Global stability for a nonlinear differential system with infinite delay and applications to BAM neural networks

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Pioneer Neural Network Models

*Pioneer Models:

- Cohen-Grossberg (1983)

$$x_i'(t) = -d_i(x_i(t))\left(b_i(x_i(t)) - \sum_{j=1}^{n} a_{ij}h_j(x_j(t))\right), \ i = 1, \dots, n. \quad (1)$$

- Hopfield (1984)

$$x_i'(t) = -b_i(x_i(t)) + \sum_{j=1}^{n} a_{ij}h_j(x_j(t)), \qquad i = 1, \dots, n. \quad (2)$$

where, $n \in \mathbb{N}$ is the number of neurons;
d_i amplification functions; b_i controller functions;
h_j activation functions; $A = [a_{ij}]$ connection matrix.

- Kosko (1988)

$$\begin{cases} x_i'(t) = -x_i(t) + \sum_{j=1}^{m} a_{ij} f(y_j(t)) + I_i \\ y_i'(t) = -y_i(t) + \sum_{j=1}^{m} b_{ij} f(x_i(t)) + J_i \end{cases}, \quad i = 1, \ldots, m, \quad (3)$$

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- Gopalsamy (2007)

$$\begin{cases} x_i'(t) = -x_i(t-\tau) + \sum_{j=1}^{m} a_{ij} f_j(y_j(t-\hat{\sigma})) + I_i \\ y_i'(t) = -y_i(t-\hat{\tau}) + \sum_{j=1}^{m} b_{ij} g_j(x_j(t-\sigma)) + J_i \end{cases}, \qquad i = 1, \ldots, m,$$

In both cases: $t \geq 0$ and $m \in \mathbb{N}$.

- Berezansky, Braverman, and Idels (2014)[1]

$$\begin{cases} x_i'(t) = r_i(t)\left(-a_i x_i(t-\tau_i(t)) + \sum_{j=1}^{m} a_{ij} f_j(y_j(t-\hat{\sigma}_{ij}(t))) + I_i\right) \\ y_i'(t) = p_i(t)\left(-b_i y_i(t-\hat{\tau}_i(t)) + \sum_{j=1}^{m} b_{ij} g_j(x_j(t-\sigma_{ij}(t))) + J_i\right) \end{cases}, \quad (4)$$

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- In [1] Berezansky et.al. studied

$$x_i'(t) = -a_i(t)x_i(t-\tau_i(t)) + \sum_{j=1}^{n} F_{ij}(t, x_j(t-\sigma_{ij}(t))), \qquad i = 1, \dots, n \quad (5)$$

where ($n \in \mathbb{N}$):

- $0 < \underline{a}_i \leq a_i(t) \leq \overline{a}_i$;
- $0 \leq \tau_i(t) \leq \overline{\tau}_i$ and $0 \leq \sigma_{ij}(t) \leq \overline{\sigma}_{ij}$; $\quad \overline{\tau} := \max\{\overline{\tau}_i, \overline{\sigma}_{ij}\}$
- $|F_{ij}(t,u)| \leq L_{ij}|u|$

[1] L.Berezansky, E. Braverman, and L. Idels, *Appl. Math. Comput.* 243 (2014) 899-910

- **Theorem 1 [1]:** If the matrix

$$C = \left[c_{ij}\right]_{i,j=1}^{n},$$

where

$$c_{ii} = 1 - \frac{\overline{a}_i(\overline{a}_i + L_{ii})\overline{\tau}_i + L_{ii}}{\underline{a}_i} \text{ and } c_{ij} = -\frac{\overline{a}_i L_{ij}\overline{\tau}_i + L_{ij}}{\underline{a}_i},\ i \neq j,$$

is a non-singular M-matrix, then system

$$x_i'(t) = -a_i(t)x_i(t-\tau_i(t)) + \sum_{j=1}^{n} F_{ij}(t, x_j(t-\sigma_{ij}(t))), \qquad i = 1, \dots, n$$

is globally exponentially stable,

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$$x_i'(t) = -a_i(t)x_i(t - \tau_i(t)) + \sum_{j=1}^{n} F_{ij}(t, x_j(t - \sigma_{ij}(t))), \qquad i = 1, \ldots, n$$

is globally exponentially stable,
i.e. there are $C \geq 1$ and $\lambda > 0$ such that

$$|x(t, t_0, \varphi) - x(t, t_0, \phi)| \leq C\mathrm{e}^{-\lambda(t-t_0)}\|\varphi - \phi\|,$$

$$\forall t_0 \geq 0, \qquad \forall \varphi, \phi \in C([-\overline{\tau}, 0]; \mathbb{R}^n)$$

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- At the end of [1], the authors provided a list with 9 open problems:

2. Study global stability for the model

$$x_i'(t) = -a_i(t)x_i(t-\tau_i(t)) + \sum_{j=1}^{n}\sum_{k=1}^{N} F_{ijk}(t, x_j(t-\sigma_{ijk}(t))), \qquad i = 1, \dots, n$$

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3. Derive sufficient stability test for

$$x_i'(t) = -a_i(t)x_i(t - \tau_i(t)) + \sum_{j=1}^{n}\int_{t-\varsigma_i}^{t} F_{ij}(s, x_j(s - \sigma_{ij}(s)))ds, \qquad i = 1, \dots, n$$

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5. Obtain sufficient stability conditions for

$$x_i'(t) = -a_i(t)x_i(t-\tau_i(t)) + \sum_{j=1}^{n}\int_{-\infty}^{t} K_{ij}(t,s)F_{ij}(s, x_j(s-\sigma_{ij}(s)))ds.$$

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- We consider the general family of DDE

$$x_i'(t) = -a_i(t)x_i(t-\tau_i(t)) + h_i\big(t, x(t-\tau_{i1}(t)), \dots, x(t-\tau_{im}(t))\big) + f_i(t, x_t),$$

for $t \geq 0$, $i = 1, \dots, n$, where $n, m \in \mathbb{N}$, where $x_t : (-\infty, 0] \to \mathbb{R}^n$ is defined by $x_t(s) = x(t+s)$ for $s \leq 0$.

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- We consider the phase space [Hale and Kato (1978)]

$$UC_g = \left\{ \phi \in C((-\infty, 0]; \mathbb{R}^n) : \sup_{s \leq 0} \frac{|\phi(s)|}{g(s)} < \infty, \frac{\phi(s)}{g(s)} \text{ unif. cont.} \right\},$$

$$\|\phi\|_g = \sup_{s \leq 0} \frac{|\phi(s)|}{g(s)} \text{ with } |x| = |(x_1, \dots, x_n)| = \max_{1 \leq i \leq n} |x_i|$$

where:
(g1) $g : (-\infty, 0] \to [1, \infty)$ non-increasing, continuous, with $g(0) = 1$;
(g2) $\displaystyle\lim_{u \to 0^-} \frac{g(s+u)}{g(s)} = 1$ uniformly on $(-\infty, 0]$;
(g3) $g(s) \to \infty$ as $s \to -\infty$.

For the general DDE

$$x_i'(t) = -a_i(t)x_i(t-\tau_i(t)) + h_i\big(t, x(t-\tau_{i1}(t)), \dots, x(t-\tau_{im}(t))\big) + f_i(t, x_t), \quad (6)$$

we assume, for each $i = 1, \dots n$ and $p = 1, \dots, m$

(A1) $a_i : [0, +\infty) \to (0, +\infty)$ is continuous such that

$$0 < \underline{a}_i \leq a_i(t);$$

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(A2) $\tau_i, \tau_{ip} : [0,+\infty) \to [0,+\infty)$ are continuous such that

$$\tau_i(t) \leq \overline{\tau}_i \qquad \text{and} \qquad \lim_{t\to\infty} \big(t - \tau_{ijp}(t)\big) = \infty;$$

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(A3) $h_i : [0, +\infty) \times \mathbb{R}^{nm} \to \mathbb{R}$ is continuous such that

$$|h_i(t, u) - h_i(t, v)| \leq H_i(t)|u - v|, \qquad t \geq 0, u, v \in \mathbb{R}^{nm};$$

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(A4) $f_i : [0,+\infty) \times UC_g \to \mathbb{R}$ is continuous such that

$$|f_i(t,\varphi) - f_i(t,\phi)| \leq F_i(t)\|\varphi - \phi\|_g, \qquad t \geq 0, \varphi, \phi \in UC_g;$$

Main result

- Consider

$$x_i'(t) = -a_i(t)x_i(t-\tau_i(t)) + h_i\big(t, x(t-\tau_{i1}(t)), \dots, x(t-\tau_{im}(t))\big) + f_i(t, x_t),$$

with bounded initial condition, i.e.

$$x_{t_0} = \varphi, \quad \text{with } t_0 \geq 0 \text{ and } \varphi \in BC \tag{7}$$

$BC = \{\varphi : (-\infty, 0] \to \mathbb{R}^n | \, \varphi \text{ is bounded and continuous}\}$
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- **Theorem 2:** Assume (A1)-(A4). If

$$\limsup_{t\to+\infty} \left(\frac{H_i(t) + F_i(t)}{a_i(t)} + \int_{t-\tau_i(t)}^{t} \big[a_i(w) + H_i(w) + F_i(w)\big]\, dw \right) < 1, \tag{8}$$

then system (6) is globally asymptotically stable, i.e. it is stable and

$$\lim_{t\to+\infty} \big[x(t, t_0, \varphi) - x(t, t_0, \tilde{\varphi})\big] = 0, \quad \forall t_0 \geq 0,\ \varphi, \tilde{\varphi} \in BC;$$

Proof (idea)

- First, we prove that $x(t, t_0, \varphi)$, the solution of (6)-(7), is defined on $\mathbb{R}$ and bounded.
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- For $t_0 \geq 0$ and $\varphi, \tilde{\varphi} \in BC$,
  let $x(t) = x(t, t_0, \varphi)$ and $y(t) = x(t, t_0, \tilde{\varphi})$ and define

$$z(t) = (z_1(t), \dots, z_n(t)),$$

where $z_i(t) = |x_i(t) - y_i(t)|$, $t \geq 0$, $i = 1, \dots, n$.

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  where $z_i(t) = |x_i(t) - y_i(t)|$, $t \geq 0$, $i = 1, \dots, n$.
- As function $z(t)$ is bounded, we define

$$\bar{z} := \max \left\{ \limsup_{t \to +\infty} z_i(t) : i = 1, \dots, n \right\}.$$

  and we have $\bar{z} \in [0, +\infty)$.

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- It remains to be prove that $\bar{z} = 0$.

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- By fluctuation lemma there is $(t_k)_{k \in \mathbb{N}}$ such that

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- Fix $\varepsilon > 0$.
- As $z(t)$ is bounded, $\lim_{t\to+\infty}(t - \tau_{ijp}(t)) = +\infty$, the properties of function g, we can prove that:

  for large $k \in \mathbb{N}$ and $\omega \in [t_k - \bar{\tau}_i, t_k]$, we have

$$\|z_\omega\|_g \leq \bar{z} + \varepsilon \qquad \text{and} \qquad |z(\omega - \tau_{ip}(\omega))| \leq \bar{z} + \varepsilon. \tag{10}$$

For large $k \in \mathbb{N}$, we have

$$\begin{aligned} z_i'(t_k) &= \operatorname{sign}(x_i(t_k) - y_i(t_k))(x_i'(t_k) - y_i'(t_k)) \\ &\leq -a_i(t_k)z_i(t_k) \\ &\quad + a_i(t_k)\Big|\big(x_i(t_k) - y_i(t_k)\big) - \big(x_i(t_k - \tau_i(t_k)) - y_i(t_k - \tau_i(t_k))\big)\Big| \\ &\quad + H_i(t_k)\max_p \left\{|z(t_k - \tau_{ip}(t_k))|\right\} + F_i(t_k)\|z_{t_k}\|_g \end{aligned}$$

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&\quad + H_i(t_k)\max_p \left\{|z(t_k - \tau_{ip}(t_k))|\right\} + F_i(t_k)\|z_{t_k}\|_g \\
&= -a_i(t_k)z_i(t_k) + a_i(t_k)\left|\int_{t_k-\tau_i(t_k)}^{t_k} (x_i'(w) - y_i'(w))\,dw\right| \\
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&= -a_i(t_k)z_i(t_k) + a_i(t_k)\left|\int_{t_k-\tau_i(t_k)}^{t_k}\big(x_i'(w) - y_i'(w)\big)\,dw\right| \\
&\quad + H_i(t_k)\max_p\left\{|z(t_k - \tau_{ip}(t_k))|\right\} + F_i(t_k)\|z_{t_k}\|_g \\
&\leq -a_i(t_k)z_i(t_k) + a_i(t_k)\int_{t_k-\tau_i(t_k)}^{t_k}\Big(a_i(w)z_i(w - \tau_i(w)) \\
&\quad + H_i(w)\max_p\left\{|z(w - \tau_{ip}(w))|\right\} + F_i(w)\|z_w\|_g\Big)dw \\
&\quad + H_i(t_k)\max_p\left\{|z(t_k - \tau_{ip}(t_k))|\right\} + F_i(t_k)\|z_{t_k}\|_g
\end{aligned}
$$

By the previous estimations (10),

$$
\begin{aligned}
z_i'(t_k) \quad \leq \quad & -a_i(t_k)z_i(t_k) + a_i(t_k)\int_{t_k-\tau_i(t_k)}^{t_k}\Bigg(a_i(w)z_i(w-\tau_i(w)) \\
& +H_i(w)\max_p\left\{|z(w-\tau_{ip}(w))|\right\} + F_i(w)\|z_w\|_g\Bigg)dw \\
& +H_i(t_k)\max_p\left\{|z(t_k-\tau_{ip}(t_k))|\right\} + F_i(t_k)\|z_{t_k}\|_g \\
\leq \quad & -a_i(t_k)z_i(t_k) + a_i(t_k)(\bar{z}+\varepsilon)\left(\int_{t_k-\tau_i(t_k)}^{t_k}\big(a_i(w)+H_i(w)+F_i(w)\big)dw\right. \\
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& + H_i(w)\max_p\left\{|z(w-\tau_{ip}(w))|\right\} + F_i(w)\|z_w\|_g\Big)dw \\
& + H_i(t_k)\max_p\left\{|z(t_k-\tau_{ip}(t_k))|\right\} + F_i(t_k)\|z_{t_k}\|_g \\
\leq \quad & -a_i(t_k)z_i(t_k) + a_i(t_k)(\bar{z}+\varepsilon)\left(\int_{t_k-\tau_i(t_k)}^{t_k}\big(a_i(w)+H_i(w)+F_i(w)\big)dw\right. \\
& \left. + H_i(t_k) + F_i(t_k)\right)
\end{aligned}$$

thus

$$\frac{z_i'(t_k)}{a_i(t_k)} \leq -z_i(t_k) + (\bar{z}+\varepsilon)\left(\int_{t_k-\tau_i(t_k)}^{t_k} a_i(w)+H_i(w)+F_i(w)dw + \frac{H_i(t_k)+F_i(t_k)}{a_i(t_k)}\right)$$

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$$
\begin{aligned}
z_i'(t_k) \quad &\leq \quad -a_i(t_k)z_i(t_k) + a_i(t_k)\int_{t_k-\tau_i(t_k)}^{t_k}\Big(a_i(w)z_i(w-\tau_i(w)) \\
&\qquad +H_i(w)\max_p\left\{|z(w-\tau_{ip}(w))|\right\} + F_i(w)\|z_w\|_g\Big)dw \\
&\qquad +H_i(t_k)\max_p\left\{|z(t_k-\tau_{ip}(t_k))|\right\} + F_i(t_k)\|z_{t_k}\|_g \\
&\leq \quad -a_i(t_k)z_i(t_k) + a_i(t_k)(\bar{z}+\varepsilon)\left(\int_{t_k-\tau_i(t_k)}^{t_k}\big(a_i(w)+H_i(w)+F_i(w)\big)dw\right. \\
&\qquad \left. +H_i(t_k)+F_i(t_k)\right)
\end{aligned}
$$

thus

$$
\frac{z_i'(t_k)}{a_i(t_k)} \leq -z_i(t_k) + (\bar{z}+\varepsilon)\left(\int_{t_k-\tau_i(t_k)}^{t_k} a_i(w)+H_i(w)+F_i(w)dw + \frac{H_i(t_k)+F_i(t_k)}{a_i(t_k)}\right)
$$

Taking $k\to+\infty$ and $\varepsilon\to 0^+$, by (A1) and (9),

$$
0 \leq -\bar{z} + \bar{z}\left(\limsup_{t\to+\infty}\left(\frac{H_i(t)+F_i(t)}{a_i(t)} + \int_{t-\tau_i(t)}^{t} a_i(w)+H_i(w)+F_i(w)dw\right)\right)
$$

Applications

- Consider

$$
\begin{aligned}
x_i'(t) = -a_i(t)x_i(t-\tau_i(t)) + \sum_{j=1}^{n}\Bigg(\sum_{k=1}^{K} h_{ijk}(t, x_j(t-\tau_{ijk}(t))) \\
+ \int_{-\infty}^{t} K_{ij}(t,s) f_{ij}(s, x_j(s-\varrho_{ij}(s)))ds\Bigg), \\
i = 1, \dots, n.
\end{aligned} \tag{11}
$$

Applications

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$$x_i'(t) = -a_i(t)x_i(t-\tau_i(t)) + \sum_{j=1}^{n}\left(\sum_{k=1}^{K} h_{ijk}(t, x_j(t-\tau_{ijk}(t))) + \int_{-\infty}^{t} K_{ij}(t,s)f_{ij}(s, x_j(s-\varrho_{ij}(s)))ds\right), \quad i = 1,\dots,n. \tag{11}$$

- Define

$$\mathcal{A} = [\alpha_{ij}]_{i,j=1}^{n},$$

by $\alpha_{ii} = 1 - \dfrac{\underline{a}_i(\overline{a}_i + \tilde{L}_{ii})\overline{\tau}_i + \tilde{L}_{ii}}{\underline{a}_i} \qquad \alpha_{ij} = -\dfrac{\underline{a}_i\tilde{L}_{ij}\overline{\tau}_i + \tilde{L}_{ij}}{\underline{a}_i}$, $i \neq j$,

where $0 < \underline{a}_i \leq a_i(t) \leq \overline{a}_i$, $|K_{ij}(t,s)| \leq \overline{\kappa}_{ij}$, and $\tilde{L}_{ij} = \sum_{k=1}^{K} H_{ijk} + \overline{\kappa}_{ij}F_{ij}$.

Applications

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- **Theorem 3:**
If $\mathcal{A}$ is a non-singular M-matrix, then (11) is globally asymptotically stable.

- Consider

$$x_i'(t) = -a_i(t)x_i(t-\tau_i(t)) + \sum_{j=1}^{n}\left(\sum_{k=1}^{K} h_{ijk}(t, x_j(t-\tau_{ijk}(t))) + \int_{-\varrho_j}^{t} K_{ij}(t,s) f_{ij}(s, x_j(s-\varrho_{ij}(s)))ds\right), \quad i = 1\dots, n \tag{12}$$

- Define

$$\mathcal{A} = \left[\alpha_{ij}\right]_{i,j=1}^{n},$$

by $\alpha_{ii} = 1 - \dfrac{\underline{a}_i(\overline{a}_i + \tilde{L}_{ii})\overline{\tau}_i + \tilde{L}_{ii}}{\underline{a}_i} \qquad \alpha_{ij} = -\dfrac{\underline{a}_i \tilde{L}_{ij}\overline{\tau}_i + \tilde{L}_{ij}}{\underline{a}_i}$, $i \neq j$,
where $0 < \underline{a}_i \leq a_i(t) \leq \overline{a}_i$, $|K_{ij}(t,s)| \leq \overline{\kappa}_{ij}$, and $\tilde{L}_{ij} = \sum_{k=1}^{K} H_{ijk} + \overline{\kappa}_{ij} F_{ij}$.

- **Corollary:** Assume $\tau_{ijk}(t)$ bounded.
If $\mathcal{A}$ is a non-singular M-matrix, then (12) is globally exponentially stable.
Proof: It is done with the change $y_i(t) = \mathrm{e}^{\lambda t} x_i(t)$

- Consider

$$x_i'(t) = -a_i(t)x_i(t-\tau_i(t)) + \sum_{j=1}^{n}\left(\sum_{k=1}^{K} h_{ijk}(t, x_j(t-\tau_{ijk}(t))) + \int_{-\varrho_j}^{t} K_{ij}(t,s)f_{ij}(s, x_j(s-\varrho_{ij}(s)))ds\right), \quad i = 1\dots,n \tag{12}$$

- Define

$$\mathcal{A} = \left[\alpha_{ij}\right]_{i,j=1}^{n}, \qquad C \leq \mathcal{A}$$

by $\alpha_{ii} = 1 - \frac{\underline{a}_i(\overline{a}_i + \tilde{L}_{ii})\overline{\tau}_i + \tilde{L}_{ii}}{\underline{a}_i}$ $\qquad \alpha_{ij} = -\frac{\underline{a}_i\tilde{L}_{ij}\overline{\tau}_i + \tilde{L}_{ij}}{\underline{a}_i}$, $i \neq j$,
where $0 < \underline{a}_i \leq a_i(t) \leq \overline{a}_i$, $|K_{ij}(t,s)| \leq \overline{\kappa}_{ij}$, and $\tilde{L}_{ij} = \sum_{k=1}^{K} H_{ijk} + \overline{\kappa}_{ij}F_{ij}$.

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  If $\mathcal{A}$ is a non-singular M-matrix, then (12) is globally exponentially stable.
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The previous results can be applied to BAM

$$\begin{cases} x_i'(t) = -\hat{b}_i(t)x_i(t-\hat{r}_i(t)) + \sum_{j=1}^{k} c_{ij}(t)h_j(y_j(t)) + \sum_{j=1}^{k} d_{ij}(t)h_j(y_j(t-r_{ij}(t))) \\ \qquad + \sum_{j=1}^{k} e_{ij}(t)\int_{-\infty}^{t} \mathcal{K}_{ij}(t-s)f_j(y_j(s-\varrho_{ij}(s)))ds + \hat{I}_i(t), \\ \qquad i \in \{1,\dots,\hat{k}\}, \\ y_j'(t) = -b_j(t)y_j(t-r_j(t)) + \sum_{i=1}^{\hat{k}} \hat{c}_{ji}(t)\hat{h}_i(x_i(t)) + \sum_{i=1}^{\hat{k}} \hat{d}_{ji}(t)\hat{h}_i(x_i(t-\hat{r}_{ji}(t))) \\ \qquad + \sum_{i=1}^{\hat{k}} \hat{e}_{ji}(t)\int_{-\infty}^{t} \hat{\mathcal{K}}_{ji}(t-s)\hat{f}_i(x_i(s-\hat{\varrho}_{ji}(s)))ds + I_j(t), \\ \qquad j \in \{1,\dots,k\} \end{cases} \tag{13}$$

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$$\begin{cases} x_i'(t) = -\hat{b}_i(t)x_i(t-\hat{r}_i(t)) + \sum_{j=1}^{k} c_{ij}(t)h_j(y_j(t)) + \sum_{j=1}^{k} d_{ij}(t)h_j(y_j(t-r_{ij}(t))) \\ \qquad + \sum_{j=1}^{k} e_{ij}(t)\int_{-\infty}^{t} \mathcal{K}_{ij}(t-s)f_j(y_j(s-\varrho_{ij}(s)))ds + \hat{I}_i(t), \\ \qquad i \in \{1,\dots,\hat{k}\}, \\ y_j'(t) = -b_j(t)y_j(t-r_j(t)) + \sum_{i=1}^{\hat{k}} \hat{c}_{ji}(t)\hat{h}_i(x_i(t)) + \sum_{i=1}^{\hat{k}} \hat{d}_{ji}(t)\hat{h}_i(x_i(t-\hat{r}_{ji}(t))) \\ \qquad + \sum_{i=1}^{\hat{k}} \hat{e}_{ji}(t)\int_{-\infty}^{t} \hat{\mathcal{K}}_{ji}(t-s)\hat{f}_i(x_i(s-\hat{\varrho}_{ji}(s)))ds + I_j(t), \\ \qquad j \in \{1,\dots,k\} \end{cases} \tag{13}$$

$$|h_j(u)-h_j(v)| \le H_j|u-v|,\ |f_j(u)-f_j(v)| \le F_j|u-v|,\ \forall u,v \in \mathbb{R}^k$$
$$|\hat{h}_i(u)-\hat{h}_i(v)| \le \hat{H}_i|u-v|,\ |\hat{f}_i(u)-\hat{f}_i(v)| \le \hat{F}_i|u-v|,\ \forall u,v \in \mathbb{R}^{\hat{k}}$$

- Assume that all coefficient functions are bounded:

$$\hat{b}_i(t) \leq \bar{\hat{b}}_i, \quad b_j(t) \leq \bar{b}_j, \quad |c_{ij}(t)| \leq \bar{c}_{ij}, \quad |\hat{c}_{ji}(t)| \leq \bar{\hat{c}}_{ji},$$
$$|d_{ij}(t)| \leq \bar{d}_{ij}, \quad |\hat{d}_{ji}(t)| \leq \bar{\hat{d}}_{ji}, \quad |e_{ij}(t)| \leq \bar{e}_{ij}, \quad \text{and } |\hat{e}_{ji}(t)| \leq \bar{\hat{e}}_{ji},$$

- Define the matrix $\mathcal{B}$ as

$$\mathcal{B} = \begin{bmatrix} \hat{\mathcal{D}} & -\mathcal{P} \\ -\hat{\mathcal{P}} & \mathcal{D} \end{bmatrix}_{(\hat{k}+k)\times(\hat{k}+k)}, \text{ with } \mathcal{P} = [p_{ij}]_{\hat{k}\times k}, \ \hat{\mathcal{P}} = [\hat{p}_{ji}]_{k\times\hat{k}}$$

where $\hat{\mathcal{D}} = diag\left(1 - \bar{\hat{b}}_1 \bar{\hat{r}}_1, \ldots, 1 - \bar{\hat{b}}_{\hat{k}} \bar{\hat{r}}_{\hat{k}}\right)$,

$$\mathcal{D} = diag\left(1 - \bar{b}_1 \bar{r}_1, \ldots, 1 - \bar{b}_k \bar{r}_k\right),$$
$$p_{ij} = \left(\bar{\hat{r}}_i + \underline{\hat{b}}_i^{-1}\right)[(\bar{c}_{ij} + \bar{d}_{ij})H_j + \bar{e}_{ij}F_j],$$
$$\hat{p}_{ji} = \left(\bar{r}_j + \underline{b}_j^{-1}\right)[(\bar{\hat{c}}_{ji} + \bar{\hat{d}}_{ji})\hat{H}_i + \bar{\hat{e}}_{ji}\hat{F}_i].$$

- **Corollary**

  If the matrix $\mathcal{B}$ is a non-singular M-matrix, then the BAM model (13) is globally asymptotically stable.

- *Numerical example*:

$$\left\{\begin{aligned} x'(t) &= -(6+\sin(t))x\left(t-\frac{|\sin(t)|}{9}\right)+c\sin(t)y(t)\\ &\quad +dy(t-10-pt)+e\int_{-\infty}^{t}\mathrm{e}^{-t+s}y(s)ds\\ y'(t) &= -(4+\cos(t))y\left(t-\frac{|\cos(t)|}{9}\right)+\hat{c}\arctan(x(t))\\ &\quad +\hat{d}\arctan\left(x(t-10-q\log(t+1))\right)\\ &\quad +\hat{e}\int_{-\infty}^{t}\mathrm{e}^{-t+s}x(s)ds \end{aligned}\right. , \quad (14)$$

where $p\in[0,1)$, $q\geq 0$, $|c|+|d|+|e|=\frac{15}{14}$, and $|\hat{c}|+|\hat{d}|+|\hat{e}|=\frac{1}{2}$

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where $p\in[0,1)$, $q\geq 0$, $|c|+|d|+|e|=\frac{15}{14}$, and $|\hat{c}|+|\hat{d}|+|\hat{e}|=\frac{1}{2}$

$$\mathcal{B}=\begin{bmatrix} \frac{2}{9} & -\frac{3}{9} \\ -\frac{2}{9} & \frac{4}{9} \end{bmatrix}, \qquad \sigma(\mathcal{B})=\left\{\frac{3-\sqrt{7}}{9},\frac{3+\sqrt{7}}{9}\right\}$$

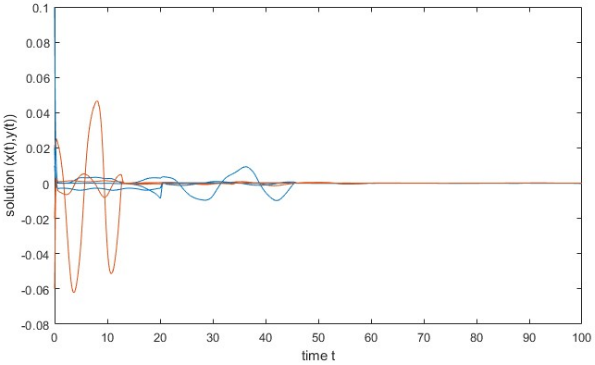

Figure: Three solutions $(x(t), y(t))$ of system (14) where $p = \frac{1}{2}$, $q = 1$, $c = \frac{1}{14}$, $d = 1$, $e = \hat{e} = 0$, and $\hat{c} = \hat{d} = \frac{1}{4}$, with initial condition $\varphi(s) = (0.01 + \sin(s), 0.01 + 0.01\cos(s))$, $\varphi(s) = (0.1\cos(s), -0.06e^{s})$, $\varphi(s) = (-0.02, 0.02)$ for $s \leq 0$, respectively, at $t_0 = 0$.

Thank you

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