Global Stability of Scalar Differential Equations with Small Delays

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Notations and Definitions

- $\tau \in \mathbb{R}^+$;

- $C := C([-\tau, 0]; \mathbb{R})$

$$\|\varphi\|_C = \sup_{\theta \in [-\tau, 0]} \|\varphi(\theta)\|;$$

- For $x \in C([a-\tau, b]; \mathbb{R})$, where $b > a \in \mathbb{R}$, and $t \in [a, b]$, we define x_t by

$$x_t(\theta) := x(t+\theta), \;\; \theta \in [-\tau, 0]$$

$$x_t \in C;$$

- For $\gamma \in \mathbb{R}$,

$C_\gamma := \{\varphi \in C : \varphi(\theta) \geq \gamma, \theta \in [-\tau, 0), \;\text{and}\; \varphi(0) > \gamma\}$

We consider the scalar functional differential equation (FDE) in general form

$$\dot{x}(t) = f(t, x_t), \quad t \in I := [0, +\infty) \tag{1}$$

where $f : I \times C \to \mathbb{R}$ is continuous with

$$f(t, 0) \equiv 0 \ \forall t \geq 0,$$

to have $x = 0$ as an equilibrium point.

We consider initial conditions

$$x_0 = \varphi, \quad \varphi \in C.$$

Main Objective

To get sufficient conditions for the global attractivity of the zero solution of (1), i.e.,

$$x(t) \to 0, \text{ as } t \to +\infty,$$

for all solutions $x(t)$ of (1).

Delayed Logistic Equation

$$\dot{x}(t) = ax(t)\left(1 - \frac{1}{k}x(t-\tau)\right), \quad t \geq 0 \quad (2)$$

$$x_0 = \varphi, \quad \varphi \in C_0,$$

with $a, \tau, k \in \mathbb{R}^+$.
Admissible solutions (i.e., positive solutions):

$$x_t \in C_0, \quad \forall t \geq 0$$

- $x(t) \equiv k$ is the positive equilibrium.

- The change of variables

$$y(t) = \frac{x(t)}{k} - 1$$

transforms (2) in the form

$$\dot{y}(t) = (1 + y(t))[-ay(t-\tau)], \quad t \geq 0 \quad (3)$$

$$x_0 = \varphi, \quad \varphi \in C_{-1},$$

Theorem [E. M. Wright, 1955]
If $a\tau \leq 3/2$, then every admissible solution $x(t)$ of (2) satisfies

$$x(t) \to k, \text{ as } t \to +\infty.$$

J. A. Yorke [1970]

$$\dot{x}(t) = f(t, x_t)$$

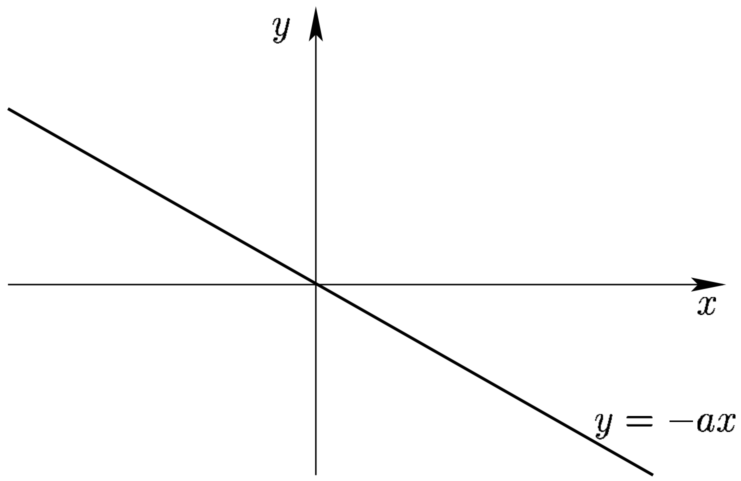
Hypotheses:

(Y1) $\forall t_n \to +\infty$, $\forall \varphi_n \in C$,
If $\varphi_n \to c \neq 0$, then $f(t_n, \varphi_n) \not\to 0$;

(Y2) $\exists a > 0, \forall t \geq 0, \forall \varphi \in C$:

$$-a\mathcal{M}(\varphi) \leq f(t, \varphi) \leq a\mathcal{M}(-\varphi),$$

where $\mathcal{M}(\varphi) := \max\{0, \max_{\theta \in [-\tau, 0]} \varphi(\theta)\}$;



Theorem
If $a\tau < 3/2$, then every solution $x(t)$ of (1) converges to zero as $t \to +\infty$.

Yorke condition (Generalizations)

T. Yoneyama [1987]
$\lambda : [0, +\infty) \to [0, \infty)$ continuous,

$$-\lambda(t)\mathcal{M}(\varphi) \le f(t, \varphi) \le \lambda(t)\mathcal{M}(-\varphi). \qquad (4)$$

$$\sup_{t \ge T} \int_{t-\tau}^{t} \lambda(s)ds < \frac{3}{2}$$

X. Zhang & J. Yan [2004]
$\lambda_i : [0, +\infty) \to [0, \infty)$, $i = 1, 2$, continuous,

$$-\lambda_1(t)\mathcal{M}(\varphi) \le f(t, \varphi) \le \lambda_2(t)\mathcal{M}(-\varphi). \qquad (5)$$

$$\alpha_i := \sup_{t \ge T} \int_{t-\tau}^{t} \lambda_i(s)ds, \; i = 1, 2$$

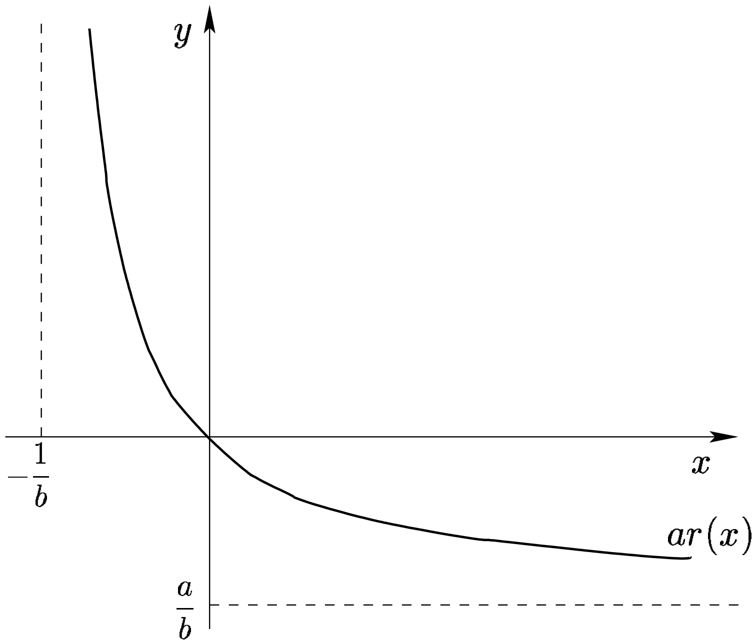
$$\min\{\alpha_1, \alpha_2\} \max\{\alpha_1^2, \alpha_2^2\} < (3/2)^3$$

E. Liz, V. Tkachenko & S. Trofimchuk [2003]

There are $a > 0$ and $b \geq 0$:

$$ar(\mathcal{M}(\varphi)) \leq f(t,\varphi) \leq ar(-\mathcal{M}(-\varphi)),$$

where $r(x) = \frac{-x}{1+bx}$, $x > -1/b$.



Notes:

• If $b = 0$, then we have the original Yorke condition.

• If $b > 0$ then, to have **bounded solutions**, we need an extra bounded condition on $f(t,\varphi)$ when $\varphi < 0$.

Our setting: hypotheses **(H)**:

(H1) There is a piecewise continuous (P.C.) function $\beta : I \to I$ such that

$$\sup_{t \geq \tau} \int_{t-\tau}^{t} \beta(s) ds < +\infty,$$

and $\forall q \in \mathbb{R}, \exists \eta(q) \in \mathbb{R}$:

$$f(t, \varphi) \leq \beta(t)\eta(q), \ \forall t \in I, \varphi \geq q;$$

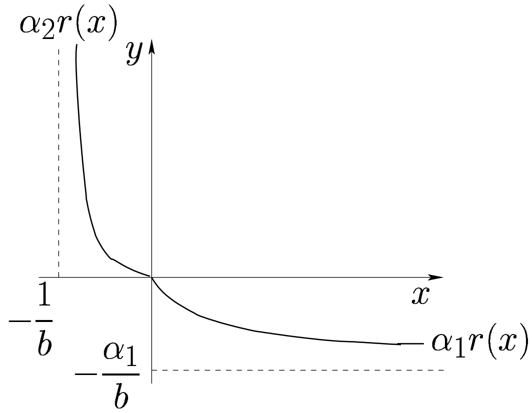
(H2) If $w : [-\tau, +\infty) \to \mathbb{R}$ is continuous and $\lim\limits_{t \to +\infty} w(t) = w^* \neq 0$, then

$$\int_{0}^{+\infty} f(s, w_s) ds \text{ diverges;}$$

(H3) There are P.C. functions $\lambda_1, \lambda_2 : I \to I$ and $b \geq 0$ such that

$$\lambda_1(t) r(\mathcal{M}(\varphi)) \leq f(t, \varphi) \leq \lambda_2(t) r(-\mathcal{M}(-\varphi)),$$

where $r(x) = \frac{-x}{1+bx}$, $x > -1/b$;



(H4) There is $T \geq \tau$ such that, for

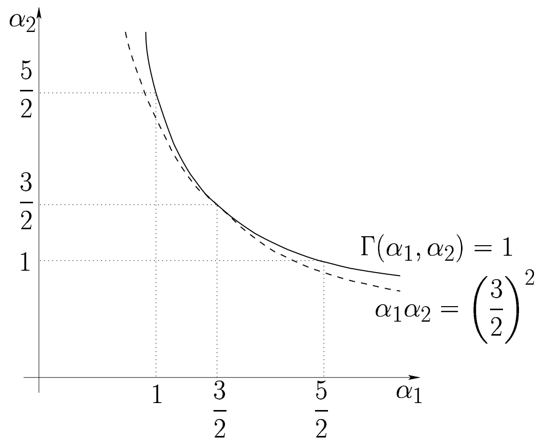
$$\alpha_i := \sup_{t \geq T} \int_{t-\tau}^{t} \lambda_i(s) ds, \ i = 1, 2,$$

we have

$$\Gamma(\alpha_1, \alpha_2) \leq 1, \tag{6}$$

where Γ is defined by

$$\Gamma(\alpha_1, \alpha_2) = \begin{cases} (\alpha_1 - 1/2)\frac{\alpha_2^2}{2}, & \alpha_1 > \frac{5}{2} \\ (\alpha_1 - 1/2)(\alpha_2 - 1/2), & \alpha_1, \alpha_2 \leq \frac{5}{2} \\ (\alpha_2 - 1/2)\frac{\alpha_1^2}{2}, & \alpha_2 > \frac{5}{2} \end{cases}$$



• If $\lambda_1(t) = \lambda_2(t)$ ($\alpha := \alpha_1 = \alpha_2$), then (6) has the form

$$\sup_{t \geq T} \int_{t-\tau}^{t} \lambda(s) ds \leq \frac{3}{2}.$$

• $\alpha_1 \alpha_2 \leq (3/2)^2$ imply $\Gamma(\alpha_1, \alpha_2) \leq 1$.

The case $b = 0$ in **(H3)**: $r(x) = -x$

(H3') There are P.C. functions $\lambda_1, \lambda_2 : I \to I$ and $h : \mathbb{R} \to \mathbb{R}$ a non-increasing function such that

$$|h(x)| < |x|, \;\; x \neq 0,$$

$$\lambda_1(t) h(\mathcal{M}(\varphi)) \leq f(t, \varphi) \leq \lambda_2(t) h(-\mathcal{M}(-\varphi)).$$

(H3')$\Rightarrow$(H3)
(H3')+(H4)$\Rightarrow$(H1)

Theorem 1
Assume **(H2)**, **(H3')** and **(H4)**. Then the zero solution of (1) is globally attractive.

Corollary
Assume **(H2)**, **(H3)** with $b = 0$ and **(H4)** with $\Gamma(\alpha_1, \alpha_2) < 1$. Then the zero solution of (1) is globally attractive.

In particular, the same conclusion holds when

$$\alpha_1 \alpha_2 \leq \left(\frac{3}{2}\right)^2 \text{ and } (\alpha_1, \alpha_2) \neq (3/2, 3/2).$$

Proof Let $x(t)$ a solution of (1)

•**(H3')** $\Rightarrow x(t)$ bounded on $[-\tau, +\infty)$.

• Case $x(t)$ is non-oscillatory:
If $x(t)$ is eventually positive, from **(H3')** $\dot{x}(t) = f(t, x_t) \leq 0$, then $x(t)$ is eventually non-increasing, so $x(t) \to u \geq 0$.
If $u > 0$, from **(H2)**

$$x(t) = x(t_0) + \int_{t_0}^{t} f(s, x_s) ds \to +\infty, \text{ as } t \to +\infty$$

a contradiction. Hence, $u = 0$.

• Case $x(t)$ is oscillatory:

$$u := \lim_{t \to +\infty} \sup x(t) \geq 0; \; -v := \lim_{t \to +\infty} \inf x(t) \leq 0$$

(I) $u \leq h(-v) \max\{\frac{1}{2}, \alpha_2 - \frac{1}{2}\}$; $u \leq h(-v) \frac{\alpha_2^2}{2}$
(II) $-v \geq h(u) \max\{\frac{1}{2}, \alpha_1 - \frac{1}{2}\}$; $-v \geq h(u) \frac{\alpha_1^2}{2}$

Using **(I)**-**(II)**, if $u \geq v$ and $u > 0$,

$$u \leq -h(u)\Gamma(\alpha_1, \alpha_2) \leq -h(u) < u,$$

a contradiction. Hence $u = 0$ and $v = 0$. Then $x(t) \to 0$ as $t \to +\infty$.

$$\dot{x}(t) = f(t, x_t)$$

Case $b > 0$ in **(H3)**: $r(x) = -\frac{x}{1+bx}$, $x > -1/b$

Theorem 2 Assume **(H1)**-**(H4)**, with $b > 0$ and $\lambda_i(t) > 0$ for t large.
If $\alpha_1 \leq \alpha_2$ then, for all solutions $x(t)$ of (1),

$$x(t) \to 0, \text{ as } t \to \infty.$$

The proof uses some arguments in the work of [E. Liz, V. Tkachenko & S. Trofimchuk 2003].

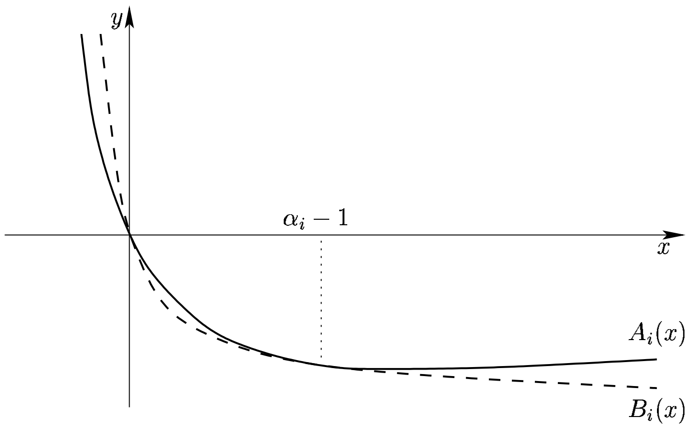
Without losing generality, we can let $b = 1$ and $\tau = 1$.

$i = 1, 2$

For $0 < \alpha_1 \leq \alpha_2$, we define A_i, B_i and D_1 by

$$A_i(x) = \begin{cases} x + \alpha_i r(x) + \dfrac{1}{r(x)} \displaystyle\int_x^0 r(t)dt, & x \neq 0 \\ 0, & x = 0 \end{cases};$$

$$B_i(x) = \begin{cases} \dfrac{1}{r(x)} \displaystyle\int_{-\alpha_i r(x)}^0 r(t)dt, & x \neq 0 \\ 0, & x = 0 \end{cases};$$



$$D_1(x) = \begin{cases} A_1(x), & 0 \leq x < \alpha_1 - 1 \\ B_1(x), & x \geq \max\{0, \alpha_1 - 1\} \end{cases}$$

For $\alpha_i > 1/2$, we define R_i by

$$R_i(x) = A_i'(0) \frac{x}{1 - \frac{x}{\nu_i}}, \quad x > \nu_i,$$

com $\nu_i := \frac{2A'(0)}{A''(0)} = -\frac{6\alpha_i - 3}{6\alpha_i - 1} < 0.$

Note: If $\alpha_1 \leq \alpha_2$, then the composition $R_2 \circ D_1$ is well defined on $[0, +\infty)$.

$$\Gamma(\alpha_1, \alpha_2) \leq 1 \Rightarrow R_2(D_1(x)) \leq x, \quad \forall x \geq 0 \quad (7)$$

Let $x(t)$ be a solution of (1)

(H1)+(H3) $\Rightarrow x(t)$ bounded on $[-\tau, +\infty)$.

- Case $x(t)$ is non-oscillatory:

(H2)+(H3)$\Rightarrow x(t) \to 0$ as $t \to +\infty$

- Case $x(t)$ is oscillatory:

$u := \lim_{t\to+\infty} \sup x(t) \geq 0$; $-v := \lim_{t\to+\infty} \inf x(t) \leq 0$

Using **(H3)**, if $v > 0$, then

$$u \leq A_2(-v) < R_2(-v) \leq R_2(D_1(u)),$$

is a contradiction by (7).

Hence $v = 0$ and $u = 0$. Then

$$x(t) \to 0 \text{ as } t \to +\infty.$$

Scalar Population Models

$$\begin{aligned} &\dot{x}(t) = x(t)f(t, x_t), \quad t \geq 0 \\ &x_0 = \varphi, \quad \varphi \in C_0, \end{aligned} \tag{8}$$

where $f : [0, +\infty) \times C \to \mathbb{R}$ is continuous and

$C_0 := \{\varphi \in C : \varphi(\theta) \geq 0, \theta \in [-\tau, 0),\ \text{and}\ \varphi(0) > 0\}$.

If $u(t)$ is the solution of (8), then the change of variables

$$\bar{x}(t) = \frac{x(t)}{u(t)} - 1$$

transform (8) in to

$$\begin{aligned} &\dot{x}(t) = (1 + x(t))F(t, x_t), \quad t \geq 0 \\ &x_0 = \varphi, \quad \varphi \in C_{-1}, \end{aligned} \tag{9}$$

where $F(t, \varphi) = f(t, u_t(1 + \varphi)) - f(t, u_t)$.

Note: To study the global stability of the solution $u(t)$ of (8) is equivalent to study the global stability of the zero solution of (9).

Consider the initial value problem (IVP)

$$\dot{x}(t) = (1 + x(t))F(t, x_t),$$
$$x_0 = \varphi, \quad \varphi \in C_{-1}.$$

For $F : [0, +\infty) \times C_{-1} \to \mathbb{R}$ we assume hypotheses **(H1)**-**(H4)** with φ restricted to C_{-1}, i.e. we suppose **(H1)**-**(H4)** hold with $\varphi \in C$ replaced by $\varphi \in C_{-1}$.

Note: If $b < 1$, then **(H3)** imply **(H1)**.

Theorem 3
For $F : [0, +\infty) \times C_{-1} \to \mathbb{R}$ continuous, assume **(H1)**-**(H4)** with φ restricted to C_{-1}. Case $b \neq \frac{1}{2}$, assume $\lambda_i(t) > 0$, for t large, and

$$\text{(i)} \quad b > \frac{1}{2} \text{ and } \alpha_1 \leq \alpha_2$$

or

$$\text{(ii)} \quad b < \frac{1}{2} \text{ and } \alpha_1 \geq \alpha_2.$$

Then the solution $x(t)$ of (9) converge to zero as $t \to +\infty$.

Proof
The change of variables $y(t) = \log(1 + x(t))$ transforms the IVP (9) in the form

$$\dot{y}(t) = f(t, y_t),$$
$$y_0 = \varphi, \quad \varphi \in C,$$

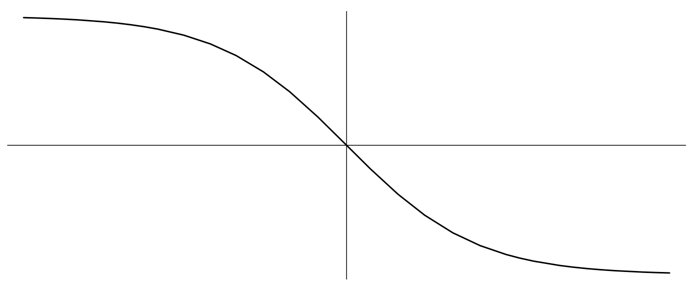
with $f(t, \varphi) := F(t, e^{\varphi} - 1)$.

For $t \geq 0$ and $\varphi \in C$,

$$\lambda_1(t) r(e^{\mathcal{M}(\varphi)} - 1) \leq f(t, \varphi) \leq \lambda_2(t) r(e^{-\mathcal{M}(-\varphi)} - 1).$$

• If $b = \frac{1}{2}$, then f satisfies **(H2)**, **(H4)** and **(H3')** with

$$h(x) := r(e^x - 1) = -2\left(1 - \frac{2}{e^x + 1}\right),$$



then, by theorem 1, we have

$$y(t) \to 0 \text{ as } t \to \infty,$$

i.e., $x(t) \to 0$ as $t \to \infty$.

• If $b > \frac{1}{2}$, then f satisfies **(H1)**, **(H2)**, **(H4)** and **(H3)** with

$$r_1(x) = \frac{-x}{1 + (b - \frac{1}{2})x}.$$

Hence, by theorem 2, if $\alpha_1 \leq \alpha_2$, then

$$y(t) \to 0 \text{ as } t \to \infty,$$

i.e., $x(t) \to 0$ as $t \to \infty$.

• If $0 < b < \frac{1}{2}$, the change $z(t) = -\log(1 + x(t))$ transforms the IVP (9) in the form

$$\dot{z}(t) = g(t, z_t),$$
$$z_0 = \varphi, \quad \varphi \in C,$$

where $g(t, \varphi) = -F(t, e^{-\varphi} - 1)$ satisfies the hypotheses **(H)**, where **(H3)** is

$$\lambda_2(t) r_2(\mathcal{M}(\varphi)) \leq g(t, \varphi) \leq \lambda_1(t) r_2(-\mathcal{M}(-\varphi)),$$

with $r_2(x) = \dfrac{-x}{1 + (\frac{1}{2} - b)x}$.

Hence, if $\alpha_2 \leq \alpha_1$, then $z(t) \to 0$, i.e., $x(t) \to 0$ as $t \to \infty$.

Example [K.Golpalsamy book, 1992; Y.Liu, 2001; T.Faria,2004]

$$\dot{N}(t) = \rho(t)N(t)\left[\frac{k - aN(t-\tau)}{k + \lambda(t)N(t-\tau)}\right]^{\alpha} \quad (10)$$
$$N_0 = \varphi, \quad \varphi \in C_0$$

with $\rho, \lambda : [0, +\infty) \to [0, +\infty)$ are continuous, $a, k, \tau > 0$ and $\alpha \geq 1$ is the ratio of two odd integers.

$$N(t) \equiv \frac{k}{a} \text{ is the positive equilibrium.}$$

Theorem Assume

$$\int_0^{+\infty} \frac{\rho(s)}{(1 + \lambda(s))^{\alpha}} ds = \infty. \quad (11)$$

For $\underline{\lambda}(t) = \min\{a, \lambda(t)\}$, let

$$\lambda_1(t) = \frac{a^{\alpha}\rho(t)}{2\underline{\lambda}(t)^{\alpha}} \text{ and } \lambda_2(t) = \frac{\rho(t)}{1 + a\underline{\lambda}(t)},$$

and assume that there is $T \geq \tau$ such that $\Gamma(\alpha_1, \alpha_2) \leq 1$, where

$$\alpha_i := \sup_{t \geq T} \int_{t-\tau}^{t} \lambda_i(s) ds, i = 1, 2. \quad (12)$$

Then the solution $N(t)$ of (10) satisfies

$$N(t) \to \frac{k}{a}, \text{ as } t \to +\infty.$$

Proof By the change $x(t) = \frac{aN(t)}{k} - 1$, the IVP (10) has the form

$$\dot{x}(t) = (1 + x(t))F(t, x_t)$$
$$x_0 = \varphi, \quad \varphi \in C_{-1},$$

with

$$F(t, x_t) = -\rho(t) \left[\frac{\varphi(-\tau)}{1 + \frac{\lambda(t)}{a}(1 + \varphi(-\tau))} \right]^{\alpha} \quad (13)$$

- (11)$\Rightarrow$**(H2)**

- **(H3)** restricted to C_{-1} holds for F with $\lambda_1(t)$, $\lambda_2(t)$ as above and $r(x) = \dfrac{-x}{1 + \frac{1}{2}x}$.

In particular, the result holds if $\alpha_1 \alpha_2 \leq (\frac{3}{2})^2$, i.e.,

$$a^{\alpha} \left(\int_{t-\tau}^{t} \frac{\rho(s)}{\underline{\lambda}(s)^{\alpha}} ds \right) \left(\int_{t-\tau}^{t} \frac{\rho(s)}{1 + a\underline{\lambda}(s)} ds \right) \leq \frac{9}{2}, \ t \text{ large}$$

Y. Liu [2001] ($k = a = 1$)

- $\lambda(t) \geq 1$ $\qquad \displaystyle\lim_{t \to +\infty} \sup \int_{t-\tau}^{t} \rho(s) ds \leq 3$

- $0 \leq \lambda(t) \leq 1$ $\qquad \displaystyle\lim_{t \to +\infty} \sup \int_{t-\tau}^{t} \frac{\rho(s)}{\lambda(s)^{\alpha}} ds \leq 3$

• $\lambda(t) \geq a$
Let $\lambda_0 := a^{-1} \inf_{t \geq 0} \lambda(t) \geq 1$.

Theorem
Assume (11).
If $\lambda(t) \geq a$, $\forall t \geq 0$, and $\Gamma(\alpha_1, \alpha_2) \leq 1$, with

$$\alpha_1 = a^{\alpha-1} \sup_{t \geq T} \int_{t-\tau}^{t} \frac{\rho(s)}{(1 + a^{-1}\lambda(s))\lambda(s)^{\alpha-1}} ds,$$
$$\alpha_2 = \frac{1}{1 + \lambda_0} \sup_{t \geq T} \int_{t-\tau}^{t} \rho(s) ds,$$

then the solution $N(t)$ of (10) satisfies

$$N(t) \to \frac{k}{a}, \text{ as } t \to +\infty.$$

Proof
(H3) restricted to C_{-1} holds for F in (13) with

$$\lambda_1(t) = \frac{a^{\alpha-1}\rho(t)}{(1 + a^{-1}\lambda(t))\lambda(t)^{\alpha-1}}, \quad \lambda_2(t) = \frac{\rho(t)}{1 + \lambda_0},$$

and $r(x) = \frac{-x}{1+bx}$, $x \geq -1$, where

$$b := \frac{\lambda_0}{1 + \lambda_0} \geq \frac{1}{2}.$$

Note that $\lambda_1(t) \leq \lambda_2(t)$, hence $\alpha_1 \leq \alpha_2$.

• $0 \leq \lambda(t) \leq a$
Let $\lambda^0 := a^{-1} \sup_{t \geq 0} \lambda(t) \leq 1$.

Theorem
Assume (11).
If $\lambda(t) \leq a$, $\forall t \geq 0$, and $\Gamma(\alpha_1, \alpha_2) \leq 1$, with

$$\alpha_1 = a^\alpha \frac{\lambda^0}{1+\lambda^0} \sup_{t \geq T} \int_{t-\tau}^{t} \frac{\rho(s)}{\lambda(s)^\alpha} ds,$$

$$\alpha_2 = \sup_{t \geq T} \int_{t-\tau}^{t} \frac{\rho(s)}{1 + a^{-1}\lambda(s)} ds,$$

then the solution $N(t)$ of (10) satisfies

$$N(t) \to \frac{k}{a}, \text{ as } t \to +\infty.$$

Proof
(H3) restricted to C_{-1} holds for F in (13) with

$$\lambda_1(t) = \frac{a^\alpha \lambda^0}{1+\lambda^0} \frac{\rho(t)}{\lambda(t)^\alpha}, \quad \lambda_2(t) = \frac{\rho(t)}{1 + a^{-1}\lambda(t)}.$$

and $r(x) = \frac{-x}{1+bx}$, $x \geq -1$, where

$$b := \frac{\lambda^0}{1+\lambda^0} \leq \frac{1}{2},$$

Note that $\lambda_1(t) \geq \lambda_2(t)$, hence $\alpha_1 \geq \alpha_2$.