Global stability for a nonlinear differential system with infinite delay and applications to BAM neural networks

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Pioneer neural network models BAM neural network models with delays

Pioneer Neural Network Models

*<u>Pioneer Models</u>:

Cohen-Grossberg (1983)

$$x'_{i}(t) = -d_{i}(x_{i}(t))\left(b_{i}(x_{i}(t)) - \sum_{j=1}^{n} a_{ij}h_{j}(x_{j}(t))\right), \ i = 1, \dots, n.$$
(1)

Hopfield (1984)

$$x'_i(t) = -b_i(x_i(t)) + \sum_{j=1}^n a_{ij}h_j(x_j(t)), \quad i = 1, ..., n.$$
 (2)

where, $n \in \mathbb{N}$ is the number of neurons; d_i amplification functions; b_i controller functions;

 h_j activation functions; $A = [a_{ij}]$ connection matrix.

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$$\begin{cases} x'_{i}(t) = -x_{i}(t) + \sum_{\substack{j=1 \ m}}^{m} a_{ij}f(y_{j}(t)) + I_{i} \\ y'_{i}(t) = -y_{i}(t) + \sum_{\substack{j=1 \ m}}^{m} b_{ij}f(x_{i}(t)) + J_{i} \end{cases}, \quad i = 1, \dots, m, (3)$$

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Gopalsamy (2007)

$$egin{aligned} & x_i'(t) = -x_i(t- au) + \sum_{j=1}^m a_{ij} f_j(y_j(t-\hat{\sigma})) + I_i \ & y_i'(t) = -y_i(t-\hat{\tau}) + \sum_{j=1}^m b_{ij} g_j(x_j(t-\sigma)) + J_i \end{aligned}, \quad i = 1, \dots, m, \end{aligned}$$

In both cases: $t \ge 0$ and $m \in \mathbb{N}$.

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Berezansky, Braverman, and Idels (2014)[1]

$$\begin{aligned} x_i'(t) &= r_i(t) \left(-a_i x_i(t - \tau_i(t)) + \sum_{j=1}^m a_{ij} f_j(y_j(t - \hat{\sigma}_{ij}(t))) + I_i \right) \\ y_i'(t) &= p_i(t) \left(-b_i y_i(t - \hat{\tau}_i(t)) + \sum_{j=1}^m b_{ij} g_j(x_j(t - \sigma_{ij}(t))) + J_i \right) \end{aligned}, \quad (4)$$

[1] L.Berezansky, E. Braverman, and L. Idels, Appl. Math. Comput. 243 (2014) 899-910

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Pioneer neural network models BAM neural network models with delays

Berezansky, Braverman, and Idels (2014)[1]

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In [1] Berezansky et.al. studied

$$x'_{i}(t) = -a_{i}(t)x_{i}(t - \tau_{i}(t)) + \sum_{j=1}^{n} F_{ij}(t, x_{j}(t - \sigma_{ij}(t))), \quad i = 1, \dots, n \quad (5)$$

where
$$(n \in \mathbb{N})$$
:
• $0 < \underline{a}_i \leq a_i(t) \leq \overline{a}_i$;
• $0 \leq \tau_i(t) \leq \overline{\tau}_i$ and $0 \leq \sigma_{ij}(t) \leq \overline{\sigma}_{ij}$; $\overline{\tau} := \max\{\overline{\tau}_i, \overline{\sigma}_{ij}\}$
• $|F_{ij}(t, u)| \leq L_{ij}|u|$

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Theorem 1 [1]: If the matrix

$$C = \left[c_{ij}\right]_{i,j=1}^n,$$

$$c_{ii} = 1 - rac{\overline{a}_i(\overline{a}_i + L_{ii})\overline{ au}_i + L_{ii}}{\underline{a}_i}$$
 and $c_{ij} = -rac{\overline{a}_i L_{ij}\overline{ au}_i + L_{ij}}{\underline{a}_i}$, $i \neq j$, is a non-singular M -matrix, then system

$$x'_i(t) = -a_i(t)x_i(t - \tau_i(t)) + \sum_{j=1}^n F_{ij}(t, x_j(t - \sigma_{ij}(t))), \quad i = 1, ..., n$$

is globally exponentially stable,

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$$x'_i(t) = -a_i(t)x_i(t - \tau_i(t)) + \sum_{j=1}^n F_{ij}(t, x_j(t - \sigma_{ij}(t))), \quad i = 1, ..., n$$

is globally exponentially stable, i.e. there are $C \ge 1$ and $\lambda > 0$ such that

$$|x(t, t_0, \varphi) - x(t, t_0, \phi)| \leq C \mathrm{e}^{-\lambda(t-t_0)} \|\varphi - \phi\|,$$

 $\forall t_0 \geq 0, \quad \forall \varphi, \phi \in C([-\overline{\tau}, 0]; \mathbb{R}^n)$

- At the end of [1], the authors provided a list with 9 open problems:
- 2. Study global stability for the model

$$x'_i(t) = -a_i(t)x_i(t - \tau_i(t)) + \sum_{j=1}^n \sum_{k=1}^N F_{ijk}(t, x_j(t - \sigma_{ijk}(t))), \quad i = 1, \dots, n$$

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3. Derive sufficient stability test for

$$x_i'(t)=-a_i(t)x_i(t- au_i(t))+\sum_{j=1}^n\int_{t-arsigma_i}^tF_{ij}(s,x_j(s-\sigma_{ij}(s)))ds, \hspace{1cm} i=1,\ldots,n$$

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5. Obtain sufficient stability conditions for

$$x_i'(t)=-a_i(t)x_i(t- au_i(t))+\sum_{j=1}^n\int_{-\infty}^t extsf{K}_{ij}(t,s) extsf{F}_{ij}(s,x_j(s- au_{ij}(s)))ds.$$

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General model Hypotheses

We consider the general family of DDE

$$x_i'(t) = -a_i(t)x_i(t - au_i(t)) + h_i(t, x(t - au_{i1}(t)), \dots, x(t - au_{im}(t))) + f_i(t, x_t),$$

for $t \ge 0$, i = 1, ..., n, where $n, m \in \mathbb{N}$, where $x_t : (-\infty, 0] \to \mathbb{R}^n$ is defined by $x_t(s) = x(t+s)$ for $s \le 0$.

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We consider the phase space [Hale and Kato (1978)]

$$UC_g = \left\{ \phi \in C((-\infty, 0]; \mathbb{R}^n) : \sup_{s \le 0} \frac{|\phi(s)|}{g(s)} < \infty, \frac{\phi(s)}{g(s)} \text{ unif. cont.} \right\},$$

$$\|\phi\|_g = \sup_{s \le 0} \frac{|\phi(s)|}{g(s)}$$
 with $|x| = |(x_1, \dots, x_n)| = \max_{1 \le i \le n} |x_i|$

where:

(g1)
$$g: (-\infty, 0] \rightarrow [1, \infty)$$
 non-increasing, continuous, with $g(0) = 1$;
(g2) $\lim_{u \to 0^-} \frac{g(s+u)}{g(s)} = 1$ uniformly on $(-\infty, 0]$;
(g3) $g(s) \rightarrow \infty$ as $s \rightarrow -\infty$.

For the general DDE

 $x'_{i}(t) = -a_{i}(t)x_{i}(t - \tau_{i}(t)) + h_{i}(t, x(t - \tau_{i1}(t)), \dots, x(t - \tau_{im}(t))) + f_{i}(t, x_{t}), \quad (6)$

we assume, for each i = 1, ..., n and p = 1, ..., m(A1) $a_i : [0, +\infty) \rightarrow (0, +\infty)$ is continuous such that

 $0 < \underline{a}_i \leq a_i(t);$

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 $0 < \underline{a}_i \leq a_i(t);$

(A2) $\tau_i, \tau_{ip}: [0, +\infty) \rightarrow [0, +\infty)$ are continuous such that

$$au_i(t) \leq \overline{ au}_i \qquad ext{and} \qquad \lim_{t o \infty} ig(t - au_{ij
ho}(t)ig) = \infty;$$

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For the general DDE

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$$au_i(t) \leq \overline{ au}_i \qquad ext{and} \qquad \lim_{t o \infty} ig(t - au_{ijp}(t)ig) = \infty;$$

(A3) $h_i: [0, +\infty) \times \mathbb{R}^{nm} \to \mathbb{R}$ is continuous such that

 $|h_i(t,u)-h_i(t,v)| \leq H_i(t)|u-v|, \quad t \geq 0, u, v \in \mathbb{R}^{nm};$

Pioneer neural network models General model General model Hypotheses Global asymptotic stability For the general DDE $x_{i}'(t) = -a_{i}(t)x_{i}(t - \tau_{i}(t)) + h_{i}(t, x(t - \tau_{i1}(t)), \dots, x(t - \tau_{im}(t))) + f_{i}(t, x_{t}), \quad (6)$ we assume, for each i = 1, ..., n and p = 1, ..., m(A1) $a_i: [0, +\infty) \to (0, +\infty)$ is continuous such that $0 < a_i < a_i(t)$: (A2) $\tau_i, \tau_{ip} : [0, +\infty) \to [0, +\infty)$ are continuous such that $au_i(t) \leq \overline{ au}_i \quad \text{and} \quad \lim_{t \to \infty} \left(t - au_{ijp}(t)\right) = \infty;$ (A3) $h_i: [0, +\infty) \times \mathbb{R}^{nm} \to \mathbb{R}$ is continuous such that $|h_i(t, u) - h_i(t, v)| < H_i(t)|u - v|, \quad t > 0, u, v \in \mathbb{R}^{nm}$ (A4) $f_i : [0, +\infty) \times UC_g \to \mathbb{R}$ is continuous such that $|f_i(t,\varphi) - f_i(t,\phi)| \leq F_i(t) \|\varphi - \phi\|_{\sigma}, \quad t \geq 0, \varphi, \phi \in UC_{\sigma};$

Main stability criterion Applications to Neural network models Numerical example

Main result

Consider

$$\begin{aligned} x_i'(t) &= -a_i(t)x_i(t - \tau_i(t)) + h_i(t, x(t - \tau_{i1}(t)), \dots, x(t - \tau_{im}(t))) + f_i(t, x_t), \\ \text{with bounded initial condition, i.e.} \end{aligned}$$

 $\begin{aligned} x_{t_0} &= \varphi, \quad \text{with } t_0 \geq 0 \text{ and } \varphi \in BC \\ BC &= \left\{ \varphi : (-\infty, 0] \to \mathbb{R}^n | \varphi \text{ is bounded and continuous} \right\} \\ BC &\subseteq UC_g \end{aligned}$ (7)

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Main result

Consider

$$\begin{aligned} x_i'(t) &= -a_i(t)x_i(t - \tau_i(t)) + h_i(t, x(t - \tau_{i1}(t)), \dots, x(t - \tau_{im}(t))) + f_i(t, x_t), \\ \text{with bounded initial condition, i.e.} \end{aligned}$$

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Theorem 2: Assume (A1)-(A4). If

$$\limsup_{t\to+\infty}\left(\frac{H_i(t)+F_i(t)}{a_i(t)}+\int_{t-\tau_i(t)}^t\left[a_i(w)+H_i(w)+F_i(w)\right]dw\right)<1,$$
 (8)

then system (6) is globally asymptotically stable, i.e. it is stable and

$$\lim_{t\to+\infty} \left[x(t,t_0,\varphi) - x(t,t_0,\tilde{\varphi}) \right] = 0, \quad \forall t_0 \ge 0, \, \varphi, \tilde{\varphi} \in BC;$$

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Main stability criterion Applications to Neural network models Numerical example

Proof (idea)

- First, we prove that x(t, t₀, φ), the solution of (6)-(7), is defined on ℝ and bounded.
- We prove that the system (6) is stable.

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Proof (idea)

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- For $t_0 \ge 0$ and $\varphi, \tilde{\varphi} \in BC$, let $x(t) = x(t, t_0, \varphi)$ and $y(t) = x(t, t_0, \tilde{\varphi})$ and define $z(t) = (z_1(t), \dots, z_n(t)),$

where $z_i(t) = |x_i(t) - y_i(t)|$, $t \ge 0$, i = 1, ..., n.

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$$z(t)=(z_1(t),\ldots,z_n(t)),$$

where $z_i(t) = |x_i(t) - y_i(t)|$, $t \ge 0$, i = 1, ..., n.

As function z(t) is bounded, we define

$$ar{z} := \max\left\{\limsup_{t \to +\infty} z_i(t) : i = 1, \dots, n
ight\}.$$

and we have $\bar{z} \in [0, +\infty)$.

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Main stability criterion Applications to Neural network models Numerical example

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and we have $\bar{z} \in [0, +\infty)$.

• It remains to be prove that $\bar{z} = 0$.

Main stability criterion Applications to Neural network models Numerical example

• Choose
$$i \in \{1, \ldots, n\}$$
 such that $\overline{z} = \limsup_{t \to +\infty} z_i(t)$;

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• Choose
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 such that $\overline{z} = \limsup_{t \to +\infty} z_i(t)$;

▶ By fluctuation lemma there is $(t_k)_{k \in \mathbb{N}}$ such that

$$t_k \nearrow \infty, \quad \lim_k z_i(t_k) = \overline{z}, \quad \text{and} \quad \lim_k z_i'(t_k) = 0.$$
 (9)

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 (9)

- Fix $\varepsilon > 0$.
- ► As z(t) is bounded, $\lim_{t\to+\infty} (t \tau_{ijp}(t)) = +\infty$, the properties of function g, we can prove that:

for large $k \in \mathbb{N}$ and $\omega \in [t_k - \bar{\tau}_i, t_k]$, we have

$$\|z_{\omega}\|_{g} \leq \bar{z} + \varepsilon$$
 and $|z(\omega - \tau_{ip}(\omega))| \leq \bar{z} + \varepsilon.$ (10)

For large $k \in \mathbb{N}$, we have

$$z'_i(t_k) = \operatorname{sign}(x_i(t_k) - y_i(t_k))(x'_i(t_k) - y'_i(t_k))$$

$$\leq -a_{i}(t_{k})z_{i}(t_{k}) \\ +a_{i}(t_{k})\Big| \Big(x_{i}(t_{k}) - y_{i}(t_{k})\Big) - \big(x_{i}(t_{k} - \tau_{i}(t_{k})) - y_{i}(t_{k} - \tau_{i}(t_{k}))\big) \\ +H_{i}(t_{k}) \max_{p} \big\{ |z(t_{k} - \tau_{ip}(t_{k}))| \big\} + F_{i}(t_{k}) ||z_{t_{k}}||_{g}$$

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$$= -a_{i}(t_{k})z_{i}(t_{k}) + a_{i}(t_{k}) \left| \int_{t_{k}-\tau_{i}(t_{k})}^{t_{k}} \left(x_{i}'(w) - y_{i}'(w) \right) dw + H_{i}(t_{k}) \max_{p} \left\{ |z(t_{k}-\tau_{ip}(t_{k}))| \right\} + F_{i}(t_{k}) ||z_{t_{k}}||_{g}$$

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$$= -a_{i}(t_{k})z_{i}(t_{k}) + a_{i}(t_{k}) \left| \int_{t_{k}-\tau_{i}(t_{k})}^{t_{k}} (x_{i}'(w) - y_{i}'(w)) dw \right| \\ + H_{i}(t_{k}) \max_{p} \left\{ |z(t_{k} - \tau_{ip}(t_{k}))| \right\} + F_{i}(t_{k}) ||z_{t_{k}}||_{g} \\ \leq -a_{i}(t_{k})z_{i}(t_{k}) + a_{i}(t_{k}) \int_{t_{k}-\tau_{i}(t_{k})}^{t_{k}} \left(a_{i}(w)z_{i}(w - \tau_{i}(w)) \right) \\ + H_{i}(w) \max_{p} \left\{ |z(w - \tau_{ip}(w))| \right\} + F_{i}(w) ||z_{w}||_{g} \right) dw \\ + H_{i}(t_{k}) \max_{p} \left\{ |z(t_{k} - \tau_{ip}(t_{k}))| \right\} + F_{i}(t_{k}) ||z_{t_{k}}||_{g}$$

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By the previous estimations (10),

$$\begin{aligned} z_{i}'(t_{k}) &\leq -a_{i}(t_{k})z_{i}(t_{k}) + a_{i}(t_{k})\int_{t_{k}-\tau_{i}(t_{k})}^{t_{k}} \left(a_{i}(w)z_{i}(w-\tau_{i}(w)) + H_{i}(w)\max_{p}\left\{|z(w-\tau_{ip}(w))|\right\} + F_{i}(w)||z_{w}||_{\mathcal{S}}\right)dw \\ &+ H_{i}(t_{k})\max_{p}\left\{|z(t_{k}-\tau_{ip}(t_{k}))|\right\} + F_{i}(t_{k})||z_{t_{k}}||_{\mathcal{S}} \\ &\leq -a_{i}(t_{k})z_{i}(t_{k}) + a_{i}(t_{k})(\bar{z}+\varepsilon)\left(\int_{t_{k}-\tau_{i}(t_{k})}^{t_{k}} (a_{i}(w) + H_{i}(w) + F_{i}(w))dw \\ &+ H_{i}(t_{k}) + F_{i}(t_{k})\right) \end{aligned}$$

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By the previous estimations (10),

$$\begin{aligned} z_{i}'(t_{k}) &\leq -a_{i}(t_{k})z_{i}(t_{k}) + a_{i}(t_{k})\int_{t_{k}-\tau_{i}(t_{k})}^{t_{k}} \left(a_{i}(w)z_{i}(w-\tau_{i}(w)) + H_{i}(w)\max_{p}\left\{|z(w-\tau_{ip}(w))|\right\} + F_{i}(w)||z_{w}||_{\mathcal{B}}\right)dw \\ &+ H_{i}(t_{k})\max_{p}\left\{|z(t_{k}-\tau_{ip}(t_{k}))|\right\} + F_{i}(t_{k})||z_{t_{k}}||_{\mathcal{B}} \\ &\leq -a_{i}(t_{k})z_{i}(t_{k}) + a_{i}(t_{k})(\overline{z}+\varepsilon)\left(\int_{t_{k}-\tau_{i}(t_{k})}^{t_{k}} (a_{i}(w) + H_{i}(w) + F_{i}(w))dw \\ &+ H_{i}(t_{k}) + F_{i}(t_{k})\right) \end{aligned}$$

thus

$$\frac{z_i'(t_k)}{a_i(t_k)} \leq -z_i(t_k) + (\bar{z} + \varepsilon) \left(\int_{t_k - \tau_i(t_k)}^{t_k} a_i(w) + H_i(w) + F_i(w) dw + \frac{H_i(t_k) + F_i(t_k)}{a_i(t_k)} \right)$$

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Taking $k \to +\infty$ and $\varepsilon \to 0^+$, by (A1) and (9),

$$0 \leq -\bar{z} + \bar{z} \left(\limsup_{t \to +\infty} \left(\frac{H_i(t) + F_i(t)}{a_i(t)} + \int_{t - \tau_i(t)}^t a_i(w) + H_i(w) + F_i(w) dw \right) \right)_{\mathbb{R}} \quad \text{for } u \in \mathbb{R}$$

José J. Oliveira

Asymptotic Stability of BAM Neural Network Models

Main stability criterion Applications to Neural network models Numerical example

Applications

Consider

$$\begin{aligned} x_{i}'(t) &= -a_{i}(t)x_{i}(t-\tau_{i}(t)) + \sum_{j=1}^{n} \left(\sum_{k=1}^{K} h_{ijk}(t,x_{j}(t-\tau_{ijk}(t))) \right. \\ &+ \int_{-\infty}^{t} K_{ij}(t,s)f_{ij}(s,x_{j}(s-\varrho_{ij}(s)))ds \right), \\ &\qquad \qquad i = 1, \dots, n. \end{aligned}$$

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Applications

Consider



$$\mathcal{A} = \left[\alpha_{ij}\right]_{i,j=1}^{n},$$

by $\alpha_{ii} = 1 - \frac{\underline{a}_{i}(\overline{a}_{i} + \widetilde{L}_{ii})\overline{\tau}_{i} + \widetilde{L}_{ii}}{\underline{a}_{i}} \quad \alpha_{ij} = -\frac{\underline{a}_{i}\widetilde{L}_{ij}\overline{\tau}_{i} + \widetilde{L}_{ij}}{\underline{a}_{i}}, i \neq j,$
where $0 < \underline{a}_{i} \leq a_{i}(t) \leq \overline{a}_{i}, |K_{ij}(t,s)| \leq \overline{\kappa}_{ij}, \text{ and } \widetilde{L}_{ij} = \sum_{k=1}^{K} H_{ijk} + \overline{\kappa}_{ij}F_{ij}.$

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Theorem 3:

If \mathcal{A} is a non-singular *M*-matrix, then (11) is globally asymptotically stable.

Main stability criterion Applications to Neural network models Numerical example



$$\begin{aligned} x_{i}'(t) &= -a_{i}(t)x_{i}(t-\tau_{i}(t)) + \sum_{j=1}^{n} \left(\sum_{k=1}^{K} h_{ijk}(t,x_{j}(t-\tau_{ijk}(t))) \right. \\ &+ \int_{-\varrho_{j}}^{t} K_{ij}(t,s)f_{ij}(s,x_{j}(s-\varrho_{ij}(s)))ds \right), \\ &i = 1...,n \end{aligned}$$
(12)

$$\mathcal{A} = \left[\alpha_{ij}\right]_{i,j=1}^n,$$

by $\alpha_{ii} = 1 - \frac{\underline{a}_i(\overline{a}_i + \widetilde{L}_{ii})\overline{\tau}_i + \widetilde{L}_{ii}}{\underline{a}_i}$ $\alpha_{ij} = -\frac{\underline{a}_i\widetilde{L}_{ij}\overline{\tau}_i + \widetilde{L}_{ij}}{\underline{a}_i}, i \neq j,$ where $0 < \underline{a}_i \le a_i(t) \le \overline{a}_i, |K_{ij}(t, s)| \le \overline{\kappa}_{ij}, \text{ and } \widetilde{L}_{ij} = \sum_{k=1}^{K} H_{ijk} + \overline{\kappa}_{ij}F_{ij}.$

Corollary: Assume τ_{ijk}(t) bounded.
 If A is a non-singular M-matrix, then (12) is globally exponentially stable.
 Proof: It is done with the change y_i(t) = e^{λt}x_i(t)

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(12)



$$\mathcal{A} = \begin{bmatrix} \alpha_{ij} \end{bmatrix}_{i,j=1}^{n}, \quad C \leq \mathcal{A}$$

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The previous results can be applied to BAM

$$\begin{aligned} \int x_{i}'(t) &= -\hat{b}_{i}(t)x_{i}(t-\hat{r}_{i}(t)) + \sum_{j=1}^{k} c_{ij}(t)h_{j}(y_{j}(t)) + \sum_{j=1}^{k} d_{ij}(t)h_{j}(y_{j}(t-r_{ij}(t))) \\ &+ \sum_{j=1}^{k} e_{ij}(t)\int_{-\infty}^{t} \mathcal{K}_{ij}(t-s)f_{j}(y_{j}(s-\varrho_{ij}(s)))ds + \hat{l}_{i}(t), \\ &\quad i \in \{1, \dots, \hat{k}\}, \\ y_{j}'(t) &= -b_{j}(t)y_{j}(t-r_{j}(t)) + \sum_{i=1}^{\hat{k}} \hat{c}_{ji}(t)\hat{h}_{i}(x_{i}(t)) + \sum_{i=1}^{\hat{k}} \hat{d}_{ji}(t)\hat{h}_{i}(x_{i}(t-\hat{r}_{ji}(t))) \\ &+ \sum_{i=1}^{\hat{k}} \hat{e}_{ji}(t)\int_{-\infty}^{t} \hat{\mathcal{K}}_{ji}(t-s)\hat{f}_{i}(x_{i}(s-\hat{\varrho}_{ji}(s)))ds + l_{j}(t), \\ &\quad j \in \{1, \dots, k\} \end{aligned}$$
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$$\begin{cases} x_{i}'(t) = -\hat{b}_{i}(t)x_{i}(t - \hat{r}_{i}(t)) + \sum_{j=1}^{k} c_{ij}(t)h_{j}(y_{j}(t)) + \sum_{j=1}^{k} d_{ij}(t)h_{j}(y_{j}(t - r_{ij}(t))) \\ + \sum_{j=1}^{k} e_{ij}(t)\int_{-\infty}^{t} \mathcal{K}_{ij}(t - s)f_{j}(y_{j}(s - \varrho_{ij}(s)))ds + \hat{l}_{i}(t), \\ i \in \{1, \dots, \hat{k}\}, \quad (13) \\ y_{j}'(t) = -b_{j}(t)y_{j}(t - r_{j}(t)) + \sum_{i=1}^{\hat{k}} \hat{c}_{ji}(t)\hat{h}_{i}(x_{i}(t)) + \sum_{i=1}^{\hat{k}} \hat{d}_{ji}(t)\hat{h}_{i}(x_{i}(t - \hat{r}_{ji}(t))) \\ + \sum_{i=1}^{\hat{k}} \hat{e}_{ji}(t)\int_{-\infty}^{t} \hat{\mathcal{K}}_{ji}(t - s)\hat{f}_{i}(x_{i}(s - \hat{\varrho}_{ji}(s)))ds + l_{j}(t), \\ j \in \{1, \dots, k\} \end{cases}$$

$$\begin{aligned} |h_{j}(u) - h_{j}(v)| &\leq H_{j}|u - v|, \ |f_{j}(u) - f_{j}(v)| &\leq F_{j}|u - v|, \ \forall u, v \in \mathbb{R}^{k} \\ \hat{h}_{i}(u) - \hat{h}_{i}(v)| &\leq \hat{H}_{i}|u - v|, \ |\hat{f}_{i}(u) - \hat{f}_{i}(v)| &\leq \hat{F}_{i}|u - v|, \ \forall u, v \in \mathbb{R}^{\hat{k}} \end{aligned}$$

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Assume that all coefficient functions are bounded:

$$egin{aligned} &\hat{b}_i(t)\leq \hat{b}_i, \quad b_j(t)\leq \overline{b}_j, \quad |c_{ij}(t)|\leq \overline{c}_{ij}, \quad |\hat{c}_{ji}(t)|\leq \overline{\hat{c}}_{ji}, \ |d_{ij}(t)|\leq \overline{\hat{d}}_{ji}, \quad |e_{ij}(t)|\leq \overline{e}_{ij}, \quad ext{and} \ |\hat{e}_{ji}(t)|\leq \overline{\hat{e}}_{ji}, \end{aligned}$$

Define the matrix B as

$$\mathcal{B} = \begin{bmatrix} \hat{\mathcal{D}} & -\mathcal{P} \\ -\hat{\mathcal{P}} & \mathcal{D} \end{bmatrix}_{(\hat{k}+k)\times(\hat{k}+k)}, \text{ with } \mathcal{P} = [p_{ij}]_{\hat{k}\times k}, \hat{\mathcal{P}} = [\hat{p}_{ji}]_{k\times \hat{k}}$$
where $\hat{\mathcal{D}} = diag \left(1 - \overline{\hat{b}}_1 \overline{\hat{r}}_1, \dots, 1 - \overline{\hat{b}}_k \overline{\hat{r}}_k\right),$

$$\mathcal{D} = diag \left(1 - \overline{b}_1 \overline{r}_1, \dots, 1 - \overline{b}_k \overline{r}_k\right),$$

$$p_{ij} = \left(\overline{\hat{r}}_i + \underline{\hat{b}}_i^{-1}\right) [(\overline{c}_{ij} + \overline{d}_{ij})H_j + \overline{e}_{ij}F_j],$$

$$\hat{p}_{ji} = \left(\overline{r}_j + \underline{b}_j^{-1}\right) [(\overline{\hat{c}}_{ji} + \overline{\hat{d}}_{ji})\hat{H}_i + \overline{\hat{e}}_{ji}\hat{F}_i].$$
Corollary

If the matrix \mathcal{B} is a non-singular M-matrix, then the BAM model (13) is globally asymptotically stable.

Pioneer neural network models General model Global asymptotic stability Main stability criterion Applications to Neural network models Numerical example

Numerical example:

$$x'(t) = -(6 + \sin(t))x\left(t - \frac{|\sin(t)|}{9}\right) + c\sin(t)y(t) + dy(t - 10 - pt) + e\int_{-\infty}^{t} e^{-t+s}y(s)ds$$
(14)

$$y'(t) = -(4 + \cos(t))y\left(t - \frac{|\cos(t)|}{9}\right) + \hat{c} \arctan(x(t))$$
, (14)
 $+\hat{d} \arctan(x(t - 10 - q\log(t + 1)))$
 $+\hat{e} \int_{-\infty}^{t} e^{-t+s}x(s)ds$

where $p \in [0,1)$, $q \ge 0$, $|c|+|d|+|e|=rac{15}{14}$, and $|\hat{c}|+|\hat{d}|+|\hat{e}|=rac{1}{2}$

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Numerical example:

$$x'(t) = -(6 + \sin(t))x\left(t - \frac{|\sin(t)|}{9}\right) + c\sin(t)y(t) + dy(t - 10 - pt) + e\int_{-\infty}^{t} e^{-t+s}y(s)ds$$

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$$+\hat{d}\arctan(x(t - 10 - q\log(t + 1)))$$
$$+\hat{c}\int_{-\infty}^{t} e^{-t+s}x(s)ds$$

where $p \in [0,1)$, $q \ge 0$, $|c|+|d|+|e|=rac{15}{14}$, and $|\hat{c}|+|\hat{d}|+|\hat{e}|=rac{1}{2}$

$$\mathcal{B} = \begin{bmatrix} \frac{2}{9} & -\frac{3}{9} \\ \\ -\frac{2}{9} & \frac{4}{9} \end{bmatrix}, \quad \sigma(\mathcal{B}) = \left\{ \frac{3 - \sqrt{7}}{9}, \frac{3 + \sqrt{7}}{9} \right\}$$

General model	Applications to Neural network models
Global asymptotic stability	Numerical example



Figure: Three solutions (x(t), y(t)) of system (14) where $p = \frac{1}{2}$, q = 1, $c = \frac{1}{14}$, d = 1, $e = \hat{e} = 0$, and $\hat{c} = \hat{d} = \frac{1}{4}$, with initial condition $\varphi(s) = (0.01 + \sin(s), 0.01 + 0.01 \cos(s))$, $\varphi(s) = (0.1 \cos(s), -0.06e^s)$, $\varphi(s) = (-0.02, 0.02)$ for $s \le 0$, respectively, at $t_0 = 0$.

Pioneer neural network models	Main stability criterion
General model	Applications to Neural network models
Global asymptotic stability	Numerical example

Thank you

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