General criterion for exponential stability of neural network models with unbounded distributed delays

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Boundedness of solutions

$$\dot{x}(t) = f(t, x_t)$$

Global exponential stability

$$\dot{x}_i(t) = -a_i(x_i(t))\,(b_i(x_i(t)) + f_i(x_t)))$$

Application

Cohen-Grossberg neural network model with unbounded delays

In

Teresa Faria and José J. Oliveira,
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the case of neural networks with bounded distributed delays was treated.

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$$UC_g = \left\{ \phi \in C((-\infty, 0]; \mathbb{R}^n) : \sup_{s \le 0} \frac{|\phi(s)|}{g(s)} < \infty, \frac{\phi(s)}{g(s)} \text{ unif. cont.} \right\}$$

where $g(s) = e^{-\alpha s}$, $s \in (-\infty, 0]$, with $\alpha > 0$, and the norm

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$$\|\phi\|_g = \sup_{s \le 0} \frac{|\phi(s)|}{g(s)};$$

- BC_g subspace of bounded continuous functions;
- $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is a **non-singular M-matrix** if $a_{ij} \le 0$, $i \ne j$ and Re $\sigma(A) > 0$.

Boundedness of solutions

- FDE in UC_g

$$\dot{x}(t) = f(t, x_t), \qquad t \geq 0, \tag{1}$$

$f = (f_1, \dots, f_n) : [0, +\infty) \times UC_g \to \mathbb{R}^n$ continuous,
$f(B)$ is bounded for all $B \subseteq [0, +\infty) \times UC_g$ closed bounded
$x_t(s) = x(t + s)$, $s \in (-\infty, 0]$,

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- **Proposition 1**

(H) $\forall t > 0, \forall \varphi \in BC_g$:

$$\forall s \in (-\infty, 0), |\varphi(s)| < |\varphi(0)| \Rightarrow \varphi_i(0) f_i(t, \varphi) < 0,$$

for some $i \in \{1, \dots, n\}$ such that $|\varphi(s)| = |\varphi_i(0)|$.
Then all solution of (1) with initial condition on BC_g is defined and bounded on $[0, +\infty)$ and

$$|x(t, 0, \varphi)| \leq \sup_{s \leq 0} |\varphi(s)|.$$

Proof of Proposition 1 (idea)

- $x(t) = x(t, 0, \varphi)$ solution on $[-\infty, a)$, $a > 0$, with $\varphi \in BC_g$
  $k := \sup_{s \leq 0} |\varphi(s)|$.

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- Suppose that $|x(t_1)| > k$ for some $t_1 > 0$ and define

$$T = \min \left\{ t \in [0, t_1] : |x(t)| = \max_{s \in [0, t_1]} |x(s)| \right\}.$$

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- We have $|x_T(s)| = |x(T + s)| < |x(T)|$, for $s < 0$.
  By **(H)** we conclude that,

$$x_i(T) f_i(T, x_T) < 0,$$

  for some $i \in \{1, \ldots, n\}$ such that $|x_i(T)| = |x(T)|$. If $x_i(T) > 0$ (analogous if $x_i(T) < 0$), then $\dot{x}_i(T) < 0$.

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- $x_i(t) \leq |x(t)| < |x(T)| = x_i(T)$, $t \in [0, T)$,

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$$\Rightarrow \dot{x}_i(T) \geq 0.$$

- Contradition. Thus $x(t)$ is defined and bounded on $[0, +\infty)$.

Global exponential stability

- Consider the general neural network model with unbounded distributed delays in UC_g, $g(s) = e^{-\alpha s}$, $\alpha > 0$,

$$\dot{x}_i(t) = -a_i(x_i(t))[b_i(x_i(t)) + f_i(x_t)] \qquad (2)$$

where $a_i : \mathbb{R} \to (0, +\infty)$, $b_i : \mathbb{R} \to \mathbb{R}$ and $f_i : UC_g \to \mathbb{R}$ are continuous functions such that

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- **(A2)** $\exists \beta_i > 0, \forall u, v \in \mathbb{R}, u \neq v$:

$$(b_i(u) - b_i(v))/(u - v) \geq \beta_i;$$

[In particular, for $b_i(u) = \beta_i u$.]

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- **(A3)** f_i is a Lipshitz function with constant l_i.

- **Theorem 1**
  Assume **(A1)**, **(A2)**, **(A3)**. If

$$\beta_i > l_i, \qquad \forall i \in \{1, \dots, n\},$$

then there is a unique equilibrium point of (2), which is globally exponentially stable, i.e. there is $\varepsilon > 0$ such that

$$|x(t, 0, \varphi)| \leq e^{-\varepsilon t} \sup_{s \leq 0} |\varphi(s)|, \qquad t \geq 0, \varphi \in BC_g.$$

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- **Proof (idea)**
Existence and uniqueness of equilibrium point
$$\begin{array}{cccc} H: & \mathbb{R}^n & \rightarrow & \mathbb{R}^n \\ & x & \mapsto & (b_i(x_i) + f_i(x))_{i=1}^n \end{array}$$
is homeomorphism.
Then there exists $x^* \in \mathbb{R}^n$, $H(x^*) = 0$, i.e. x^* is the equilibrium.

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- $\beta_i > l_i \Rightarrow \varepsilon - \underline{a}(\beta_i - l_i) < 0$ for some $\varepsilon \in (0, \alpha)$
- Let $x(t, 0, \varphi)$ a solution of (2), $\varphi \in BC_g$.
  The change of variables

$$z(t) = e^{\varepsilon t} x(t)$$

transforms (2) into

$$\dot{z}_i(t) = F_i(t, z_t), \qquad i = 1, \dots, n, \tag{3}$$

where

$$F_i(t, \phi) = \varepsilon \phi_i(0) - a_i(e^{-\varepsilon(t+\cdot)}\phi)e^{\varepsilon t}\left[b_i(e^{-\epsilon t}\phi_i(0)) + f_i(e^{-\varepsilon(t+\cdot)}\phi)\right]$$

- Let $\phi \in BC_g$ such that $|\phi(s)| < |\phi(0)|$, for $s \in (-\infty, 0)$. Consider $i \in \{1, \ldots, n\}$ such that $|\phi_i(0)| = |\phi(0)|$.

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- If $\phi_i(0) > 0$ ($\phi_i(0) < 0$ is analogous)
  From the hypotheses we conclude that

$$\begin{aligned} F_i(t,\phi) &\leq \varepsilon\phi_i(0) - \underline{a}e^{\varepsilon t}\left[b_i(e^{-\varepsilon t}\phi_i(0)) - b_i(0) + \right. \\ &\quad \left. + f_i(e^{-\varepsilon(t+\cdot)}\phi) - f_i(0)\right] \\ &\leq \varepsilon\phi_i(0) - \underline{a}\left[\beta_i\phi_i(0) - l_i \sup_{s\leq 0} e^{(\alpha-\varepsilon)s}|\phi(s)|\right] \\ &\leq \phi_i(0)[\varepsilon - \underline{a}(\beta_i - l_i)] < 0. \end{aligned}$$

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 Then $F = (F_1, \dots, F_n)$ satisfies **(H)**
- From Proposition 1, $z(t)$ is defined on $[0, +\infty)$ and

$$|x(t,0,\varphi)| = |e^{-\varepsilon t} z(t,0,e^{\varepsilon \cdot}\varphi)| \leq e^{-\varepsilon t} \sup_{s\leq 0} |\varphi(s)|.$$

Generalized Cohen-Grossberg Model

Cohen-Grossberg model with unbounded distributed delays (4)

$$\dot{x}_i(t) = -a_i(x_i(t)) \left[b_i(x_i(t)) + \sum_{j=1}^{n} \sum_{p=1}^{P} f_{ij}^{(p)} \left(\int_{-\infty}^{0} g_{ij}^{(p)}(x_j(t+s)) d\eta_{ij}^{(p)}(s) \right) \right]$$

- $a_i : \mathbb{R} \to (0, +\infty)$, are continuous satisfying **(A1)**;

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- $\eta_{ij}^{(p)} : (-\infty, 0] \to \mathbb{R}$ are non-decreasing bounded normalized functions such that

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$$\exists \gamma > 0 : \quad \int_{-\infty}^{0} e^{-\gamma s} d\eta_{ij}^{(p)} < \infty;$$

- $N := diag(\beta_1, \dots, \beta_n) - [l_{ij}]$, where $l_{ij} = \sum_{p=1}^{P} \mu_{ij}^{(p)}\sigma_{ij}^{(p)}$.

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- **Proof** (idea)
  N non-singular M-matrix $\Rightarrow \exists d = (d_1, \dots, d_n) > 0$: $Nd > 0$
  $\Rightarrow \exists \delta > 0$:

$$\beta_i > d_i^{-1} \sum_{j=1}^{n} l_{ij}(1+\delta)d_j, \qquad i = 1, \dots, n; \tag{5}$$

- There is $\alpha \in (0, \gamma)$ such that

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- Taking $g(s) = e^{-\alpha s}$, $s \in (-\infty, 0]$, we consider UC_g the phase space of (4).

- The change of variables

$$y_i(t) = d_i^{-1} x_i(t)$$

transforms (4) into

$$\dot{y}_i(t) = -\bar{a}_i(y_i(t)) \left[\bar{b}_i(y_i(t)) + \bar{f}_i(y_t) \right], \qquad (6)$$

where

$$\bar{f}_i(\phi) = d_i^{-1} \sum_{j=1}^{n} \sum_{p=1}^{P} f_{ij}^{(p)} \left(\int_{-\infty}^{0} g_{ij}^{(p)}(d_j \phi_j(s)) d\eta_{ij}^{(p)}(s) \right), \phi \in UC_g$$

$$\bar{b}_i(u) = d_i^{-1} b_i(d_i(u)), \qquad \bar{a}_i = a_i(d_i(u)), \qquad u \in \mathbb{R}.$$

- After some computations,

$$|\bar{f}_i(\phi)-\bar{f}_i(\psi)| \leq \left(d_i^{-1} \sum_{j=1}^{n} l_{ij}(1+\delta)d_j \right) \|\phi-\psi\|_g, \qquad \phi,\psi \in UC_g,$$

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- Once $\bar{b}_i$ satisfies **(A2)** with $\bar{\beta}_i = \beta_i$, the conclusion follows from (5) and Theorem 1.

Example

Cohen-Grossberg neural network with unbounded distributed delays

$$\dot{x}_i(t) = -a_i(x_i(t)) \left[b_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j \left(\int_{-\infty}^{0} k_{ij}(-s) x_j(t+s) ds \right) \right] \qquad (7)$$

- $a_{ij} \in \mathbb{R}$, $a_i : \mathbb{R} \to (0, +\infty)$ and $b_i : \mathbb{R} \to \mathbb{R}$ are continuous satisfying **(A1)** and **(A2)** respectively;

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- $f_j : \mathbb{R} \to \mathbb{R}$ are Lipschitzian with constant μ_j;
- The delay kernel functions $k_{ij} : [0, +\infty) \to \mathbb{R}_0^+$ satisfy

$$\int_0^{+\infty} k_{ij}(t)dt = 1, \qquad \exists\gamma > 0 : \int_0^{+\infty} k_{ij}(t)e^{\gamma t}dt < \infty;$$

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- $a_{ij} \in \mathbb{R}$, $a_i : \mathbb{R} \to (0, +\infty)$ and $b_i : \mathbb{R} \to \mathbb{R}$ are continuous satisfying **(A1)** and **(A2)** respectively;
- $f_j : \mathbb{R} \to \mathbb{R}$ are Lipschitzian with constant μ_j;
- The delay kernel functions $k_{ij} : [0, +\infty) \to \mathbb{R}_0^+$ satisfy

$$\int_0^{+\infty} k_{ij}(t) dt = 1, \qquad \exists \gamma > 0 : \int_0^{+\infty} k_{ij}(t) e^{\gamma t} dt < \infty;$$

- $N := diag(\beta_1, \dots, \beta_n) - [l_{ij}]$, where $l_{ij} = |a_{ij}| \mu_j$.

- **Corollary**
  If N is a non-singular M-matrix, then there is a unique equilibrium point of (7), which is globally exponentially stable.

[1] W. Wu, B.T. Cui, X.Y. Lou, Global exponential stability of Cohen-Grossberg neural networks with distributed delays, Math. Comput. Modelling, 47 (2008) 868-873.

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  System (4) reduces to (7) if $P = 1$, $f_{ij}^{(1)}(u) = a_{ij} f_j(u)$ and

$$\eta_{ij}^{(1)}(s) = \int_{-\infty}^{s} k_{ij}(-\zeta) d\zeta,$$

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  then the result follows from Theorem 2.
- In [1] assumed the additional conditions:
  $0 < \underline{a}_i \leq a_i(u) \leq \bar{a}_i$;
  $\underline{N} := B\underline{A} - \overline{A}[l_{ij}]$ is a non-singular M-matrix, where
  $\underline{A} = diag(\underline{a}_1, \ldots, \underline{a}_n)$, $\overline{A} = diag(\bar{a}_1, \ldots, \bar{a}_n)$.

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Thank you