# Global stability of a Cohen-Grossberg neural network model with both time-varying and continuous distributed delays

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### **Neural Network Models**

\*Pioneer Models:

Cohen-Grossberg (1983)

$$\dot{x}_i(t) = -a_i(x_i(t)) \left( b_i(x_i(t)) - \sum_{j=1}^n c_{ij} f_j(x_j(t)) \right), \ i = 1, \dots, n.$$
 (1)

Hopfield (1984)

$$\dot{x}_i(t) = -b_i(x_i(t)) + \sum_{j=1}^n c_{ij}f_j(x_j(t)), \quad i = 1, ..., n.$$
 (2)

where

- *a<sub>i</sub>* amplification functions;  $b_i$  controller functions;
- $f_i$  activation functions;

 $C = [c_{ii}]$  conection matrix. (本部) (本語) (本語)

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#### \*Cohen-Grossberg neural network model

$$\dot{x}_{i}(t) = -a_{i}(x_{i}(t)) \left[ b_{i}(x_{i}(t)) + \sum_{j=1}^{n} \sum_{p=1}^{P} \left( h_{ij}^{(p)}(x_{j}(t - \tau_{ij}^{(p)}(t))) + f_{ij}^{(p)}\left( \int_{-\infty}^{0} g_{ij}^{(p)}(x_{j}(t + s)) d\eta_{ij}^{(p)}(s) \right) \right) \right]$$
(3)

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▶  $a_i : \mathbb{R} \to (0, +\infty)$ , are continuous functions;

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- ▶  $a_i : \mathbb{R} \to (0, +\infty)$ , are continuous functions;
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- ▶  $au_{ij}^{(p)}: [0, +\infty) \to [0, +\infty)$  are continuous functions;
- ▶  $\eta_{ij}^{(p)}$  :  $(-\infty, 0] \to \mathbb{R}$  are non-decreasing bounded normalized functions so that

$$\eta_{ij}^{(p)}(0) - \eta_{ij}^{(p)}(-\infty) = 1.$$

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\*Phase Space "strong fading memory"

$$UC_g = \left\{ \phi \in C((-\infty, 0]; \mathbb{R}^n) : \sup_{s \le 0} \frac{|\phi(s)|}{g(s)} < \infty, \frac{\phi(s)}{g(s)} \text{ unif. cont.} \right\},$$

$$\|\phi\|_g = \sup_{s \le 0} \frac{|\phi(s)|}{g(s)}$$
 with  $|x| = |(x_1, \dots, x_n)| = \max_{1 \le i \le n} |x_i|$ 

where:

(g1) 
$$g: (-\infty, 0] \rightarrow [1, +\infty)$$
 non-increasing, continuous,  $g(0) = 1$ ;  
(g2)  $\lim_{u \rightarrow 0^{-}} \frac{g(s+u)}{g(s)} = 1$  uniformly on  $(-\infty, 0]$ ;  
(g3)  $g(s) \rightarrow +\infty$  as  $s \rightarrow -\infty$ .

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(g1)  $g: (-\infty, 0] \rightarrow [1, +\infty)$  non-increasing, continuous, g(0) = 1; (g2)  $\lim_{u \to 0^{-}} \frac{g(s+u)}{g(s)} = 1$  uniformly on  $(-\infty, 0]$ ; (g3)  $g(s) \rightarrow +\infty$  as  $s \rightarrow -\infty$ .

Example:  $g(s) = e^{-lpha s}$ ,  $s \in (-\infty, 0]$ , with lpha > 0

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Arguing as in [1], we conclude that there is  $g : (-\infty, 0] \rightarrow [1, +\infty)$  continuous such that **(g1)-(g3)** hold and

$$\int_{-\infty}^0 g(s) d\eta_{ij}^{(p)} < +\infty.$$

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\* Initial Condition

$$x_0 = \varphi, \quad \varphi \in BC_g$$
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where  $BC_g := \{ \varphi \in C((-\infty, 0]; \mathbb{R}^n) : \varphi \text{ is bounded} \} \leq UC_g.$ 

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A equilibrium point  $x^* \in \mathbb{R}^n$  is said global attractive if

$$x(t,0,arphi) o x^*$$
 as  $t \to \infty$ , for all  $arphi \in BC_g$ .

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\*  $A = [a_{ij}] \in \mathbb{R}^{n \times n}$  is a non-singular M-matrix if  $a_{ij} \leq 0$ ,  $i \neq j$ and Re  $\sigma(A) > 0$ .

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For (3) we assume the following hypotheses:

▶ (A1) 
$$\exists \beta_i > 0, \forall u, v \in \mathbb{R}, u \neq v$$
:

$$(b_i(u) - b_i(v))/(u - v) \geq \beta_i;$$

[In particular, for  $b_i(u) = \beta_i u$ .]

Cohen-Grossberg neural network model Model Phase Space Global asymptotic stability Hypotheses Application Preliminaries lemmas For (3) we assume the following hypotheses: ▶ (A1)  $\exists \beta_i > 0, \forall u, v \in \mathbb{R}, u \neq v$ :  $(b_i(u) - b_i(v))/(u - v) > \beta_i$ [In particular, for  $b_i(u) = \beta_i u$ .] • (A2)  $f_{ij}^{(p)}, g_{ij}^{(p)}, h_{ij}^{(p)}$  are Lipshitz function with Lipshitz constant  $\gamma_{ii}^{(\vec{p})}, \mu_{ii}^{(\vec{p})}$ , and  $\sigma_{ii}^{(p)}$  respectively;

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is a non-singular M-matrix.

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Lemma A [2] If H : ℝ<sup>n</sup> → ℝ<sup>n</sup> is continuous and injective such that

$$\lim_{|x|\to+\infty}|H(x)|=+\infty$$

than H(x) is a homeomorphism of  $\mathbb{R}^n$  onto itself.

[2] M. Forti, A. Tesi, IEEE Trans.Circuits Syst. 42 (1995) 354-366.

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• FDE with  $\infty$  delay in  $UC_g$ 

$$\begin{aligned} \dot{x}(t) &= f(t, x_t), \quad t \ge 0 \end{aligned} \tag{5}$$
$$x_t \in UC_g, \quad x_t(s) = x(t+s), s \le 0 \end{aligned}$$
with  $f = (f_1, \dots, f_n) : [0, +\infty) \times UC_g \to \mathbb{R}^n$  continuous

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#### Lemma B

Assume that f transforms closed bounded sets of  $(-\infty, 0] \times UC_g$  into bounded sets of  $\mathbb{R}^n$ . If **(H)**  $\forall t > 0, \forall \varphi \in BC_g$ :  $\forall s \in (-\infty, 0), |\varphi(s)| < |\varphi(0)| \Rightarrow \varphi_i(0)f_i(t, \varphi) < 0,$ for some  $i \in \{1, ..., n\}$  such that  $|\varphi(s)| = |\varphi_i(0)|,$ 

then the solution  $x(t) = x(t, 0, \varphi)$ ,  $\varphi \in BC_g$ , of (5) is defined and bounded on  $[0, +\infty)$  and

$$|x(t,0,\varphi)| \leq ||\varphi||_{\infty}.$$

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#### \*Proof of Lemma B (idea)

► 
$$x(t) = x(t, 0, \varphi)$$
 solution on  $[-\infty, a)$ ,  $a > 0$ , with  $\varphi \in BC_g$   
 $k := \sup_{s \le 0} |\varphi(s)|$ .

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►  $x(t) = x(t, 0, \varphi)$  solution on  $[-\infty, a)$ , a > 0, with  $\varphi \in BC_g$  $k := \sup_{s \le 0} |\varphi(s)|.$ 

Suppose that  $|x(t_1)| > k$  for some  $t_1 > 0$  and define

$$T = \min \left\{ t \in [0, t_1] : |x(t)| = \max_{s \in [0, t_1]} |x(s)| \right\}.$$

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• We have  $|x_T(s)| = |x(T+s)| < |x(T)|$ , for s < 0. By **(H)** we conclude that,

$$x_i(T)f_i(T,x_T) < 0,$$

for some  $i \in \{1, ..., n\}$  such that  $|x_i(T)| = |x(T)|$ . If  $x_i(T) > 0$  (analogous if  $x_i(T) < 0$ ), then  $\dot{x}_i(T) < 0$ .

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$$\Rightarrow \dot{x}_i(T) \geq 0.$$

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$$\Rightarrow \dot{x}_i(T) \ge 0.$$

▶ Contradition. Thus x(t) is defined and bounded on  $[0, \pm \infty)_{\pm}$ 

## **Global asymptotic stability**

• Consider equation (3) in  $UC_g$ ,

$$egin{aligned} \dot{x}_i(t) &= -a_i(x_i(t)) iggl[ b_i(x_i(t)) + \sum_{j=1}^n \sum_{p=1}^P iggl(h_{ij}^{(p)}(x_j(t- au_{ij}^{(p)}(t))) + \ + f_{ij}^{(p)} iggl(\int_{-\infty}^0 g_{ij}^{(p)}(x_j(t+s)) d\eta_{ij}^{(p)}(s) iggr) iggr) iggr] \end{aligned}$$

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► Theorem Assume (A1)-(A4). Then there is an unique equilibrium point x\* ∈ ℝ<sup>n</sup> of (3) which is globally attractive.

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Proof (idea)

N non-singular M-matrix  $\Rightarrow \exists d = (d_1, \ldots, d_n) > 0$ :

$$\beta_i > d_i^{-1} \sum_{j=1}^n I_{ij} d_j, \quad i = 1, \dots, n;$$
 (6)

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Proof (idea)

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 (6)

• The change of variables  $y_i(t) = d_i^{-1} x_i(t)$  transforms (3) into

$$\dot{y}_i(t) = -\bar{a}_i(y_i(t)) \left[ \bar{b}_i(y_i(t)) + \bar{h}_i(t, y_t) \right], \qquad (7)$$

where, for  $t\geq$  0,  $u\in\mathbb{R}$ , and  $\phi\in\mathit{UC}_{g}$ ,

$$ar{h}_i(t,\phi) = d_i^{-1} \sum_{j=1}^n \sum_{p=1}^P igg[ h_{ij}^{(p)}(d_j\phi_j(- au_{ij}^{(p)}(t))) + \ + f_{ij}^{(p)} \left( \int_{-\infty}^0 g_{ij}^{(p)}(d_j\phi_j(s)) d\eta_{ij}^{(p)}(s) 
ight) igg], \ ar{a}_i = a_i(d_i(u)), \quad ar{b}_i(u) = d_i^{-1}b_i(d_i(u)).$$

Existence and uniqueness of equilibrium point Using Lemma A the continuous function  $H : \mathbb{R}^n \to \mathbb{R}^n$  defined by

$$H(z) = \left(\bar{b}_i(z_i) + d_i^{-1} \sum_{j=1}^n \sum_{p=1}^P (h_{ij}^{(p)}(d_j z_j) + f_{ij}^{(p)}(g_{ij}^{(p)}(d_j z_j)))\right)_{i=1}^n$$

is a homeomorphism.

Then there exists  $x^* \in \mathbb{R}^n$ ,  $H(x^*) = 0$ , i.e.  $x^*$  is the equilibrium.

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Then there exists  $x^* \in \mathbb{R}^n$ ,  $H(x^*) = 0$ , i.e.  $x^*$  is the equilibrium.

• By translation, we may assume that  $x^* = 0$ , i.e.,

$$ar{b}_i(0)+ar{h}_i(t,0)=0, \quad orall i=1,\ldots,n, t\geq 0.$$

▶ Bounded solution on  $\mathbb{R}$ For  $\varphi \in BC_g$  such that  $|\varphi(s)| < |\varphi(0)| = \varphi_i(0) > 0$  for s < 0 (analogous if  $\varphi_i(0) < 0$ ),

$$\begin{split} \bar{b}_{i}(\varphi_{i}(0)) + \bar{h}_{i}(t,\varphi) &= [\bar{b}_{i}(\varphi_{i}(0)) - \bar{b}_{i}(0)] + [\bar{h}_{i}(t,\varphi) - \bar{h}_{i}(t,0)] \\ &\geq \beta_{i}\varphi_{i}(0) - d_{i}^{-1}\sum_{j=1}^{n}\sum_{p=1}^{P} \left(\gamma_{ij}^{(p)}d_{j}|\varphi_{j}(-\tau_{ij}^{(p)}(t))| + \right. \\ &+ \mu_{ij}^{(p)}\sigma_{ij}^{(p)}d_{j}\int_{-\infty}^{0} |\varphi_{j}(s)|d\eta_{ij}^{(p)}(s)\right) \\ &\geq \beta_{i}\varphi_{i}(0) - d_{i}^{-1}\sum_{j=1}^{n}l_{ij}d_{j}\sup_{s\leq 0} |\varphi(s)| = (\beta_{i} - d_{i}^{-1}\sum_{j=1}^{n}l_{ij}d_{j})\varphi_{i}(0) > 0. \end{split}$$

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$$egin{aligned} ar{b}_i(arphi_i(0)) + ar{h}_i(t,arphi) &= [ar{b}_i(arphi_i(0)) - ar{b}_i(0)] + [ar{h}_i(t,arphi) - ar{h}_i(t,0)] \ &\geq eta_iarphi_i(0) - d_i^{-1}\sum_{j=1}^n\sum_{p=1}^p \left( \gamma_{ij}^{(p)} d_j |arphi_j(- au_{ij}^{(p)}(t))| + \ &+ \mu_{ij}^{(p)} \sigma_{ij}^{(p)} d_j \int_{-\infty}^0 |arphi_j(s)| d\eta_{ij}^{(p)}(s) 
ight) \ &\geq eta_iarphi_i(0) - d_i^{-1}\sum_{j=1}^n l_{ij} d_j \sup_{s\leq 0} |arphi(s)| = (eta_i - d_i^{-1}\sum_{j=1}^n l_{ij} d_j) arphi_i(0) > 0. \end{aligned}$$

► Then (H) holds and from Lemma B we conclude that all solutions of (7) are defined and bounded on [0, +∞).

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### • It remains to prove that x = 0 is global attractive.

José J. Oliveira Global Stability of a general Cohen-Grossberg model

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- ▶ Let  $y(t) = y(t, 0, \varphi) = (y_i(t))_{i=1}^n$  a solution of (7), with  $\varphi \in BC_g$ , and define

 $-v_i = \liminf_{t \to +\infty} y_i(t), \quad u_i = \limsup_{t \to +\infty} y_i(t)$  $v = \max_i \{v_i\}, \quad u = \max_i \{u_i\},$  $u, v \in \mathbb{R}, -v \le u.$  We have to show that  $\max(u, v) = 0.$ 

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- Suppose  $|v| \le u$ .  $(|u| \le v$  is similar) Let  $i \in \{1, ..., n\}$  such that  $u_i = u$ .
- ▶ By computations, we can show that exists  $(t_k)_{k \in \mathbb{N}}$  such that

$$t_k \nearrow +\infty, \quad y_i(t_k) 
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$$t_k \nearrow +\infty, \quad y_i(t_k) \to u, \quad \text{and} \quad \overline{b}_i(y_i(t_k)) + \overline{h}_i(t_k, y_{t_k}) \to 0.$$

Now, assume that u > 0 (to get a contradiction). Fix  $\varepsilon > 0$  and  $T = T(\varepsilon) > 0$  such that  $|y(t)| < u_{\varepsilon} := u + \varepsilon$ , and  $\int_{-\infty}^{-T} d\eta_{ij}^{(p)}(s) < \varepsilon/||y_0||_{\infty}$ .

▶ There is a large  $k_0 \in \mathbb{N}$  such that, for  $k \ge 0$ 

$$\begin{split} \bar{b}_{i}(y_{i}(t_{k})) + \bar{h}_{i}(t_{k}, y_{t_{k}}) &\geq \beta_{i}y_{i}(t_{k}) - d_{i}^{-1}\sum_{j=1}^{n}\sum_{\rho=1}^{P}\left(\gamma_{ij}^{(\rho)}d_{j}u_{\varepsilon} + \mu_{ij}^{(\rho)}\sigma_{ij}^{(\rho)}d_{j}\left(\int_{-\infty}^{-T}|y_{j}(t_{k}+s)|d\eta_{ij}^{(\rho)}(s) + \int_{-T}^{0}|y_{j}(t_{k}+s)|d\eta_{ij}^{(\rho)}(s)\right) \geq \\ &\geq \beta_{i}y_{i}(t_{k}) - d_{i}^{-1}\sum_{j=1}^{n}\sum_{\rho=1}^{P}\left(\gamma_{ij}^{(\rho)}d_{j}u_{\varepsilon} + \mu_{ij}^{(\rho)}\sigma_{ij}^{(\rho)}d_{j}\left(\varepsilon + u_{\varepsilon}\int_{-T}^{0}d\eta_{ij}^{(\rho)}(s)\right)\right) \\ &\geq \beta_{i}y_{i}(t_{k}) - d_{i}^{-1}\sum_{j=1}^{n}d_{j}l_{ij}u_{2\varepsilon}. \end{split}$$

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▶ Taking  $k \to +\infty$  and  $\varepsilon \to 0$ , we get

$$\liminf_{k\to\infty} [\bar{b}_i(y_i(t_k)) + \bar{h}_i(t_k, y_{t_k})] \geq \left(\beta_i - d_i^{-1}\sum_{j=1}^n l_{ij}d_j\right)u > 0.$$

which is a contradiction and we conclude u = v = 0.

# Application

Cohen-Grossberg model with infinite discrete time-varying delays [3]

$$\dot{x}_{i}(t) = -a_{i}(x_{i}(t)) \left[ b_{i}(x_{i}(t)) - \sum_{j=1}^{n} c_{ij}g_{j}(x_{j}(t)) - \sum_{j=1}^{n} d_{ij}f_{j}(x_{j}(t - \tau_{ij}(t))) \right]$$
(8)

▶  $a_i : \mathbb{R} \to (0, +\infty)$ , are continuous functions;

[3] T. Huang, A. Chan, Y. Huang, Neural Netw. 20 (2007) 868-873. ← □ → ← ∂→ ← ≧ → ← ≧ → ■ ≧

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- ▶  $\tau_{ij} : [0,\infty) \to [0,\infty)$  are continuous satisfying (A3);
- $N := diag(\beta_1, \dots, \beta_n) [I_{ij}]$ , with  $I_{ij} = |c_{ij}|\gamma_j + |d_{ij}|\mu_j$  is a M-matrix.
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#### Corollary

There is a unique equilibrium point of (8), which is globally attractive.

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#### Corollary

There is a unique equilibrium point of (8), which is globally attractive.

In [3] assumed the following <u>additional conditions</u>:
 b<sub>i</sub>(u) are differentiable such that

 $b_i'(u) \geq \beta_i > 0,$ 

[clearly a strong hypothesis than **(A1)**]; There are  $\underline{a}_i, \overline{a}_i > 0$  such that

$$\underline{a}_i \leq a_i(u) \leq \overline{a}_i, \quad u \in \mathbb{R}.$$

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#### Thank you

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