$$|h_{ij}(m,u) - h_{ij}(m,v)| \leqslant H_{ij}(m)|u-v|, \;\; \forall u, v \in \mathbb{R},\; m \in \mathbb{N}_0.$$

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Generalizes:

$$x_i(m+1) = x_i(m)\,\mathrm{e}^{-a_i(m)h} + \theta_i(m)\left(\sum_{j=1}^{N} b_{ij}(m)f_j(x_j(m-\tau(m))) + I_i(m)\right)$$

where $\theta_i(m) = \frac{1-\mathrm{e}^{-a_i(m)h}}{a_i(m)}$.

H. Xu, R. Wu, Periodicity and exponential stability of discrete-time neural networks with variable coefficients and delays, Adv. Difference Equ. (2013) 2013:226.

- $r \in \mathbb{N}_0$; $N \in \mathbb{N}$
- $I_{\mathbb{Z}} = I \cap \mathbb{Z}$, where I is a real interval
- Y - Banach space with norm $|\cdot|$
- X - space of $\alpha : [-r, 0]_{\mathbb{Z}} \to Y$, with the norm $\|\alpha\| = \max_{j \in [-r,0]_{\mathbb{Z}}} |\alpha(j)|$
- X^N and Y^N equipped with the supremum norm
- Given $m \in \mathbb{N}_0$ and

$$\begin{array}{rcl} \overline{x} : [-r, +\infty[_{\mathbb{Z}} & \to & Y^N \\ s & \mapsto & (x_1(s), \ldots, x_N(s)) \end{array}$$

define $x_{i,m} \in X$ and $\overline{x}_m \in X^N$ respectively by

$$x_{i,m}(j) = x_i(m+j), \quad j = -r, -r+1, \ldots, 0, \quad i = 1, \ldots, N,$$

$$\overline{x}_m(j) = \overline{x}(m+j), \quad j = -r, -r+1, \ldots, 0.$$

Consider the general nonautonomous delay difference equation

$$x_i(m+1) = L^{(i)}(m)x_{i,m} + f^{(i)}(m, \overline{x}_m), \quad m \in \mathbb{N}_0,\ i = 1, \dots, N$$

- $L^{(i)}(m)\colon X \to Y$ are bounded linear operators;
- $f^{(i)}(m, \cdot) : X^N \to Y$ are Lip perturbations with $f^{(i)}(m, 0) = 0$;
- given $n \in \mathbb{N}_0$ and $\overline{\alpha} = (\alpha_1, \dots, \alpha_N) \in X^N$, there is a unique solution $\overline{x}(\cdot, n, \overline{\alpha})\colon [n - r, +\infty)_{\mathbb{Z}} \to Y^N$ such that $\overline{x}_n = \overline{\alpha}$;
- for $m, n \in \mathbb{N}_0$ with $m \geqslant n$, define the evolution operator $\overline{\mathcal{F}}_{m,n} : X^N \to X^N$ by

$$\overline{\mathcal{F}}_{m,n}(\overline{\alpha}) = \overline{x}_m(\cdot, n, \overline{\alpha}), \quad \overline{\alpha} \in X^N.$$

$$\boxed{v_i(m+1) = L^{(i)}(m)v_{i,m}, \quad i = 1,\dots,N}$$

- $v_{i,m}(j) = v_i(m+j), \quad j = -r, -r+1, \dots, 0;$
- for $n \in \mathbb{N}_0$ and $\alpha_i \in X$, we obtain a unique solution $v_i(\cdot, n, \alpha_i) : [n-r, +\infty[_{\mathbb{Z}} \to Y$ such that $v_{i,m}(\cdot, n, \alpha_i) = \alpha_i$;
- for $m, n \in \mathbb{N}_0$ with $m \geqslant n$, $i = 1, \dots, N$, define evolution operator $\mathcal{A}^{(i)}_{m,n} : X \to X$ by

$$\mathcal{A}^{(i)}_{m,n}\alpha_i = v_{i,m}(\cdot, n, \alpha_i), \ \alpha_i \in X.$$

Lemma | Properties of $\mathcal{A}^{(i)}(m,n)$

- $\mathcal{A}^{(i)}_{m,n}$ is linear for $m \geqslant n$;
- $\mathcal{A}^{(i)}_{m,m} = \mathrm{Id}_X$;
- $\mathcal{A}^{(i)}_{l,m}\mathcal{A}^{(i)}_{m,n} = \mathcal{A}^{(i)}_{l,n}$ for $l \geqslant m$ and $m \geqslant n$.

- $\Gamma\colon Y \to X$ defined by $\Gamma u\colon [-r,0]_{\mathbb{Z}} \to Y$ where

$$\Gamma u(j) = \begin{cases} u & \text{if } j = 0, \\ 0 & \text{if } j < 0, \end{cases}$$

- For all $\overline{\alpha} = (\alpha_1, \ldots, \alpha_N) \in X^N$,

$$\mathcal{F}^{(i)}_{m,n}(\overline{\alpha}) = \mathcal{A}^{(i)}_{m,n}\alpha_i + \sum_{k=n}^{m-1} \mathcal{A}^{(i)}_{m,k+1}\Gamma f^{(i)}(k, \overline{x}_k), \quad i = 1, \ldots, N$$

Lemma (Barreira & Valls 2007) | equation for $\mathcal{F}^{(i)}(m,n)$

$$\overline{\mathcal{F}}_{m,n}(\overline{\alpha}) = \left(\mathcal{F}^{(1)}_{m,n}(\overline{\alpha}), \ldots, \mathcal{F}^{(N)}_{m,n}(\overline{\alpha})\right)$$

Theorem | Abstract result

Assume

- $f^{(i)}(m,\cdot)\colon X^N \to Y^N$ are Lip functions with $f^{(i)}(m,0)=0,\ \forall i$
- $\|\mathcal{A}^{(i)}_{m,n}\| \leqslant a^{(i)}_{m,n} \leqslant a'_{m,n}, \quad m \geqslant n,\ i=1\dots,N$
- $\lambda := \max_{i=1,\dots,N}\left[\sup_{m\geqslant n}\left\{\frac{1}{a'_{m,n}}\sum_{k=n}^{m-1} a^{(i)}_{m,k+1}\,\mathrm{Lip}(f^{(i)}(k,\cdot))a'_{k,n}\right\}\right] < 1$

Then

$$\|\overline{\mathcal{F}}_{m,n}(\overline{\alpha})\| \leqslant \frac{1}{1-\lambda}\,a'_{m,n}\|\overline{\alpha}\|$$

for every $\overline{\alpha}=(\alpha_1,\dots,\alpha_N)\in X^N$ and $m \geqslant n \geqslant 0$.

Proof

Banach fixed point theorem in suitable complete metric space...

Coming back to the Hopfield model

$$x_i(m+1) = c_i(m)x_i(m) + \sum_{j=1}^{N} h_{ij}\big(m, x_j(m-\tau_{ij}(m))\big), \quad i = 1, \dots, N$$

Theorem | General stability for Hopfield model

Assume

- $a^{(i)}_{m,n} := \prod_{s=n}^{m-1} c_i(s) \leqslant a'_{m,n}, \qquad i = 1, \dots, N, \ \ m \geqslant n \geqslant 0$
- $\lambda := \max_{i=1,\dots,N} \left[\sup_{m \geqslant n} \left\{ \frac{1}{a'_{m,n}} \sum_{k=n}^{m-1} a^{(i)}_{m,k+1} a'_{k,n} \sum_{j=1}^{N} H_{ij}(k) \right\} \right] < 1$

Then, for every $\overline{\alpha}, \overline{\alpha}^* : [-r, 0]_{\mathbb{Z}} \to \mathbb{R}^N$ and $m \geqslant n \geqslant 0$

$$\|\overline{x}_m(\cdot, n, \overline{\alpha}) - \overline{x}_m(\cdot, n, \overline{\alpha}^*)\| \leqslant \frac{1}{1-\lambda}\, a'_{m,n} \|\overline{\alpha} - \overline{\alpha}^*\|$$

Proof

- The change $\overline{y}(m) = \overline{x}(m, n, \overline{\alpha}) - \overline{x}(m, n, \overline{\alpha}^*)$ transforms
  $x_i(m+1) = c_i(m)x_i(m) + \sum_{j=1}^{N} h_{ij}(m, x_j(m - \tau_{ij}(m)))$ into
  $y_i(m+1) = c_i(m)y_i(m) + \sum_{j=1}^{N} \widetilde{h}_{ij}(m, y_j(m - \tau_{ij}(m)))$
- $\text{Lip}\Big(\widetilde{h}_{ij}(m, \cdot)\Big) = H_{ij}(m)$ and $\overline{y} = 0$ is an equilibrium point
- For $Y = \mathbb{R}$, by the abstract result (previous Theorem)

$$\|\overline{y}_m(\cdot, n, \overline{\beta})\| \leqslant \frac{1}{1-\lambda}\, a'_{m,n}\|\overline{\beta}\|$$

- Setting $\overline{\beta} = \overline{\alpha} - \overline{\alpha}^*$, we conclude that

$$\|\overline{x}_m(\cdot, n, \overline{\alpha}) - \overline{x}_m(\cdot, n, \overline{\alpha}^*)\| = \|\overline{y}_m(\cdot, n, \overline{\alpha} - \overline{\alpha}^*)\| \leqslant \frac{1}{1-\lambda}\, a'_{m,n}\|\overline{\alpha} - \overline{\alpha}^*\|$$

$$x_i(m+1) = x_i(m)\,\mathrm{e}^{-a_i(m)h} + \theta_i(m)\left(\sum_{j=1}^{N} b_{ij}(m) f_j(x_j(m-\tau(m))) + I_i(m)\right)$$

$a_i^- = \inf_m a_i(m) \quad b_{ij}^+ = \sup_m |b_{ij}(m)| \quad \theta_i^+ = \sup_m \theta_i(m) \quad F_j$ Lip. constant of f_j

Corollary | Stability

Assume

$a_i^- > \sum_{j=1}^N b_{ij}^+ F_j, \quad i = 1, \dots, N$

Then

model Xu-Wu is globally exponentially stable, i.e., there are $\mu > 0$ and $C > 1$ such that, for every $\overline{\alpha}, \overline{\alpha}^* : [-\tau, 0]_{\mathbb{Z}} \to \mathbb{R}^N$, $m \geqslant n \geqslant 0$,

$$\|\overline{x}_m(\cdot, n, \overline{\alpha}) - \overline{x}_m(\cdot, n, \overline{\alpha}^*)\| \leqslant C\,\mathrm{e}^{-\mu(m-n)}\,\|\overline{\alpha} - \overline{\alpha}^*\|,$$

Proof

- $c_i(m) = \mathrm{e}^{-a_i(m)h} \Rightarrow a^{(i)}_{m,n} = \prod_{s=n}^{m-1} \mathrm{e}^{-a_i(s)h} \leqslant \mathrm{e}^{-\nu_i(m-n)} \leqslant \mathrm{e}^{-\mu(m-n)} = a'_{m,n}$
  where $\nu_i := a_i^- h$ and $0 < \mu < \min_i \nu_i$ such that
- $\dfrac{\mathrm{e}^{\nu_i-\mu} - 1}{\mathrm{e}^{\nu_i} - 1} a_i^- > \sum_{j=1}^{N} b_{ij}^+ F_j$, for all $i = 1, \dots N$
- by computations we have

$$\lambda = \max_{i=1,\dots,N} \left[\sup_{m \geqslant n} \left\{ \frac{1}{a'_{m,n}} \sum_{k=n}^{m-1} a^{(i)}_{m,k+1} a'_{k,n}\, \theta_i(k) \sum_{j=1}^{N} |b_{ij}(k)| F_j \right\} \right]$$
$$< \max_{i=1,\dots,N} \left[\sup_{m \geqslant n} \left\{ \sum_{k=n}^{m-1} \mathrm{e}^{(\nu_i-\mu)(k-m)} \right\} \mathrm{e}^{\nu_i} \; \frac{1-\mathrm{e}^{-\nu_i}}{a_i^-} \frac{\mathrm{e}^{\nu_i-\mu} - 1}{\mathrm{e}^{\nu_i} - 1} a_i^- \right]$$
$$= \max_{i=1,\dots,N} \left[\sup_{m \geqslant n} \left\{ \frac{1-\mathrm{e}^{(\nu_i-\mu)(n-m)}}{\mathrm{e}^{\nu_i-\mu} - 1} \right\} (\mathrm{e}^{\nu_i-\mu} - 1) \right] = 1$$

Corollary | Stability

Assume

$\mathcal{M} = diag(a_1^-, \ldots, a_N^-) - \left[b_{ij}^+ F_j\right]$ is an M-matrix

Then

model Xu-Wu is global exponential stable, i.e., there are $\mu > 0$, $C > 1$ such that

$$\|\overline{x}_m(\cdot, n, \overline{\alpha}) - \overline{x}_m(\cdot, n, \overline{\alpha}^*)\| \leqslant C\,\mathrm{e}^{-\mu(m-n)}\,\|\overline{\alpha} - \overline{\alpha}^*\|.$$

for every $\overline{\alpha}, \overline{\alpha}^* : [-\tau, 0]_{\mathbb{Z}} \to \mathbb{R}^N$ and $m \geqslant n \geqslant 0$.

Proof

- $\mathcal{M}$ is an M-matrix $\Leftrightarrow$ there is $\overline{d} = (d_1, \ldots, d_N) > 0$ such that $\mathcal{M}\,\overline{d} > 0$, i.e.,

$$d_i a_i^- > \sum_{j=1}^{N} d_j b_{ij}^+ F_j$$

- The change $y_i(m) = d_i^{-1} x_i(m) \Rightarrow$

$$y_i(m+1) = y_i(m)\,\mathrm{e}^{-a_i(m)h} + \theta_i(m) \left[\sum_{j=1}^{N} \tilde{b}_{ij}(m) \tilde{f}_j(y_j(m-\tau(m))) + \tilde{I}_i(m) \right]$$

where $\tilde{b}_{ij}(m) = d_i^{-1} b_{ij}(m)$, $\tilde{f}_j(u) = f_j(d_j u)$, and $\tilde{I}_i(m) = d_i^{-1} I_i(m)$

- f_j Lip with constant $F_j \Rightarrow \tilde{f}_j$ Lip with constant $\tilde{F}_j = d_j F_j$
- $a_i^- > \sum_{j=1}^{N} d_i^{-1} b_{ij}^+ d_j F_j \quad \Leftrightarrow \quad a_i^- > \sum_{j=1}^{N} \tilde{b}_{ij}^+ \tilde{F}_j$

Theorem | Existence and stability of periodic solution

Assume

- a_i, b_{ij}, I_i, τ are ω-periodic functions
- $\mathcal{M} = diag(a_1^-, \dots, a_N^-) - \left[b_{ij}^+ F_j\right]$ is an M-matrix

Then

the ω-periodic Xu-Wu model has a unique ω-periodic solution which is globally exponentially stable.

Xu-Wu assume:

- a_i, b_{ij}, I_i, τ are ω-periodic functions
- $\exists \bar{d} > 0 : \; d_i a_i^- > \sum_{j=1}^{N} d_j b_{ij}^+ F_j, \; i = 1, \dots, N \quad (\Leftrightarrow \mathcal{M} \text{ is an } M\text{-matrix})$
- $\hat{a}_i > F_i \sum_{j=1}^{N} |\hat{b}_{ji}|$, $i = 1, \dots, N$, where $\hat{a}_i := \frac{1}{\omega} \sum_{n=1}^{\omega-1} a_i(n)$, $\hat{b}_{ij} := \frac{1}{\omega} \sum_{n=1}^{\omega-1} b_{ij}(n)$

Proof

- Previous Corollary $\Rightarrow$ there are $\mu > 0$,$C > 1$ such that

$$\|\overline{x}_m(\cdot, n, \overline{\alpha}) - \overline{x}_m(\cdot, n, \overline{\alpha}^*)\| \leqslant C\,\mathrm{e}^{-\mu(m-n)}\,\|\overline{\alpha} - \overline{\alpha}^*\|$$

- choose $k \in \mathbb{N}$ such that $C\,\mathrm{e}^{-\mu k\omega} < 1$
- Define $P : X^N \to X^N$ by $P(\overline{\alpha}) = \overline{x}_{n+\omega}(\cdot, n, \overline{\alpha})$
- $\|P^k(\overline{\alpha}) - P^k(\overline{\alpha}^*)\| \leqslant C\,\mathrm{e}^{-\mu k\omega}\,\|\overline{\alpha} - \overline{\alpha}^*\|$
- P^k contraction on X^N $\Rightarrow$ there is a unique $\overline{\varphi} \in X^N$ such that $P^k(\overline{\varphi}) = \overline{\varphi} \Leftrightarrow \overline{x}_{n+\omega}(\cdot, n, \overline{\varphi}) = \overline{\varphi}$
- $\overline{x}(m, n, \overline{\varphi}) = \overline{x}(m, n, \overline{x}_{n+\omega}(\cdot, n, \overline{\varphi})) = \overline{x}(m+\omega, n, \overline{\varphi})$
- $\overline{x}(m, n, \overline{\varphi})$ is a ω-periodic solution and all other solutions converge to it with exponential rates.

Thank you

A.Bento, J.J.Oliveira, C.Silva, *Nonuniform behavior and stability of Hopfield neural networks with delays* Nonlinearity **30** (2017) 3088-3103.