Unity in structural proof theory and structural extensions of the λ-calculus

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Motivation

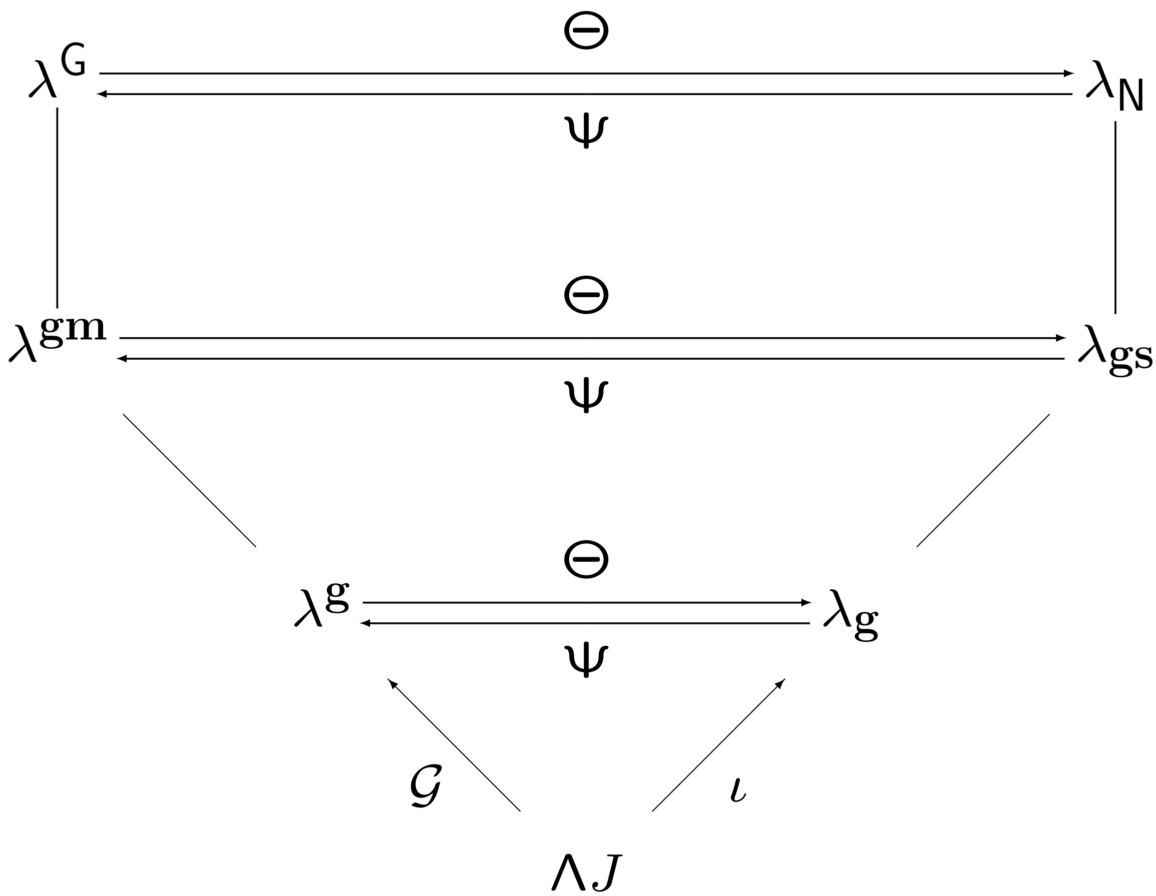
Two closely related problems

1. Unity in structural proof theory
2. Structural extensions of the λ-calculus

Goal: fully realize the programme suggested by

1. von Plato's natural deduction with general elimination rules
2. Herbelin's "λ-calculus structure" for sequent calculus
3. JES' isomorphism between a fragment of LJT and an extension of natural deduction

Roadmap



Sequent calculi (left) and natural deduction systems (right)

The λ^{gm}-calculus (1)

Expressions

(Terms)	t, u, v	$::=$	$x \mid \lambda x.t \mid tl$
(Applicative contexts)	l	$::=$	$u \cdot (x)v \mid u :: l$

Abbreviations $[u] \equiv u \cdot (x)x$

Extracting hidden information

Match

$$t(u_1 :: ... :: u_m \cdot (x)v) \qquad (m \geq 1)$$

with $\Psi(F, u_m \cdot (x)v)$ for some F, or

with $\Psi(F', u_{m-1} :: u_m \cdot (x)v)$ for some F', etc.

The λ^{gm}-calculus (2)

Sequents

$\Gamma \vdash t : A \qquad \Gamma ; A \vdash l : B$

Typing rules

$$\frac{}{\Gamma, x : A \vdash x : A}\, Axiom$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \supset B}\, Right$$

$$\frac{\Gamma \vdash t : A \quad \Gamma ; A \vdash l : B}{\Gamma \vdash tl : B}\, h - Cut$$

$$\frac{\Gamma \vdash u : A \quad \Gamma ; B \vdash l : C}{\Gamma ; A \supset B \vdash u :: l : C}\, lc - Left$$

$$\frac{\Gamma \vdash u : A \quad \Gamma, x : B \vdash v : C}{\Gamma ; A \supset B \vdash u \cdot (x)v : C}\, l - Left$$

The λ^{gm}-calculus (3)

Reduction rules

$$
\begin{array}{rrcl}
(\beta 1) & (\lambda x.t)(u \cdot (y)v) & \rightarrow & s(s(u,x,t),y,v) \\
(\beta 2) & (\lambda x.t)(u :: l) & \rightarrow & s(u,x,t)l \\
(\pi) & \Psi(F, u \cdot (x)v)l & \rightarrow & \Psi(F, u \cdot (x)vl) \\
(\mu) & u \cdot (x)xl & \rightarrow & u :: l, \text{ if } x \notin l
\end{array}
$$

t is in $\boxed{\beta\pi\text{-nf}}$ iff every cut occurring in t is of the form xl (morally a left introduction)

1) For typable terms, reduction is SN

2) Any combination of the reduction rules is confluent

Proof: Because this holds of $\lambda\mathbf{Jm}$. Think of

$$t(u_1 :: ... :: u_m \cdot (x)v)$$

as

$$t(u_1, u_2 :: ... :: u_m :: [], (x)v)$$

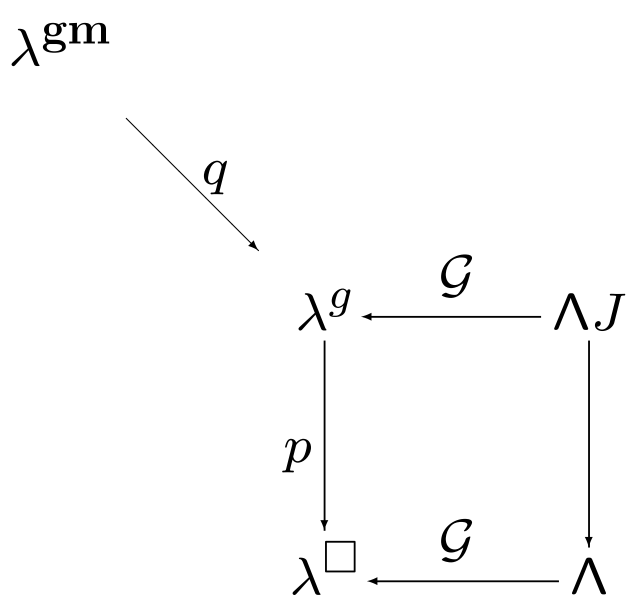
The λ^{gm}-calculus (4)

Permutation rules

$$\begin{array}{llll} (p) & \Psi(F, u \cdot (x)v) & \rightarrow & s(\Psi(F,[u]), x, v), \text{if } v \neq x \\ (q) & \Psi(F, u :: l) & \rightarrow & \Psi(F,[u])l \end{array}$$

t is in $\boxed{q\text{-nf}}$ iff every cut occurring in t is of the form $t(u \cdot (x)v)$ (morally a general application)

t is in $\boxed{pq\text{-nf}}$ iff every cut occurring in t is of the form $t[u] \equiv t(u \cdot (x)x)$ (morally an ordinary application)



The λ_{gs}-calculus (1)

Expressions

(Terms)	M, N, P	$::=$	$x \mid \lambda x.M \mid app(F, N, (x)P)$
(Functions)	F	$::=$	$hd(M) \mid FN$

Abbreviations $\quad app(F, N) \equiv app(F, N, (x)x)$

Extracting hidden information

Match

$app(hd(M)N_1 ... N_{m-1}, N_m, (x)P) \qquad (m \geq 1)$

with $\Theta(hd(M), l)$ for some l, or

with $\Theta(hd(M)N_1, l')$ for some l', etc.

The λ_{gs}-calculus (2)

Sequents

$\Gamma \vdash M : A \qquad \Gamma \rhd F : A$

Typing rules

$$\frac{}{\Gamma, x : A \vdash x : A}\, Assumption$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \supset B}\, Intro$$

$$\frac{\Gamma \rhd F : A \supset B \quad \Gamma \vdash N : A}{\Gamma \rhd FN : B}\, inner-Elim$$

$$\frac{\Gamma \rhd F : A \supset B \quad \Gamma \vdash N : A \quad \Gamma, x : B \vdash P : C}{\Gamma \vdash app(F, N, (x)P) : C}\, outer-Elim$$

$$\frac{\Gamma \vdash M : A}{\Gamma \rhd hd(M) : A}\, Coercion$$

The λ_{gs}-calculus (3)

<u>Reduction rules</u>

$$
\begin{array}{lrcl}
(\beta 1) & app(hd(\lambda x.M), N, (y)P) & \rightarrow & [[N/x]M/y]P \\
(\beta 2) & hd(\lambda x.M)N & \rightarrow & hd([N/x]M)
\end{array}
$$

$$
\begin{array}{ll}
(\pi) & \ominus(hd(app(F, N, (x)P), l) \rightarrow \\
 & \qquad \rightarrow app(F, N, (x)\ominus(hd(P), l)) \\
(\mu) & app(F, N, (x)\ominus(hd(x), l)) \rightarrow \ominus(FN, l), \text{ if } x \notin l
\end{array}
$$

M is in $\boxed{\beta\pi\text{-nf}}$ iff every coercion occurring in M is of the form $hd(x)$

A derivation $\mathcal{D}$ in $\lambda_{\mathbf{gs}}$ is $\boxed{\beta\pi\text{-normal}}$ iff every coercion formula occurring in $\mathcal{D}$ is an assumption

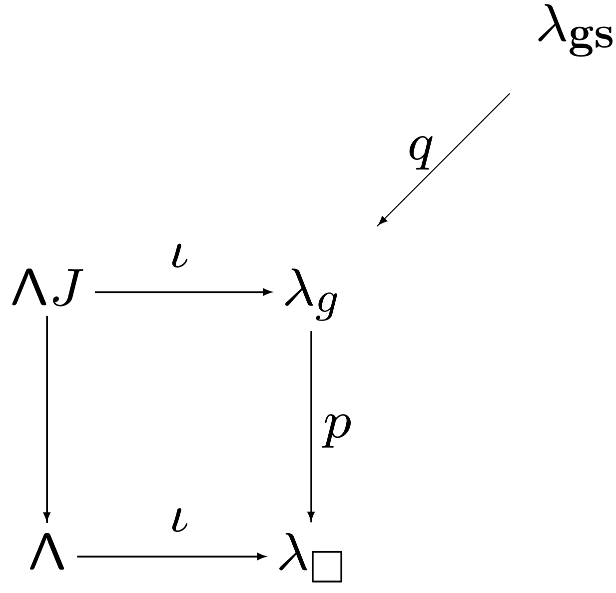
The λ_{gs}-calculus (4)

Permutation rules

$$
\begin{array}{llcl}
(p) & app(F,N,(x)P) & \rightarrow & [app(F,N)/x]P,\ P \neq x \\
(q) & FN & \rightarrow & hd(app(F,N))
\end{array}
$$

M is in q-nf iff every gs-application occurring in M is of the form $app(hd(M),N,(x)P)$ (morally a general application)

M is in pq-nf iff every gs occurring in M is of the form $app(hd(M),N) \equiv app(hd(M),N,(x)x)$



Isomorphism (1)

Idea:

$$t(u_1 :: ... :: u_m \cdot (x)v)$$

$$\sim$$

$$app(hd(M)N_1...N_{m-1}, N_m, (x)P)$$

as long as $t \sim M$, $u_i \sim N_i$ and $v \sim P$

Case $m > 1$:

1) Sequent calculus: right-associativity, head at the surface

2) Natural deduction: left-associativity, tail at the surface

3) To each occurrence of $u :: l$ corresponds one occurrence of FN

4) Inversion of associativity of applicative terms

Isomorphism (2)

Case $m = 1$

$$t(u \cdot (x)v)$$

$$\sim$$

$$app(hd(M), N, (x)P)$$

1) Translation between notational variants of generalised application, inducing a translation between two copies of ΛJ

2) Neutral associativity, both head and tail at the surface

3) ΛJ (and hence the Λ) are neutral w.r.t. the characterization of sequent calculus and natural deduction in terms of associativity

Isomorphism (3)

$$\lambda^{\mathbf{gm}} \underset{\Theta}{\overset{\Psi}{\rightleftarrows}} \lambda_{\mathbf{gs}}$$

$$\begin{array}{rcl} \Psi(app(F,N,(x)P)) & = & \Psi'(F, \Psi N \cdot (x)\Psi P) \\ \Psi'(FN,l) & = & \Psi'(F, \Psi N :: l) \\ \Psi'(hd(M),l) & = & (\Psi M)l \end{array}$$

$$\begin{array}{rcl} \Theta(tl) & = & \Theta'(hd(\Theta t), l) \\ \Theta'(F, u :: l) & = & \Theta'(F\Theta u, l) \\ \Theta'(F, u \cdot (x)v) & = & app(F, \Theta u, (x)\Theta v) \end{array}$$

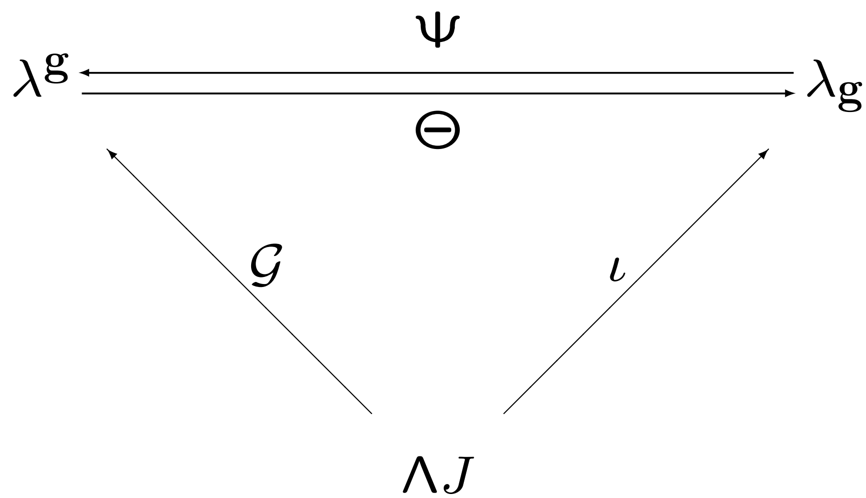
Mappings Ψ and Θ are sound, mutually inverse bijections. For each $R \in \{\beta 1, \beta 2, \pi, \mu, p, q\}$:

1) $M \to_R M'$ in $\lambda_{\mathbf{gs}}$ iff $\Psi M \to_R \Psi M'$ in $\lambda^{\mathbf{gm}}$.

2) $t \to_R t'$ in $\lambda^{\mathbf{gm}}$ iff $\Theta t \to_R \Theta t'$ in $\lambda_{\mathbf{gs}}$.

Corollary: $\lambda_{\mathbf{gs}}$ inherits the properties of $\lambda^{\mathbf{gm}}$

Isomorphism (4)

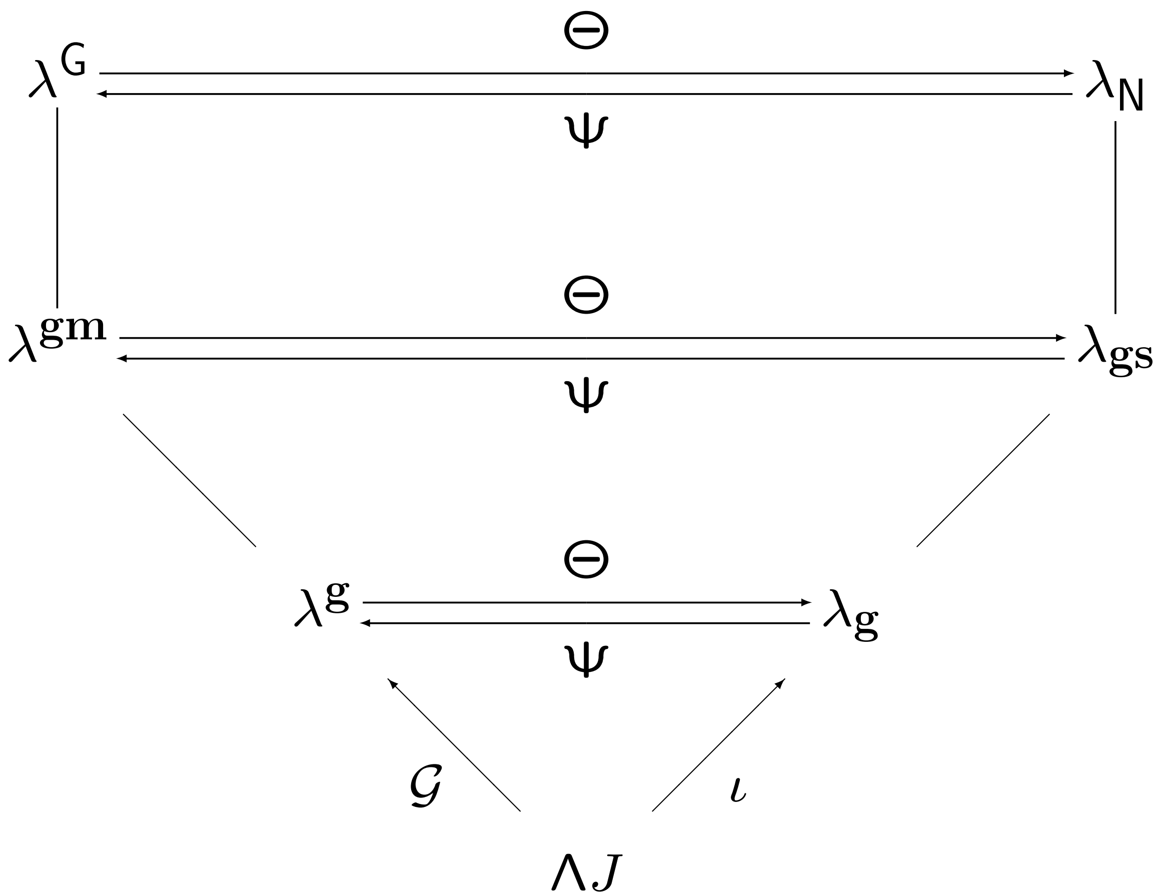


$$\Psi(app(hd(M), N, (x)P)) \;=\; (\Psi M)(\Psi N \cdot (x)\Psi P)$$

$$\Theta(t(u \cdot (x)v)) \;=\; app(\Theta(t), \Theta(u), (x)\Theta(v))$$

Mappings Ψ and Θ translate between two copies of ΛJ

Roadmap again



The λ^{G}-calculus (1)

Expressions

$$
\begin{array}{lrcl}
\text{(Terms)} & t, u, v & ::= & x \,|\, \lambda x.t \,|\, tk \\
\text{(Contexts)} & k & ::= & (x)v \,|\, u :: k
\end{array}
$$

Abbreviations $\quad [u] = u :: (z)z$

Typing rules

$$\frac{}{\Gamma, x : A \vdash x : A}\, Axiom$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \supset B}\, Right$$

$$\frac{\Gamma \vdash u : A \quad \Gamma; B \vdash k : C}{\Gamma; A \supset B \vdash u :: k : C}\, Left$$

$$\frac{\Gamma \vdash t : A \quad \Gamma; A \vdash k : B}{\Gamma \vdash tk : B}\, Cut$$

$$\frac{\Gamma, x : A \vdash v : B}{\Gamma; A \vdash (x)v : B}\, Selection$$

The λ^{G}-calculus (2)

Reduction rules

$$
\begin{array}{llcl}
(\sigma) & t((x)v) & \to & s(t,x,v) \\
(\beta) & (\lambda x.t)(u :: k) & \to & u((x)t)k \\
(\pi) & \Psi(E,(x)v)(u :: k) & \to & \Psi(E,(x)v(u :: k)) \\
(\mu) & (x)xk & \to & k,\ \text{if } x \notin k
\end{array}
$$

Permutation rules

$$
\begin{array}{llcl}
(p) & \Psi(EN,(x)v) & \to & s(\Psi(E,[\Psi N]),x,v),\ \text{if } v \neq x \\
(q) & \Psi(EN,u :: k) & \to & \Psi(E,[\Psi N])(u :: k)
\end{array}
$$

t is in $\boxed{pq\text{-nf}}$ iff every cut occurring in t is of the form $t[u] \equiv t(u :: (x)x)$ or $t((x)v)$. Hence, t is in pq-nf iff t is morally a $\lambda\mathrm{x}$-term

The λ_N-calculus (1)

Expressions

(Terms)	M, N, P	$::=$	$x \mid \lambda x.M \mid \{E/x\}P$
(FAP- -Expressions)	E	$::=$	$hd(M) \mid EN$

Abbreviations $ap(E) = \{E/z\}z$

Typing rules

$$\frac{}{\Gamma, x : A \vdash x : A}\,Assumption$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \supset B}\,Intro$$

$$\frac{\Gamma \rhd E : A \supset B \quad \Gamma \vdash N : A}{\Gamma \rhd EN : B}\,Elim$$

$$\frac{\Gamma \rhd E : A \quad \Gamma, x : A \vdash P : B}{\Gamma \vdash \{E/x\}P : B}\,Subst$$

$$\frac{\Gamma \vdash M : A}{\Gamma \rhd hd(M) : A}\,Coercion$$

The λ_N-calculus (2)

Reduction rules

$$
\begin{array}{lrcl}
(\sigma) & \{hd(M)/x\}P & \to & [M/x]P \\
(\beta) & hd(\lambda x.M)N & \to & hd(\{hd(N)/x\}M)
\end{array}
$$

$$
\begin{array}{ll}
(\pi) & \Theta(hd(\{E/x\}P), u :: k) \to \\
 & \qquad \to \{E/x\}\Theta(hd(P), u :: k) \\
(\mu) & \{E/x\}\Theta(hd(x), k) \to \Theta(E, k), \text{ if } x \notin k
\end{array}
$$

Permutation rules

$$
\begin{array}{lrcl}
(p) & \{EN/x\}P & \to & [ap(EN)/x]P, \text{ if } P \neq x \\
(q) & ENN' & \to & hd(ap(EN))N'
\end{array}
$$

A derivation $\mathcal{D}$ in λ_N is **$\beta\pi\sigma$-normal** iff every coercion formula occurring in $\mathcal{D}$ is an assumption and the main premiss of an elimination

M is in **pq-nf** iff every substitution occurring in M is of the form $ap(hd(M)M) \equiv \{hd(M)N/x\}x$ or $\{hd(M)/x\}P$. Hence, M is in pq-nf iff M is morally a $\lambda\mathrm{x}$-term

Decompositions of general elimination

$$\frac{\Gamma \vdash M : A \supset B \quad \Gamma \vdash N : A \quad \Gamma, x : B \vdash P : C}{\Gamma \vdash M(N, x.P) : C}$$

From $\Lambda J = \lambda^{\mathbf{g}} \subset \lambda^{\mathbf{gm}} \subset \lambda^{\mathsf{G}}$

$$\frac{\Gamma \vdash t : A \supset B \quad \dfrac{\Gamma \vdash u : A \quad \dfrac{\Gamma, x : B \vdash v : C}{\Gamma; B \vdash (x)v}(1)}{\Gamma; A \supset B \vdash u :: (x)v : C}(2)}{\Gamma \vdash t(u :: (x)v) : C}(3)$$

(1) Selection (2) Left (3) Cut

From $\Lambda J = \lambda_{\mathbf{g}} \subset \lambda_{\mathbf{gs}} \subset \lambda_{\mathsf{N}}$

$$\frac{\dfrac{\dfrac{\Gamma \vdash M : A \supset B}{\Gamma \rhd hd(M) : A \supset B}(1) \quad \Gamma \vdash N : A}{\Gamma \rhd hd(M)N : B}(2) \quad \Gamma, x : B \vdash C}{\Gamma \vdash \{hd(M)N/x\}P : C}(3)$$

(1) Coercion (2) Elim (3) Subst

Conclusions and future work

1) Natural deduction **isomorphic** to full sequent calculus

2) Both proof system presented as (meaningful) extensions of the simply typed λ-calculus

3) Difference reduced to **associativity** of applicative terms

4) Isomorphism: at the logical level

$$\begin{array}{rcl} \text{left-introduction} & \sim & \text{elimination} \\ \text{cut} & \sim & \text{primitive substitution} \end{array}$$

5) Isomorphism: at the λ-calculi level, **inversion** of associativity

6) Future work: **unification** of sequent calculus and natural deduction