Global exponential stability of nonautonomous neural network models with delays

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Neural Network Models

*Pioneer Models:

- Cohen-Grossberg (1983)

$$x_i'(t) = -a_i(x_i(t))\left(b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij} f_j(x_j(t))\right), \quad i = 1, \dots, n. \qquad (1)$$

- Hopfield (1984)

$$x_i'(t) = -b_i(x_i(t)) + \sum_{j=1}^{n} c_{ij} f_j(x_j(t)), \qquad i = 1, \dots, n. \qquad (2)$$

where
a_i amplification functions; b_i controller functions;
f_j activation functions; $C = [c_{ij}]$ conection matrix.

* Nonautonomous system of delay differential equations

$$x_i'(t) = -a_i(t, x_t)\left[b_i(t, x_i(t)) + f_i(t, x_t)\right], \qquad t \geq 0,\ i = 1, \ldots, n \quad (3)$$

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* **Phase Space:** $C_n := C([-\tau, 0]; \mathbb{R}^n)$ for $\tau > 0$,

$$\|\phi\| = \sup_{s \in [-\tau, 0]} |\phi(s)| \qquad \text{with } |x| = |(x_1, \dots, x_n)| = \max_{1 \leq i \leq n} |x_i|$$

$$x_t \in C_n, \qquad x_t(s) = x(t+s),\ s \in [-\tau, 0]$$

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$$x_t \in C_n, \qquad x_t(s) = x(t+s),\ s \in [-\tau, 0]$$

- $a_i : [0, +\infty) \times C_n \to (0, +\infty)$ are continuous functions;
- $b_i : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ are continuous functions;
- $f_i : [0, +\infty) \times C_n \to \mathbb{R}$ are continuous functions.

* **Initial Condition**

$$x_0 = \bar{\varphi}, \qquad \bar{\varphi} \in C_n \qquad (4)$$

* **Definition**

The solution $x(t, 0, \bar{\varphi})$ of (3)-(4) is said *globally exponentially stable* if $\exists\delta > 0$ and $M \geq 1$

$$|x(t, 0, \varphi) - x(t, 0, \bar{\varphi})| \leq Me^{-\delta t}\|\varphi - \bar{\varphi}\|,$$

for all $t \geq 0$, $\varphi \in C_n$.

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for all $t \geq 0$, $\varphi_1, \varphi_2 \in C_n$.

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- **(A1)** $\exists\, x^* = (x_1^*, \ldots, x_n^*) \in \mathbb{R}^n$ a equilibrium point of (3);
- **(A2)** $\exists\, \underline{\rho}_i > 0$:

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- **(A3)** $\exists\, \beta_i : [0, +\infty) \to (0, +\infty)$, $\forall u, v \in \mathbb{R}$ $u \neq v$:
$$(b_i(t, u) - b_i(t, v))/(u - v) \geq \beta_i(t), \qquad \forall t \geq 0;$$

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- **(A4)** $\exists\, l_i : [0,+\infty) \to (0,+\infty)$
$$|f_i(t,\varphi) - f_i(t,\psi)| \leq l_i(t)||\varphi - \psi||, \qquad \forall t \geq 0,\ \forall \varphi, \psi \in C_n;$$

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$$|f_i(t,\varphi) - f_i(t,\psi)| \leq l_i(t)||\varphi - \psi||, \qquad \forall t \geq 0,\ \forall \varphi, \psi \in C_n;$$

- **(A5)** $\exists \varepsilon > 0$ and $\lambda : \mathbb{R} \to (0,+\infty)$ a continuous function:

$$\underline{\rho}_i \left(\beta_i(t) - l_i(t) e^{\int_{t-\tau}^{t} \lambda(s)\, ds} \right) > \lambda(t) \text{ and } \int_0^t \lambda(s)\, ds \geq \varepsilon t,\ \forall t \geq 0.$$

- In C_n, consider the FDE

$$x'(t) = f(t, x_t), \qquad t \geq 0 \tag{5}$$

with $f = (f_1, \ldots, f_n) : [0, +\infty) \times C_n \to \mathbb{R}^n$ continuous

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* **Lemma:**[1] If
(H) $\forall t > 0, \forall \varphi \in C_n$:

$$\forall s \in [-\tau, 0), |\varphi(s)| < |\varphi(0)| \Rightarrow \varphi_i(0) f_i(t, \varphi) < 0,$$

for some $i \in \{1, \dots, n\}$ such that $|\varphi(s)| = |\varphi_i(0)|$,

then the solution $x(t) = x(t, 0, \varphi)$, $\varphi \in C_n$, of (5) is defined and bounded on $[-\tau, +\infty)$ and

$$|x(t, 0, \varphi)| \leq ||\varphi||.$$

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*Proof of Lemma (idea)

- $x(t) = x(t, 0, \varphi)$ solution on $[-\tau, \alpha)$, $\alpha > 0$, with $\varphi \in C_n$
- Suppose that $|x(t_1)| > \|\varphi\|$ for some $t_1 > 0$ and define

$$T = \min\left\{t \in [0, t_1] : |x(t)| = \max_{s\in[0,t_1]} |x(s)|\right\}.$$

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- We have $|x_T(s)| = |x(T + s)| < |x(T)|$, for $s \in [-\tau, 0)$. By **(H)** we conclude that,

$$x_i(T) f_i(T, x_T) < 0,$$

for some $i \in \{1, \ldots, n\}$ such that $|x_i(T)| = |x(T)|$. If $x_i(T) > 0$ (analogous if $x_i(T) < 0$), then $x_i'(T) < 0$.

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- Contradition. Thus $x(t)$ is defined and bounded on $[0, +\infty)$.

Global exponential stability

Theorem 1: Assume **(A1)-(A5)**
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- The change of variables $z(t) = e^{\int_0^t \lambda(u)\,du} x(t)$ transforms (3) into

$$z_i'(t) = g_i(t, z_t), \qquad t \geq 0 \tag{6}$$

with

$$g_i(t,\varphi) = \lambda(t)\varphi_i(0) - a_i(t,\psi(t))e^{\int_0^t \lambda(u)\,du}[b_i(t,\psi(t)_i(0)) + f_i(t,\psi(t))]$$

and $\psi(t)(s) = e^{-\int_0^{t+s} \lambda(u)\,du}\varphi(s)$, $s \in [-\tau, 0]$

- $\psi(t) \in C_n$

- Take $t \geq 0$ and $\varphi \in C_n$ such that

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$$\begin{aligned} g_i(t,\varphi) &= \lambda(t)\varphi_i(0) - a_i(t,\psi(t))e^{\int_0^t \lambda(u)\,du}[b_i(t,\psi(t)_i(0)) - b_i(t,0) \\ &\quad + f_i(t,\psi(t)) - f_i(t,0)] \\ &\leq \lambda(t)\varphi_i(0) - a_i(t,\psi(t))e^{\int_0^t \lambda(u)\,du} \cdot \\ &\quad \cdot \left[\beta_i(t)e^{-\int_0^t \lambda(u)\,du}\varphi_i(0) - l_i(t)e^{-\int_0^{t-\tau} \lambda(u)\,du}\|\varphi\|\right] \\ &\leq \varphi_i(0)\left[\lambda(t) - \underline{\rho}_i\left(\beta_i(t) - l_i(t)e^{\int_{t-\tau}^t \lambda(u)\,du}\right)\right] < 0 \end{aligned}$$

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- Then **(H)** holds and, from the Lemma,

$$\begin{aligned} |x(t,0,\varphi)| &= \left|e^{-\int_0^t \lambda(u)\,du} z(t,0,\psi(0))\right| \\ &\leq e^{-\varepsilon t}\left|z\left(t,0,e^{-\int_\cdot^0 \lambda(u)\,du}\varphi\right)\right| \leq e^{-\varepsilon t}\|\varphi\|. \end{aligned}$$

Corollary 1: Assume $a_i(t,\varphi)=1$ and **(A3)-(A5)**
Then the system

$$x_i'(t) = -b_i(t, x_i(t)) + f_i(t, x_t), \qquad t \geq 0, \tag{7}$$

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* Proof (idea)
- Let $\bar{x}(t)=x(t,0,\bar{\varphi})$. The change of variables $z(t)=x(t)-\bar{x}(t)$ transforms (7) into

$$z_i'(t)=-\bar{b}_i(t,z_i(t))+\bar{f}_i(t,z_t),\qquad t\geq 0 \tag{8}$$

with

$$\bar{b}_i(t,u)=b_i(t,u+\bar{x}_i(t)) \text{ and } \bar{f}_i(t,\varphi)=f_i(t,\varphi+\bar{x}_t)+b_i(t,\bar{x}_i(t))-f_i(t,\bar{x}_t)$$

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- Zero is an equilibrium of (8) and from the Theorem 1

$$|x(t)-\bar{x}(t)| = |z(t)| \leq e^{-\varepsilon t}\|z_0\| = e^{-\varepsilon t}\|\varphi-\bar{\varphi}\|, \qquad \forall t \geq 0.$$

Corollary 2: Assume **(A3)** and **(A4)**

- If $l_i(t)$ are bounded and there exists $\alpha > 0$:

$$\beta_i(t) - l_i(t) > \alpha, \quad \forall t \geq 0, \tag{9}$$

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* (Proof) For $l_i(t) < L_i$, from (9), we have

$$\beta_i(t) - l_i(t)\left(1 + \frac{\alpha}{2nL_i}\right) > \frac{\alpha}{2}.$$

Taking $\varepsilon_i^* = \frac{1}{\tau} \log\left(1 + \frac{\alpha}{2nL_i}\right) > 0$ and $\varepsilon = \min_i\{\frac{\alpha}{2}, \varepsilon_i^*\}$,

$$\beta_i(t) - l_i(t)e^{\varepsilon\tau} > \varepsilon.$$

With $\lambda(t) = \varepsilon$, the condition **(A5)** holds.

Periodic systems

Assume that the system

$$x_i'(t) = -b_i(t, x_i(t)) + f_i(t, x_t), \qquad t \geq 0 \qquad (7)$$

is ω-periodic, $\omega > 0$, that is:

$$b_i(t, u) = b_i(t + \omega, u), \qquad \forall t \geq 0, \ \forall u \in \mathbb{R};$$

$$f_i(t, \varphi) = f_i(t + \omega, \varphi), \qquad \forall t \geq 0, \ \forall \varphi \in C_n.$$

Theorem 2: Assume **(A3)**, **(A4)**, and

$$\beta_i(t) - l_i(t) > 0, \qquad \forall t \in [0, \omega].$$

Then (7) has a ω-periodic solution which is globally exponentially stable.

* (Proof) Show the existence of a periodic solution.
From Corollary 2

$$\|x_t(\varphi) - x_t(\bar{\varphi})\| \leq e^{-\varepsilon(t-\tau)}\|\varphi - \bar{\varphi}\|, \quad \forall t \geq \tau, \ \forall \varphi, \bar{\varphi} \in C_n.$$

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- Let $k \in \mathbb{N}$ such that $e^{-(k\omega-\tau)} \leq \frac{1}{2}$ and define
$P : C_n \to C_n$ by $P(\varphi) = x_\omega(\varphi)$.

$$\|P^k(\varphi) - P^k(\bar{\varphi})\| = \|x_{k\omega}(\varphi) - x_{k\omega}(\bar{\varphi})\| \leq \frac{1}{2}\|\varphi - \bar{\varphi}\|,$$

P^k is a contraction map on Banach space C_n. Thus, P^k has a unique fixed point $\varphi^* \in C_n$: $P^k(\varphi^*) = \varphi^*$.

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  P^k is a contraction map on Banach space C_n. Thus, P^k has a unique fixed point $\varphi^* \in C_n$: $P^k(\varphi^*) = \varphi^*$.
- As $P^k(P(\varphi^*)) = P(P^k(\varphi^*)) = P(\varphi^*)$, then

$$P(\varphi^*) = \varphi^* \Leftrightarrow x_\omega(\varphi^*) = \varphi^*$$

  and $x(t, 0, \varphi^*)$ is the periodic solution of (7).

Hopfield neural network model [2]

$$x_i'(t) = -b_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) + I_i(t) \quad (10)$$

[2] Q. Zhang, X. Wei, J. Xu, *Chaos Solitons & Fractals* 39 (2009) 1152-1157.

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- $b_i, a_{ij}, b_{ij}, I_i : [0, +\infty) \to \mathbb{R}$, $\tau_{ij}(t) \geq 0$ are continuous;
- $f_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with constant l_j;
- $b_i(t) - \sum_{j=1}^{n} l_j \left(|a_{ij}(t)| + |b_{ij}(t)| e^{\int_{t-\tau_{ij}}^{t} \lambda(s)ds} \right) > \lambda(t), \quad \forall i$

  and $\int_0^t \lambda(s)ds \geq \varepsilon t$, for some $\varepsilon > 0$ and some function $\lambda(t)$.
  Then system (10) is globally exponentially stable.

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  and $\int_0^t \lambda(s)ds \geq \varepsilon t$, for some $\varepsilon > 0$ and some function $\lambda(t)$.
  Then system (10) is globally exponentially stable.
- In [2], a different hypotheses set is assumed to get the same conclusion.

[2] Q. Zhang, X. Wei, J. Xu, *Chaos Solitons & Fractals* 39 (2009) 1152-1157.

For the periodic model:

$$x_i'(t) = -b_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) + I_i(t) \quad (11)$$

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- $b_i, a_{ij}, b_{ij}, I_i : [0,+\infty) \to \mathbb{R}$, $\tau_{ij}(t) \geq 0$ are ω-periodic continuous;
- $f_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with constant l_j;
- $b_i(t) - \sum_{j=1}^{n} l_j(|a_{ij}(t)| + |b_{ij}(t)|) > 0, \quad \forall i, \forall t \in [0,\omega].$

  Then (11) has a global exponential stable ω-periodic solution.

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For the periodic model:

$$x_i'(t) = -b_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) + I_i(t) \quad (11)$$

- $b_i, a_{ij}, b_{ij}, I_i : [0,+\infty) \to \mathbb{R}$, $\tau_{ij}(t) \geq 0$ are ω-periodic continuous;
- $f_j : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with constant l_j;
- $b_i(t) - \sum_{j=1}^{n} l_j(|a_{ij}(t)| + |b_{ij}(t)|) > 0, \quad \forall i, \forall t \in [0,\omega].$

  Then (11) has a global exponential stable ω-periodic solution.
- In [3] assumed the additional hypothesis

$$b_j(t) - \sum_{i=1}^{n} l_j(|a_{ij}(t)| + |b_{ij}(t)|) > 0, \quad \forall j, \forall t \in [0,\omega],$$

[3] M. Tan, Y. Tan, *Appl. Math. Model.* 33 (2009) 373-385.

Thank you

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