Global stability for impulsive delay differential equations and application to a periodic Lasota-Wazewska model

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Impulsive delayed differential equation

- Impulsive functional differential equation:

$$\begin{cases} x'(t) = -a(t)x(t) + f(t, x_t), & 0 \le t \neq t_k \\ \Delta x(t_k) := x(t_k^+) - x(t_k) = I_k(x_{t_k}), & k = 1, 2, \cdots \end{cases} \tag{1}$$

where
  - $a : [0, \infty) \to [0, \infty)$, $I_k : \mathbb{R} \to \mathbb{R}$ are continuous;
  - $0 < t_1 < t_2 < \cdots < t_k \to \infty$;
  - $x_t(s) = x(t + s)$, for $s \in (-\infty, 0]$;
  - $f : [0, +\infty) \times \mathcal{BS} \to \mathbb{R}$ is continuous or piecewise continuous.

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With $\mathcal{BS}$ a convenient Banach space of functions $\varphi : (-\infty, 0] \to \mathbb{R}$.
- We consider Bounded Initial Conditions:

$$x_0 = \varphi \in B\mathcal{BS}, \text{(bounded functions on } \mathcal{BS}). \tag{2}$$

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- $[\gamma, \beta]$ compact interval of $\mathbb{R}$, $PC([\gamma, \beta]; \mathbb{R})$ space of functions $\phi : [\gamma, \beta] \to \mathbb{R}$ continuous except for a finite points s, $\phi(s^-), \phi(s^+)$ exist, and $\phi(s^-) = \phi(s)$;

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with sup norm $\|\varphi\| = \sup_{s \leq 0} |\varphi(s)|$.
- For $t \geq 0$, we define $PC(t) := PC([-\tau(t), 0], \mathbb{R})$, where $\tau : [0, \infty) \to [0, \infty)$ is a continuous function such that $\lim_{t \to \infty} (t - \tau(t)) = \infty$.

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(H1) $\exists$ $(a_k)_k$, $(b_k)_k$ positive sequences:

$$b_k x^2 \le x[x + I_k(x)] \le a_k x^2, \qquad x \in \mathbb{R}, k \in \mathbb{N};$$

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(H3) $\exists\ \lambda_1, \lambda_2 : [0,\infty) \to [0,\infty)$ piecewise continuous:

$$-\lambda_1(t)\mathcal{M}_t(\varphi) \leq f(t,\varphi) \leq \lambda_2(t)\mathcal{M}_t(-\varphi),\ t \geq 0,\ \varphi \in PC(t), \quad (3)$$

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(H4) $\exists\ T > 0$ such that

$$\alpha_1\alpha_2 < 1,$$

where $\alpha_j = \sup\limits_{t\geq T} \displaystyle\int_{t-\tau(t)}^{t} \lambda_j(s)\, e^{-\int_s^t a(u)\,du}\, B(s)\, ds, j = 1, 2,$

with $B(t) := \max_{\theta\in[-\tau(t),0]} \left(\prod_{k:t+\theta\leq t_k<t} b_k^{-1} \right)$.

Main result

(H1)+(H3) $\Longrightarrow$ $x = 0$ is an equilibrium point of (1).

- **Theorem 1**
  Assume (H1)-(H4). Then the zero solution of (1) is globally asymptotically stable.

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- **Theorem 1**
  Assume (H1)-(H4). Then the zero solution of (1) is globally asymptotically stable.
- Yan's work [1]
  - (A1) $b_k x^2 \leq x[x + I_k(x)] \leq x^2, \quad x \in \mathbb{R}, k \in \mathbb{N};$
  - (A2) (H2)+extra condition to deal with non-oscillatory solutions;
  - (A3) Yorke's condition with $\lambda(t) := \lambda_1(t) = \lambda_2(t)$;
  - (A4) 3/2-type condition

$$\overline{\alpha} = \sup_{T \geq 0} \int_{t-\tau(t)}^{t} \lambda(s)\, e^{\int_{t-\tau(t)}^{s} a(u)\, du} B(s)\, ds < \frac{3}{2}.$$

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* **Lemma 1** [1] If $x(t)$ is solution of (1) then

$$y(t) = \prod_{k:0\leq t_k<t} J_k(x(t_k))x(t), \qquad (4)$$

with $J_k(u) = \frac{u}{u+I_k(u)}$ $(u \neq 0)$, is continuous and satisfies

$$y'(t) + a(t)y(t) = \prod_{k:0\leq t_k<t} J_k(x(t_k))f(t,x_t), \qquad t \neq t_k. \qquad (5)$$

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* **Remark**

$$\text{(H1)} \Rightarrow a_k^{-1} \le J_k(u) \le b_k^{-1} \quad \forall\ u \neq 0,\ k \in \mathbb{N} \tag{6}$$

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* **Definition**
A solution $x(t)$ is *oscillatory* if it is not eventually zero and it has arbitrarily large zeros; otherwise $x(t)$ is *non-oscillatory*.
* **Notation**

$$A(t) := \int_0^t a(u)du$$

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* **Proof of main result** (idea)
<u>*Case 1*</u>: $x(t)$ is non-oscillatory (assume $x(t) > 0$ for large t)
$x(t) > 0 \Rightarrow y(t) > 0$ and from Yorke's condition (H3)

$$y'(t) \leq y'(t) + a(t)y(t) \leq 0.$$

Thus $y(t) \searrow c$ and $e^{A(t)}y(t) \searrow w$, for some $c, w > 0$.
Now,

$$x(t) = y(t) \prod_{k:0 \leq t_k < t} J_k^{-1}(x(t_k)) \leq y(t) \prod_{k:0 \leq t_k < t} a_k \leq y(t)M,$$

with $M = \max_m \left(\prod_{k=1}^m a_k\right)$.
(H2) $\Rightarrow \lim\limits_{t\to\infty} A(t) = \infty$, consequently

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0.$$

Case 2: $x(t)$ is oscillatory

* **Lemma 2** Assume (H1), (H3), and (H4)
  Let $t_0 > T$ such that $y(t_0) = 0$. For any $\eta > 0$:
  (i) If $-\eta \leq y(t) \leq \eta\alpha_2$ for all $t \in [t_0 - \tau(t_0), t_0]$, then
  $$-\eta \leq y(t) \leq \eta\alpha_2 \text{ for all } t > t_0;$$
  (ii) If $-\eta\alpha_1 \leq y(t) \leq \eta$ for all $t \in [t_0 - \tau(t_0), t_0]$, then
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  $-\eta\alpha_1 \leq y(t) \leq \eta$ for all $t > t_0$.

- *Proof of (i)* (idea).
  By contradiction, suppose that there exists $T_0 > t_0$:

$$y(T_0) > \eta\alpha_2 \text{ and } -\eta \leq y(t) < y(T_0), \ \forall t < T_0.$$

  It is easy to show that exists $\xi_0 \in [T_0 - \tau(T_0), T_0]$ such that

$$y(\xi_0) = 0 \text{ and } y(t) > 0 \text{ for all } t \in (\xi_0, T_0].$$

  Clearly, $t_0 \leq \xi_0$.

- By hypotheses, for $s \in [\xi_0, T_0] \setminus \{t_k\}$,

$$
\begin{aligned}
\left(e^{A(s)}y(s)\right)' &= \prod_{k:0\le t_k<s} J_k(x(t_k))e^{A(s)}f(s,x_s)\\
&\le e^{A(s)}\lambda_2(s)\prod_{k:0\le t_k<s} J_k(x(t_k))\mathcal{M}_s(-x_s)
\end{aligned}
$$

$$
= e^{A(s)}\lambda_2(s)\max\left\{0,\sup_{\theta\in[-\tau(s),0]}\left(-y(s+\theta)\prod_{k:s+\theta\le t_k<s} J_k(x(t_k))\right)\right\}
$$

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\le e^{A(s)}\lambda_2(s)B(s)\mathcal{M}_s(-y_s).
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$$\le e^{A(s)}\lambda_2(s)B(s)\mathcal{M}_s(-y_s).$$

- As $-y_s \le \eta$ for all $s \in [\xi_0, T_0]$, thus

$$\left(e^{A(s)}y(s)\right)' \le \eta e^{A(s)}\lambda_2(s)B(s).$$

- Integrating over $[\xi_0, T_0]$, we get

$$y(T_0) \leq \eta e^{-A(T_0)} \int_{\xi_0}^{T_0} e^{A(s)} \lambda_2(s) B(s)\, ds$$
$$= \eta \int_{\xi_0}^{T_0} e^{-\int_s^{T_0} a(u)\, du} \lambda_2(s) B(s)\, ds \leq \alpha_2 \eta,$$

which contradicts the definition of T_0.

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- Analogously, we obtain a contradiction if we assume that there exists $T_0 > t_0$:

$$y(T_0) < -\eta \text{ and } y(T_0) \leq y(t) < \eta\alpha_2,\ \forall t < T_0.$$

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- The proof of (ii) is similar.□

- By Lemma 2, $y(t)$ is bounded.
  $x(t)$ oscillatory, implies $y(t)$ oscillatory. Thus

$$-v := \liminf_{t\to\infty} y(t), \qquad u := \limsup_{t\to\infty} y(t)$$

  with $0 \leq u, v < \infty$. We have to show that $u = v = 0$.

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- Fix $\varepsilon > 0$. We have

$$-v_\varepsilon := -(v+\varepsilon) < y(t) < u + \varepsilon := u_\varepsilon \text{ for large } t \tag{7}$$

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- $y(t)$ is continuous, there exists $s_n \nearrow \infty$:
  $y(s_n) > 0, y(s_n)$ are local maxima, and $y(s_n) \to u,$ as $n \to \infty$.
  As above, there exists $\xi_n \in [s_n - \tau(s_n), s_n)$ such that

$$y(\xi_n) = 0 \text{ and } y(s) > 0 \text{ for all } s \in (\xi_n, s_n].$$

  By (7), we have

$$y(s) > -v_\varepsilon, \;\; \forall s \in [\xi_n - \tau(\xi_n), s_n] \text{ for large } n.$$

- Arguing as in the proof of Lemma 2 (i), we conclude that

$$y(s_n) \leq \alpha_2 v_\varepsilon.$$

Letting $n \to \infty$ and $\varepsilon \to 0^+$, we obtain

$$u \leq \alpha_2 v.$$

- Arguing as in the proof of Lemma 2 (i), we conclude that

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- Similar arguments lead to

$$v \leq \alpha_1 u.$$

Now, we have

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Now, we have

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- Finally, hypotheses (H4), i.e. $\alpha_1 \alpha_2 < 1$, implies $u = v = 0$.$\square$

- Theorem 1 can be slightly improved form models

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where $f_i(t, x_t^i) = f_i(t, x_{|_{[t-\tau_i(t), t]}})$ and $\tau_i(t)$ are delay functions.

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- **Theorem 2** Assume (H1), (H2), there exist piecewise continuous $\lambda_{1,i}, \lambda_{2,i} : [0, \infty) \to [0, \infty)$ such that

$$-\lambda_{1,i}(t)\mathcal{M}_t^i(\varphi) \le f_i(t, \varphi_{|_{[-\tau_i(t),0]}}) \le \lambda_{2,i}(t)\mathcal{M}_t^i(-\varphi), \quad (9)$$

  and $\alpha_1\alpha_2 < 1$ with

$$\alpha_j(T) := \sup_{t \ge T} \int_{t-\tau(t)}^{t} \sum_{i=1}^{n} \lambda_{j,i}(s) e^{-\int_s^t a(u)du} B_i(s)\, ds, \ j = 1, 2,$$

  and $B_i(t) := \max_{\theta \in [-\tau_i(t),0]} \left(\prod_{k: t+\theta \le t_k < t} b_k^{-1} \right), \ i = 1, \ldots, n.$

  Then the zero solution of (8) is globally asymptotically stable.

* **Corollary 1** Assume (H1), (H2), f_i satisfy (9), and

$$\sum_{i=1}^{n} \lambda_{j,i}(t) B_i(t) \leq c_j a(t), \qquad j = 1, 2,$$

with $c_1, c_2 > 0$. If either
(i) $\mathcal{A} := \limsup_{t\geq 0} \int_{t-\tau(t)}^{t} a(u)\,du < \infty$ with $c_1 c_2 < \frac{1}{(1-e^{-\mathcal{A}})^2}$
or
(ii) $\mathcal{A} = \infty$ with $c_1 c_2 < 1$,
then the zero solution of (8) is globally asymptotically stable.

[2] X.H. Tang, *J. Math. Anal. Appl.* 302 (2005) 342-359.

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- Considering the model (1), $n = 1$, without impulses, $I_k(u) \equiv 0$, and taking $c_1 = c_2 = 1$ in Corollary 1, we obtain the criterion presented by X.H. Tang [2].

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Periodic Lasota-Wazewska model

$$\begin{cases} N'(t) + a(t)N(t) = \sum_{i=1}^{n} b_i(t)e^{-\beta_i(t)N(t-\tau_i(t))}, & t \neq t_k, \\ \Delta N(t_k) = I_k(N(t_k)), & k = 1, 2, \ldots, \end{cases} \tag{10}$$

where

(f_0) $a(t), b_i(t), \beta_i(t), \tau_i(t)$ are continuous, positive and ω-periodic;

(i_0) $I_k : [0, \infty) \to \mathbb{R}$ are continuous, $u + I_k(u) > 0$, and $\exists p \in \mathbb{N}$

$$t_{k+p} = t_k + \omega, \qquad I_{k+p}(u) = I_k(u), \qquad k \in \mathbb{N}, u > 0;$$

(i_1) $\exists\ a_1, \ldots, a_p,\ b_1, \ldots, b_p$, with $b_k > -1$, such that

$$b_k \leq \frac{I_k(x) - I_k(y)}{x - y} \leq a_k, \qquad x, y \geq 0, x \neq y, k = 1, \ldots, p;$$

(i_2) $\prod_{k=1}^{p}(1 + a_k) \leq 1.$

- Some criteria for the existence of an ω-periodic solution to (10) have been established. For example in Li et al.[3]. We assume that there exists a positive ω-periodic solution $N^*(t)$.

[3] X. Li, X. Lin, D. Jiang, X. Zhang, *Nonlinear Anal.* 62 (2005) 683-701.

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- With the change of variables $x(t) = N(t) - N^*(t)$, model (10) is transformed into

$$\begin{cases} x'(t) = -a(t)x(t) + f(t, x_t), & 0 \le t \neq t_k \\ \Delta x(t_k) = \tilde{I}_k(x_{t_k}), & k \in \mathbb{N}, \cdots \end{cases} \tag{11}$$

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- where

$$\begin{aligned} f(t,\varphi) &= \sum_{i=1}^{n} b_i(t) e^{-c_i(t)} \left[e^{-\beta_i(t)\varphi(-\tau_i(t))} - 1 \right], \\ c_i(t) &= \beta_i(t) N^*(t - \tau_i(t)), \qquad i = 1, \ldots, n, \\ \tilde{I}_k(u) &= I_k\big(N^*(t_k) + u\big) - I_k\big(N^*(t_k)\big), \qquad k = 1, \ldots, p. \end{aligned} \tag{12}$$

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- From (i_0) and (i_1) we have

$$\tilde{b}_k := b_k + 1 \le \frac{u + \tilde{I}_k(u)}{u} \le a_k + 1 =: \tilde{a}_k, \quad u \neq 0, u > -N^*(t_k),$$

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- The model (11) satisfies the York's condition (9) with

$$\lambda_{1,i}(t) := \beta_i(t) b_i(t) e^{-\beta_i(t) N^*(t-\tau_i(t))},$$

$$\lambda_{2,i}(t) = \beta_i(t) b_i(t),$$

  for all $t \geq 0$, and $i = 1 \dots n$. This means that (H3) holds.

- **Theorem 3** Assume (f_0), (i_0)–(i_2), and that there is a positive ω-periodic solution $N^*(t)$ of (10). If there is $T > 0$ such that $\alpha_1\alpha_2 < 1$ with

$$\alpha_1 := \sup_{t\geq T} \int_{t-\tau(t)}^{t} \sum_{i=1}^{n} \beta_i(s) b_i(s) e^{-\beta_i(s) N^*(s-\tau_i(s))} B_i(s) e^{-\int_s^t a(u)\,du}\, ds$$

$$\alpha_2 := \sup_{t\geq T} \int_{t-\tau(t)}^{t} \sum_{i=1}^{n} \beta_i(s) b_i(s) B_i(s) e^{-\int_s^t a(u)\,du}\, ds,$$

where $B_i(t) = \max_{\theta\in[-\tau_i(t),0]} \left(\prod_{k:t+\theta\leq t_k<t} (1+b_k)^{-1} \right)$, then $N^*(t)$ is globally asymptotically stable i.e., it is stable and any positive solution $N(t)$ of (10) satisfies

$$\lim_{t\to\infty} (N(t) - N^*(t)) = 0.$$

- **Theorem 4** Consider $\tau_i(t) \equiv m_i\omega$ ($m_i \in \mathbb{N}$ and $\bar{m} = \max_{1\le i\le n} m_i$). Assume (f_0), (i_0)–(i_2) with $I_k(0) = 0$, and that there is a positive ω-periodic solution $N^*(t)$. If

$$B^{\bar{m}}\left(\overline{\beta}\,\overline{N^*}(e^{\overline{\beta}\,\overline{N^*}}-1)\right)^{\frac{1}{2}}\left(1-e^{-\bar{m}\int_0^\omega a(u)\,du}\right)\cdot$$
$$\cdot\left[1-\left(1-e^{-\int_0^\omega a(u)\,du}\right)^{-1}\sum_{k=1}^{p}\min(b_k,0)\right] < 1,$$

where $\overline{\beta} = \max_{t\in[0,\omega]}\beta(t)$, $\overline{N^*} = \max_{t\in[0,\omega]} N^*(t)$, and

$B = \max_{1\le l,j\le p}\prod_{k=1}^{j}(1+b_{l+k})^{-1}$, then $N^*(t)$ attracts any positive solution $N(t)$ of (10).

Thank you

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