Global stability of delayed differential equations with applications

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December 19, 2014

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Standard notations

- The delay: $\tau > 0$
- Banach space: $C_n := C([-\tau, 0]; \mathbb{R}^n)$

$$\|\varphi\| = \sup_{\theta \in [-\tau, 0]} |\varphi(\theta)|,$$

with $|x| = |(x_1, \ldots, x_n)| = \max_i |x_i|$ for $x \in \mathbb{R}^n$.

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- For $x \in C([a - \tau, b]; \mathbb{R}^n)$, where $a < b$, and $t \in [a, b]$, we define $x_t \in C_n$ by

$$x_t(\theta) = x(t + \theta), \quad \theta \in [-\tau, 0].$$

Delay differential equation

- The delay differential equation

$$x'(t) = f(t, x_t), \qquad t \geq 0, \tag{1}$$

where $f : [0, +\infty) \times C_n \to \mathbb{R}^n$ is a function and C_n is the phase space. ($\mathbb{R}^n$ is not the phase space)

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$$x_0 = \varphi, \qquad \varphi \in C_n. \tag{2}$$

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- Why do we need the delay?
  * Biological models:
    To take into account the maturation period of the species.
  * Neural network models:
    To take into account the communication time between neurons.

- (Existence) If $f(t,\varphi)$ is continuous then there is a solution $x:[-\tau,a)\to\mathbb{R}^n$ of IVP (1)-(2), for some $a>0$.

- (Uniqueness) If $f(t,\varphi)$ is continuous and locally Lipschitz on φ, then the solution of (1)-(2) is unique.
  We denote the solution by $x(t,0,\varphi)$.

- (Noncontinuable solution) If $f(t,\varphi)$ is completely continuous, then a maximal solution of (1) either exists on $[-\tau,+\infty)$ or becomes unbounded at some finite time.

Global stability

* **Definition:** A solution $\bar{x}(t)$ of (1) is said *globally attractive* on $\mathcal{X} \subseteq C_n$ if

$$|x(t,0,\varphi) - \bar{x}(t)| \longrightarrow 0, \quad \text{as} \quad t \to +\infty, \quad \forall \varphi \in \mathcal{X};$$

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$$|x(t,0,\varphi_1) - x(t,0,\varphi_2)| \leq M\,\mathrm{e}^{-\delta t}\,\|\varphi_1 - \varphi_2\|,$$

for all $t \geq 0$, $\varphi_1, \varphi_2 \in C_n$.

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- Two kinds of hypotheses to get the global stability:
  - Small delays;
  - The growth function $f(t,\varphi)$ has a dominant undelayed part.

* Scalar Logistic equation with delay

$$y'(t) = ay(t)\left(1 - \frac{1}{k}y(t-\tau)\right), \qquad t \geq 0 \tag{3}$$

where $a > 0$ is the growth rate, $k > 0$ is the carrying capacity of the ecosystem, and $\tau > 0$ is the maturation period.

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- The change $x(t) = -1 + y(t)/k$ transforms (3) into

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with characteristic equation $\lambda + a\,\mathrm{e}^{-\lambda\tau} = 0$.

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with characteristic equation $\lambda + a\,e^{-\lambda\tau} = 0$.

- If $a\tau > \frac{\pi}{2}$, then the equilibrium $y(t) \equiv k$ of (3) is unstable. If $a\tau < \frac{\pi}{2}$, then the equilibrium $y(t) \equiv k$ of (3) is locally asymptotically stable.

* **Theorem [Wright, 1955]:** If

$$a\tau < \frac{3}{2},$$

then the equilibrium $y(t) \equiv k$ of (3) globally attractive on positive solutions.

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* **Wright's conjecture, [1955](remains open):**

The result holds with $a\tau < \frac{\pi}{2}$ instead of $a\tau < \frac{3}{2}$.

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* **Wright's conjecture, [1955](remains open):**

The result holds with $a\tau < \dfrac{\pi}{2}$ instead of $a\tau < \dfrac{3}{2}$.

* **Theorem [Yoneyama, 1987]:** For any $\alpha > \frac{3}{2}$, there exists $a(t) \leq \alpha$ for all $t \geq 0$ such that the equilibrium $y(t) \equiv k$ of

$$y'(t) = a(t)y(t)\left(1 - \frac{1}{k}y(t-1)\right), \quad t \geq 0 \tag{6}$$

is unstable. If $\alpha < \frac{3}{2}$ then $y(t) \equiv k$ of (6) is globally attractive on positive solutions.

[1] E.M. Wright, *J. Reine Angew. Math.* 194 (1955) 66-87.

3/2 conditions

- **Main contributions**

  * [1] E.M. Wright, *J. Reine Angew. Math.* 194 (1955) 66-87.
  * [2] J.A. Yorke, J. Differential Equations 7 (1970) 189-202.
  * [3] T. Yoneyama, J. Math. Anal. Appl. 125 (1987) 161-173.
  * [4] J. Sugie, Proc. Roy. Soc. Edinburgh 120A (1992) 179-184.
  * [5] E. Liz, V. Tkachenko, & S. Trofimchuk, Discrete Contin. Dyn. Syst. (Suppl.) (2003) 580-589.

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  * [5] E. Liz, V. Tkachenko, & S. Trofimchuk, Discrete Contin. Dyn. Syst. (Suppl.) (2003) 580-589.

- **Small contributions**

  * [6] J.J. Oliveira, T. Faria, E. Liz, & S. Trofimchuk, Discrete Contin. Dyn. Syst. 12 (2005) 481-500.
  * [7] J.J. Oliveira & T. Faria, J. Math. Anal. Appl. 329 (2007) 1397-1420.

- Hopfield neural network model (1984)

$$x_i'(t) = -b_i(x_i(t)) + \sum_{j=1}^{n} c_{ij} f_j(x_j(t)), \qquad i = 1, \dots, n, \tag{7}$$

where b_i are controller functions, f_j are activation functions, and $C = [c_{ij}]$ is the conection matrix.

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- Generalized Hopfield's model with delays

$$x_i'(t) = -b_i(t, x_i(t)) + f_i(t, x_t), \qquad i = 1, \dots, n, \qquad (8)$$

where

* $b_i : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ are continuous functions;
* $f_i : [0, +\infty) \times C_n \to \mathbb{R}$ are continuous functions.

- In C_n, consider the DDE

$$x_i'(t) = F_i(t, x_t), \qquad t \geq 0, \qquad i = 1, \ldots, n, \tag{9}$$

with $F = (F_1, \ldots, F_n) : [0, +\infty) \times C_n \to \mathbb{R}^n$ continuous

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* **Lemma [TF, JJO, 8]:** If
(H) $\forall t > 0, \forall \varphi \in C_n$:

$$\forall \theta \in [-\tau, 0), |\varphi(\theta)| < |\varphi(0)| \Rightarrow \varphi_i(0) F_i(t, \varphi) < 0,$$

for some $i \in \{1, \dots, n\}$ such that $\|\varphi\| = |\varphi(0)| = |\varphi_i(0)|$,

then the solution $x(t) = x(t, 0, \varphi)$, $\varphi \in C_n$, of (9) is defined and bounded on $[-\tau, +\infty)$ and

$$|x(t, 0, \varphi)| \leq ||\varphi||.$$

[8] T. Faria, J.J. Oliveira, *J. Diff. Equ.* (2008) 1049-1079.

*Proof of Lemma (idea)

- $x(t) = x(t, 0, \varphi)$ solution on $[-\tau, \alpha)$, $\alpha > 0$, with $\varphi \in C_n$
- Suppose that $|x(t_1)| > \|\varphi\|$ for some $t_1 > 0$ and define

$$T = \min\left\{ t \in [0, t_1] : |x(t)| = \max_{s \in [0, t_1]} |x(s)| \right\}.$$

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$$T=\min\left\{t\in[0,t_1]:|x(t)|=\max_{s\in[0,t_1]}|x(s)|\right\}.$$

- We have $|x_T(\theta)|=|x(T+\theta)|<|x(T)|$, for $\theta\in[-\tau,0)$. By **(H)** we conclude that,

$$x_i(T)F_i(T,x_T)<0,$$

for some $i\in\{1,\dots,n\}$ such that $|x_i(T)|=|x(T)|$. If $x_i(T)>0$ (analogous if $x_i(T)<0$), then $x_i'(T)<0$.

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- $x_i(t) \leq |x(t)| < |x(T)| = x_i(T)$, $t \in [0, T)$,

$$\Rightarrow x_i'(T) \geq 0.$$

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$$\Rightarrow x_i'(T) \geq 0.$$

- Contradition. Thus $x(t)$ is defined and bounded on $[0, +\infty)$.

For (8) we assume the following hypotheses:

For each $i = 1, \ldots, n$,

- **(A1)** $\exists\, x^* = (x_1^*, \ldots, x_n^*) \in \mathbb{R}^n$ a equilibrium point of (8);

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- **(A1)** $\exists\, x^* = (x_1^*, \dots, x_n^*) \in \mathbb{R}^n$ a equilibrium point of (8);
- **(A2)** $\exists\, \beta_i : [0, +\infty) \to (0, +\infty)$, $\forall u, v \in \mathbb{R}\ u \neq v$:

$$(b_i(t,u) - b_i(t,v))/(u - v) \geq \beta_i(t), \qquad \forall t \geq 0;$$

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- **(A3)** $\exists\, l_i : [0, +\infty) \to (0, +\infty)$

$$|f_i(t,\varphi) - f_i(t,\psi)| \leq l_i(t) ||\varphi - \psi||, \quad \forall t \geq 0,\ \forall \varphi, \psi \in C_n;$$

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- **(A3)** $\exists\ l_i:[0,+\infty)\to(0,+\infty)$

$$|f_i(t,\varphi)-f_i(t,\psi)|\leq l_i(t)||\varphi-\psi||,\qquad\forall t\geq 0,\ \forall\varphi,\psi\in C_n;$$

- **(A4)** $\exists\varepsilon>0$ and $\lambda:\mathbb{R}\to(0,+\infty)$ a continuous function:

$$\beta_i(t)-l_i(t)e^{\int_{t-\tau}^{t}\lambda(s)\,ds}>\lambda(t)\text{ and }\int_0^t\lambda(s)\,ds\geq\varepsilon t,\ \forall t\geq 0.$$

Global exponential stability

Theorem [SE, EG, JJO 9]: Assume **(A1)-(A4)**
Then the equilibrium of (8) is globally exponentially stable.

[9] S. Esteves, E. Gökmen, J.J. Oliveira, *Appl. Math. Comput.* 219 (2013) 2861-2870.

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* Proof of Theorem (idea)

- The change of variables $z(t) = e^{\int_0^t \lambda(u)\, du} x(t)$ transforms (8) into

$$z_i'(t) = g_i(t, z_t), \qquad t \geq 0 \tag{10}$$

with

$$g_i(t, \varphi) = \lambda(t)\varphi_i(0) - e^{\int_0^t \lambda(u)\, du}[b_i(t, \psi(t)_i(0)) - f_i(t, \psi(t))]$$

and $\psi(t)(\theta) = e^{-\int_0^{t+\theta} \lambda(u)\, du} \varphi(\theta)$, $\theta \in [-\tau, 0]$

- $\psi(t) \in C_n$

[9] S. Esteves, E. Gökmen, J.J. Oliveira, *Appl. Math. Comput.* 219 (2013) 2861-2870.

- Take $t \geq 0$ and $\varphi \in C_n$ such that

$$|\varphi(\theta)| < |\varphi(0)| = \|\varphi\| = \varphi_i(0) > 0, \qquad \forall \theta \in [-\tau, 0).$$

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- From the hypotheses and assuming $x^* = 0$

$$\begin{aligned} g_i(t,\varphi) &= \lambda(t)\varphi_i(0) - e^{\int_0^t \lambda(u)\,du}[b_i(t,\psi(t)_i(0)) - b_i(t,0) \\ &\quad + f_i(t,\psi(t)) - f_i(t,0)] \\ &\leq \lambda(t)\varphi_i(0) - e^{\int_0^t \lambda(u)\,du} \cdot \\ &\quad \cdot \left[\beta_i(t)e^{-\int_0^t \lambda(u)\,du}\varphi_i(0) - l_i(t)e^{-\int_0^{t-\tau} \lambda(u)\,du}\|\varphi\|\right] \\ &\leq \varphi_i(0)\left[\lambda(t) - \left(\beta_i(t) - l_i(t)e^{\int_{t-\tau}^t \lambda(u)\,du}\right)\right] < 0 \end{aligned}$$

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- Then **(H)** holds and, from the Lemma,

$$\begin{aligned} |x(t,0,\varphi)| &= \left|e^{-\int_0^t \lambda(u)\,du} z(t,0,\psi(0))\right| \\ &\leq e^{-\varepsilon t}\left|z\left(t,0,e^{-\int_\cdot^0 \lambda(u)\,du}\varphi\right)\right| \leq e^{-\varepsilon t}\|\varphi\|. \end{aligned}$$

Corollary 1: Assume **(A2)-(A4)**
Then the system (8)

$$x_i'(t) = -b_i(t, x_i(t)) + f_i(t, x_t), \qquad t \geq 0,$$

is globally exponentially stable.

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* Proof (idea)

- Let $\bar{x}(t) = x(t, 0, \bar{\varphi})$. The change of variables $z(t) = x(t) - \bar{x}(t)$ transforms (11) into

$$z_i'(t) = -\bar{b}_i(t, z_i(t)) + \bar{f}_i(t, z_t), \qquad t \geq 0 \qquad (11)$$

with

$$\bar{b}_i(t, u) = b_i(t, u + \bar{x}_i(t)) \text{ and } \bar{f}_i(t, \varphi) = f_i(t, \varphi + \bar{x}_t) + b_i(t, \bar{x}_i(t)) - f_i(t, \bar{x}_t)$$

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$$\bar{b}_i(t, u) = b_i(t, u + \bar{x}_i(t)) \text{ and } \bar{f}_i(t, \varphi) = f_i(t, \varphi + \bar{x}_t) + b_i(t, \bar{x}_i(t)) - f_i(t, \bar{x}_t)$$

- Zero is an equilibrium of (11) and from the Theorem [SE, EG, JJO 9]

$$|x(t) - \bar{x}(t)| = |z(t)| \leq e^{-\varepsilon t}\|z_0\| = e^{-\varepsilon t}\|\varphi - \bar{\varphi}\|, \qquad \forall t \geq 0.$$

* **Corollary 2:** Assume **(A2)** and **(A3)**.
If $l_i(t)$ are bounded and there exists $\alpha > 0$:

$$\beta_i(t) - l_i(t) > \alpha, \quad \forall t \geq 0, \tag{12}$$

then the system (8) is globally exponentially stable.

* **Corollary 2:** Assume **(A2)** and **(A3)**.
If $l_i(t)$ are bounded and there exists $\alpha > 0$:

$$\beta_i(t) - l_i(t) > \alpha, \quad \forall t \geq 0, \tag{12}$$

then the system (8) is globally exponentially stable.

- Proof (idea)
By computation, it is easy to see that (12) implies that there exists $\varepsilon > 0$ such that **(A4)** holds with $\lambda(t) = \varepsilon$, i.e.

$$\beta_i(t) - l_i(t)e^{\varepsilon\tau} > \varepsilon.$$

Periodic systems

Assume that the system (8)

$$x_i'(t) = -b_i(t, x_i(t)) + f_i(t, x_t), \qquad t \geq 0$$

is ω-periodic, $\omega > 0$, that is:

$$b_i(t, u) = b_i(t + \omega, u), \qquad \forall t \geq 0, \ \forall u \in \mathbb{R};$$

$$f_i(t, \varphi) = f_i(t + \omega, \varphi), \qquad \forall t \geq 0, \ \forall \varphi \in C_n.$$

Theorem [SE, EG, JJO 9]: Assume **(A2)**, **(A3)**, and

$$\beta_i(t) - l_i(t) > 0, \qquad \forall t \in [0, \omega].$$

Then (8) has a ω-periodic solution which is globally exponentially stable.

[9] S. Esteves, E. Gökmen, J.J. Oliveira, *Appl. Math. Comput.* 219 (2013) 2861-2870.

* (Proof) Show the existence of a periodic solution.
  From Corollary 2

$$\|x_t(\varphi) - x_t(\bar{\varphi})\| \leq e^{-\varepsilon(t-\tau)}\|\varphi - \bar{\varphi}\|, \quad \forall t \geq \tau, \ \forall \varphi, \bar{\varphi} \in C_n.$$

* (Proof) Show the existence of a periodic solution.
  From Corollary 2

$$\|x_t(\varphi) - x_t(\bar{\varphi})\| \leq e^{-\varepsilon(t-\tau)}\|\varphi - \bar{\varphi}\|, \quad \forall t \geq \tau, \ \forall \varphi, \bar{\varphi} \in C_n.$$

- Let $k \in \mathbb{N}$ such that $e^{-(k\omega-\tau)} \leq \frac{1}{2}$ and define $P : C_n \to C_n$ by $P(\varphi) = x_\omega(\varphi)$.

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- As $P^k(P(\varphi^*)) = P(P^k(\varphi^*)) = P(\varphi^*)$, then

$$P(\varphi^*) = \varphi^* \Leftrightarrow x_\omega(\varphi^*) = \varphi^*$$

  and $x(t, 0, \varphi^*)$ is the periodic solution of (11).

Example: For the periodic model:

$$x_i'(t) = -b_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t - \tau_{ij}(t))) + I_i(t) \quad (13)$$

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- $b_i, a_{ij}, b_{ij}, I_i : [0, +\infty) \to \mathbb{R}$, $\tau_{ij}(t) \geq 0$ are ω-periodic continuous;
- $f_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with constant l_j;
- $b_i(t) - \sum_{j=1}^{n} l_j(|a_{ij}(t)| + |b_{ij}(t)|) > 0, \qquad \forall i, \forall t \in [0, \omega].$

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- In [10] assumed the additional hypothesis

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Thank you