Global stability of a Cohen-Grossberg neural network model with both time-varying and continuous distributed delays

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Neural Network Models

*Pioneer Models:

- Cohen-Grossberg (1983)

$$\dot{x}_i(t) = -a_i(x_i(t))\left(b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij} f_j(x_j(t))\right), \quad i = 1, \dots, n. \quad (1)$$

- Hopfield (1984)

$$\dot{x}_i(t) = -b_i(x_i(t)) + \sum_{j=1}^{n} c_{ij} f_j(x_j(t)), \qquad i = 1, \dots, n. \quad (2)$$

where
a_i amplification functions; b_i controller functions;
f_j activation functions; $C = [c_{ij}]$ conection matrix.

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$$\dot{x}_i(t) = -a_i(x_i(t))\Bigg[b_i(x_i(t)) + \sum_{j=1}^{n}\sum_{p=1}^{P}\Bigg(h_{ij}^{(p)}(x_j(t-\tau_{ij}^{(p)}(t)))+ \\ + f_{ij}^{(p)}\left(\int_{-\infty}^{0} g_{ij}^{(p)}(x_j(t+s))d\eta_{ij}^{(p)}(s)\right)\Bigg)\Bigg] \qquad (3)$$

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- $\tau_{ij}^{(p)} : [0, +\infty) \to [0, +\infty)$ are continuous functions;
- $\eta_{ij}^{(p)} : (-\infty, 0] \to \mathbb{R}$ are non-decreasing bounded normalized functions so that

$$\eta_{ij}^{(p)}(0) - \eta_{ij}^{(p)}(-\infty) = 1.$$

***Phase Space** "strong fading memory"

$$UC_g = \left\{ \phi \in C((-\infty, 0]; \mathbb{R}^n) : \sup_{s \le 0} \frac{|\phi(s)|}{g(s)} < \infty, \frac{\phi(s)}{g(s)} \text{ unif. cont.} \right\},$$

$$\|\phi\|_g = \sup_{s \le 0} \frac{|\phi(s)|}{g(s)} \text{ with } |x| = |(x_1, \dots, x_n)| = \max_{1 \le i \le n} |x_i|$$

where:

(g1) $g : (-\infty, 0] \to [1, +\infty)$ non-increasing, continuous, $g(0) = 1$;

(g2) $\lim_{u \to 0^-} \frac{g(s+u)}{g(s)} = 1$ uniformly on $(-\infty, 0]$;

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Example: $g(s) = e^{-\alpha s}$, $s \in (-\infty, 0]$, with $\alpha > 0$

Arguing as in [1], we conclude that there is $g : (-\infty, 0] \to [1, +\infty)$ continuous such that **(g1)-(g3)** hold and

$$\int_{-\infty}^{0} g(s) d\eta_{ij}^{(p)} < +\infty.$$

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* **Initial Condition**

$$x_0=\varphi, \qquad \varphi\in BC_g \tag{4}$$

where $BC_g := \{\varphi\in C((-\infty,0];\mathbb{R}^n) : \varphi \text{ is bounded}\} \leq UC_g$.

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* **Definition**

A equilibrium point $x^* \in \mathbb{R}^n$ is said *global attractive* if

$$x(t, 0, \varphi) \to x^* \text{ as } t \to \infty, \text{ for all } \varphi \in BC_g.$$

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* $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is a **non-singular M-matrix** if $a_{ij} \leq 0$, $i \neq j$ and Re $\sigma(A) > 0$.

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For (3) we assume the following hypotheses:

- **(A1)** $\exists \beta_i > 0, \forall u, v \in \mathbb{R}, u \neq v$:

$$(b_i(u) - b_i(v))/(u - v) \geq \beta_i;$$

[In particular, for $b_i(u) = \beta_i u$.]

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- **(A2)** $f_{ij}^{(p)}, g_{ij}^{(p)}, h_{ij}^{(p)}$ are Lipshitz function with Lipshitz constant $\gamma_{ij}^{(p)}, \mu_{ij}^{(p)}$, and $\sigma_{ij}^{(p)}$ respectively;

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- **(A4)** The matrix

$$N = diag(\beta_1, \ldots, \beta_n) - [l_{ij}], \quad \text{with } l_{ij} = \sum_{p=1}^{P} \gamma_{ij}^{(p)} + \mu_{ij}^{(p)} \sigma_{ij}^{(p)},$$

  is a non-singular M-matrix.

- **Lemma A** [2] If $H : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and injective such that

$$\lim_{|x| \to +\infty} |H(x)| = +\infty$$

than $H(x)$ is a homeomorphism of $\mathbb{R}^n$ onto itself.

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- FDE with ∞ delay in UC_g

$$\dot{x}(t) = f(t, x_t), \qquad t \geq 0 \tag{5}$$

$$x_t \in UC_g, \qquad x_t(s) = x(t+s), s \leq 0$$

with $f = (f_1, \ldots, f_n) : [0, +\infty) \times UC_g \to \mathbb{R}^n$ continuous

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- **Lemma B**

  Assume that f transforms closed bounded sets of $(-\infty, 0] \times UC_g$ into bounded sets of $\mathbb{R}^n$.
  If
  (H) $\forall t > 0, \forall \varphi \in BC_g$:

$$\forall s \in (-\infty, 0), |\varphi(s)| < |\varphi(0)| \Rightarrow \varphi_i(0) f_i(t, \varphi) < 0,$$

  for some $i \in \{1, \dots, n\}$ such that $|\varphi(s)| = |\varphi_i(0)|$,

  then the solution $x(t) = x(t, 0, \varphi)$, $\varphi \in BC_g$, of (5) is defined and bounded on $[0, +\infty)$ and

$$|x(t, 0, \varphi)| \leq ||\varphi||_\infty.$$

*Proof of Lemma B (idea)

- $x(t) = x(t, 0, \varphi)$ solution on $[-\infty, a)$, $a > 0$, with $\varphi \in BC_g$
  $k := \sup_{s \leq 0} |\varphi(s)|$.

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- $x(t)=x(t,0,\varphi)$ solution on $[-\infty,a)$, $a>0$, with $\varphi\in BC_g$
  $k:=\sup_{s\leq 0}|\varphi(s)|$.
- Suppose that $|x(t_1)|>k$ for some $t_1>0$ and define

$$T=\min\left\{t\in[0,t_1]:|x(t)|=\max_{s\in[0,t_1]}|x(s)|\right\}.$$

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$$T = \min \left\{ t \in [0, t_1] : |x(t)| = \max_{s \in [0, t_1]} |x(s)| \right\}.$$

- We have $|x_T(s)| = |x(T + s)| < |x(T)|$, for $s < 0$. By **(H)** we conclude that,

$$x_i(T) f_i(T, x_T) < 0,$$

for some $i \in \{1, \ldots, n\}$ such that $|x_i(T)| = |x(T)|$. If $x_i(T) > 0$ (analogous if $x_i(T) < 0$), then $\dot{x}_i(T) < 0$.

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$$\Rightarrow \dot{x}_i(T) \geq 0.$$

- Contradition. Thus $x(t)$ is defined and bounded on $[0, +\infty)$.

Global asymptotic stability

- Consider equation (3) in UC_g,

$$\dot{x}_i(t) = -a_i(x_i(t))\Bigg[b_i(x_i(t)) + \sum_{j=1}^{n}\sum_{p=1}^{P}\Bigg(h_{ij}^{(p)}(x_j(t-\tau_{ij}^{(p)}(t)))+ \\ + f_{ij}^{(p)}\left(\int_{-\infty}^{0} g_{ij}^{(p)}(x_j(t+s))d\eta_{ij}^{(p)}(s)\right)\Bigg)\Bigg]$$

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- **Theorem** Assume **(A1)-(A4)**.
  Then there is an unique equilibrium point $x^* \in \mathbb{R}^n$ of (3) which is globally attractive.

- **Proof (idea)**
  N non-singular M-matrix $\Rightarrow \exists d = (d_1, \dots, d_n) > 0$:

$$\beta_i > d_i^{-1} \sum_{j=1}^{n} l_{ij} d_j, \qquad i = 1, \dots, n; \tag{6}$$

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  $$\beta_i > d_i^{-1}\sum_{j=1}^{n} l_{ij}d_j, \qquad i=1,\dots,n; \tag{6}$$
- The change of variables $y_i(t)=d_i^{-1}x_i(t)$ transforms (3) into
  $$\dot{y}_i(t) = -\bar{a}_i(y_i(t))\left[\bar{b}_i(y_i(t))+\bar{h}_i(t,y_t)\right], \tag{7}$$
  where, for $t\geq 0$, $u\in\mathbb{R}$, and $\phi\in UC_g$,
  $$\bar{h}_i(t,\phi)=d_i^{-1}\sum_{j=1}^{n}\sum_{p=1}^{P}\Big[h_{ij}^{(p)}(d_j\phi_j(-\tau_{ij}^{(p)}(t)))+$$
  $$+f_{ij}^{(p)}\left(\int_{-\infty}^{0} g_{ij}^{(p)}(d_j\phi_j(s))d\eta_{ij}^{(p)}(s)\right)\Big],$$
  $$\bar{a}_i=a_i(d_i(u)), \qquad \bar{b}_i(u)=d_i^{-1}b_i(d_i(u)).$$

- Existence and uniqueness of equilibrium point

  Using Lemma A the continuous function $H:\mathbb{R}^n \to \mathbb{R}^n$ defined by

$$H(z) = \left(\bar{b}_i(z_i) + d_i^{-1} \sum_{j=1}^{n} \sum_{p=1}^{P} (h_{ij}^{(p)}(d_j z_j) + f_{ij}^{(p)}(g_{ij}^{(p)}(d_j z_j))) \right)_{i=1}^{n}$$

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  is a homeomorphism.
  Then there exists $x^*\in\mathbb{R}^n$, $H(x^*)=0$, i.e. x^* is the equilibrium.
- By translation, we may assume that $x^*=0$, i.e.,

$$\bar{b}_i(0)+\bar{h}_i(t,0)=0,\qquad \forall i=1,\dots,n,t\geq 0.$$

- Bounded solution on $\mathbb{R}$
  For $\varphi \in BC_g$ such that $|\varphi(s)| < |\varphi(0)| = \varphi_i(0) > 0$ for $s < 0$ (analogous if $\varphi_i(0) < 0$),

$$\begin{aligned}
\bar{b}_i(\varphi_i(0)) + \bar{h}_i(t,\varphi) &= [\bar{b}_i(\varphi_i(0)) - \bar{b}_i(0)] + [\bar{h}_i(t,\varphi) - \bar{h}_i(t,0)] \\
&\geq \beta_i \varphi_i(0) - d_i^{-1} \sum_{j=1}^{n} \sum_{p=1}^{P} \left(\gamma_{ij}^{(p)} d_j |\varphi_j(-\tau_{ij}^{(p)}(t))| + \right. \\
&\qquad \left. + \mu_{ij}^{(p)} \sigma_{ij}^{(p)} d_j \int_{-\infty}^{0} |\varphi_j(s)| d\eta_{ij}^{(p)}(s) \right) \\
&\geq \beta_i \varphi_i(0) - d_i^{-1} \sum_{j=1}^{n} l_{ij} d_j \sup_{s \leq 0} |\varphi(s)| = (\beta_i - d_i^{-1} \sum_{j=1}^{n} l_{ij} d_j) \varphi_i(0) > 0.
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$$\bar{b}_i(\varphi_i(0)) + \bar{h}_i(t,\varphi) = [\bar{b}_i(\varphi_i(0)) - \bar{b}_i(0)] + [\bar{h}_i(t,\varphi) - \bar{h}_i(t,0)]$$

$$\geq \beta_i\varphi_i(0) - d_i^{-1}\sum_{j=1}^{n}\sum_{p=1}^{P}\Bigg(\gamma_{ij}^{(p)} d_j|\varphi_j(-\tau_{ij}^{(p)}(t))| +$$

$$+\mu_{ij}^{(p)}\sigma_{ij}^{(p)} d_j \int_{-\infty}^{0}|\varphi_j(s)|d\eta_{ij}^{(p)}(s)\Bigg)$$

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- Then **(H)** holds and from Lemma B we conclude that all solutions of (7) are defined and bounded on $[0, +\infty)$.

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$$-v_i = \liminf_{t \to +\infty} y_i(t), \qquad u_i = \limsup_{t \to +\infty} y_i(t)$$

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- Suppose $|v| \leq u$. ($|u| \leq v$ is similar)
  Let $i \in \{1, \ldots, n\}$ such that $u_i = u$.
- By computations, we can show that exists $(t_k)_{k \in \mathbb{N}}$ such that

$$t_k \nearrow +\infty, \qquad y_i(t_k) \to u, \qquad \text{and} \qquad \bar{b}_i(y_i(t_k)) + \bar{h}_i(t_k, y_{t_k}) \to 0.$$

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- Now, assume that $u > 0$ (to get a contradiction).
  Fix $\varepsilon > 0$ and $T = T(\varepsilon) > 0$ such that
  $|y(t)| < u_\varepsilon := u + \varepsilon$, and $\int_{-\infty}^{-T} d\eta_{ij}^{(p)}(s) < \varepsilon / ||y_0||_\infty$.

- There is a large $k_0 \in \mathbb{N}$ such that, for $k \geq 0$

$$\bar{b}_i(y_i(t_k)) + \bar{h}_i(t_k, y_{t_k}) \geq \beta_i y_i(t_k) - d_i^{-1} \sum_{j=1}^{n} \sum_{p=1}^{P} \Bigg(\gamma_{ij}^{(p)} d_j u_\varepsilon +$$

$$\mu_{ij}^{(p)} \sigma_{ij}^{(p)} d_j \left(\int_{-\infty}^{-T} |y_j(t_k + s)| d\eta_{ij}^{(p)}(s) + \int_{-T}^{0} |y_j(t_k + s)| d\eta_{ij}^{(p)}(s) \right) \geq$$

$$\geq \beta_i y_i(t_k) - d_i^{-1} \sum_{j=1}^{n} \sum_{p=1}^{P} \left(\gamma_{ij}^{(p)} d_j u_\varepsilon + \mu_{ij}^{(p)} \sigma_{ij}^{(p)} d_j \left(\varepsilon + u_\varepsilon \int_{-T}^{0} d\eta_{ij}^{(p)}(s) \right) \right)$$

$$\geq \beta_i y_i(t_k) - d_i^{-1} \sum_{j=1}^{n} d_j l_{ij} u_{2\varepsilon}.$$

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$$\geq \beta_i y_i(t_k) - d_i^{-1} \sum_{j=1}^{n} d_j l_{ij} u_{2\varepsilon}.$$

- Taking $k \to +\infty$ and $\varepsilon \to 0$, we get

$$\liminf_{k \to \infty} [\bar{b}_i(y_i(t_k)) + \bar{h}_i(t_k, y_{t_k})] \geq \Bigg(\beta_i - d_i^{-1} \sum_{j=1}^{n} l_{ij} d_j \Bigg) u > 0.$$

which is a contradiction and we conclude $u = v = 0$.

Application

Cohen-Grossberg model with infinite discrete time-varying delays [3]

$$\dot{x}_i(t) = -a_i(x_i(t)) \left[b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij} g_j(x_j(t)) - \sum_{j=1}^{n} d_{ij} f_j(x_j(t - \tau_{ij}(t))) \right] \quad (8)$$

- $a_i : \mathbb{R} \to (0, +\infty)$, are continuous functions;

[3] T. Huang, A. Chan, Y. Huang, Neural Netw. 20 (2007) 868-873.

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- $\tau_{ij} : [0, \infty) \to [0, \infty)$ are continuous satisfying **(A3)**;
- $N := diag(\beta_1, \dots, \beta_n) - [l_{ij}]$, with $l_{ij} = |c_{ij}|\gamma_j + |d_{ij}|\mu_j$ is a M-matrix.

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- **Corollary**
  There is a unique equilibrium point of (8), which is globally attractive.

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- **Corollary**
  There is a unique equilibrium point of (8), which is globally attractive.
- In [3] assumed the following additional conditions:
  $b_i(u)$ are differentiable such that

$$b_i'(u) \geq \beta_i > 0,$$

[clearly a strong hypothesis than **(A1)**];
There are $\underline{a}_i, \overline{a}_i > 0$ such that

$$\underline{a}_i \leq a_i(u) \leq \overline{a}_i, \qquad u \in \mathbb{R}.$$

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Thank you

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