Unity in structural proof theory and structural extensions of the λ-calculus

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Abstract

This talk is a contribution, in the context of intuitionistic implicational logic, to the solution of two closely related problems:

(1) Unity in structural proof theory. This is the problem of understanding the differences (and recognizing the similarities) between sequent calculus (a system based on left-introduction and cut) and natural deduction (a system based on elimination and substitution). The problem permanently attracted attention since the birth of structural proof theory [4] until the present day, and is about the following entanglement found in most comparisons: an elimination is seen as a combination of cut and left-introduction, and a left-introduction is seen as a combination of substitution and elimination.

(2) Structural extensions of the λ-calculus. This is the problem of extending the simply-typed λ-calculus so as to obtain a meaningful Curry-Howard correspondence with sequent calculus. The extension is structural because types are not touched, only the structure of terms is changed in order to achieve a correspondence with another formalism of structural proof theory. An explicit, early statement of this problem is found in [3].

In [5], Herbelin proved that there is a "λ-calculus structure" for a specific sequent calculus and suggested that the difference between sequent calculus and natural deduction can be reduced to the way applicative terms are organized, with the head at the surface of the term, in the former case. Problem: the way applicative terms are organized is certainly relevant, but the difference can be reduced to that *only if* natural deduction is extended [1, 2]. In [6] von Plato defined natural deduction with general elimination rules and showed that bijections preserving the structure of proofs could be obtained between the set of deductions and the set of sequent calculus derivations. Problem: an elimination is still seen as a combination of cut and left-introduction. In [1, 2] natural deduction was extended in yet another way, with a two-sorted syntax, so that

isomorphisms with fragments of Herbelin's sequent calculus were obtained, and that at both ends of the isomorphism were systems just differing in the way applicative terms are organized. Problem: in those fragments, restrictions are imposed on left-introductions and cuts.

In this talk we explain how to fully realize Herbelin's seminal suggestion. Specifically, we show how to define natural deduction so that: (i) an isomorphism is obtained with a sequent calculus that has a primitive rule of cut and where no constraints are imposed on left-introductions and cuts (here "isomorphism" means sound bijection between the sets of derivations that makes the cut-elimination relation and the normalisation relation isomorphic); (ii) the essence of the isomorphism is the replacement, in derivations, of left-introductions by eliminations (or vice versa); and (iii) the isomorphism can be described as an inversion of associativity.

Our system of natural deduction has primitive rules of *modus ponens* (= Gentzen's elimination rule) and substitution, plus an auxiliary sort of derivations to be used as main premiss of eliminations or "actual parameter" of substitutions. Normalisation forces these auxiliary derivations to start with an assumption. The system is also equipped with permutation rules, and a derivation corresponds to a normal natural deduction in the traditional sense iff it is irreducible w.r.t. both normalisation and permutation rules. Natural deduction is thus defined in an extended way that follows the suggestion and the methods of our previous work. Simultaneously, the system extends and refines von Plato's system because the general elimination rule is decomposed into a substitution and an ordinary elimination.

In addition, both sequent calculus and our system of natural deduction are presented as extensions λ^{G} and λ_{N}, respectively, of the simply-typed λ-calculus. This is possible because there are two parallel, meaningful ways of extending the λ-calculus, both elaborating the notion of multiple application, so that a Curry-Howard kind of correspondence with our proof-systems obtains. Under this presentation, the isomorphism (and the difference) between the two kinds of systems can be described as a mere inversion of the associativity of applicative terms, as desired.

The isomorphism $\lambda^{\mathsf{G}} \cong \lambda_{\mathsf{N}}$ holds, in two simpler versions, between fragments of λ^{G} and λ_{N}. In the simplest version, it holds between two copies (one as a fragment of λ^{G}, the other as a fragment of λ_{N}) of the λ-calculus with general applications. Because of this, we reach the paradoxal conclusion that von Plato's system is in the intersection between sequent calculus and natural deduction, if natural deduction is defined as we propose.

The intermediate version of the isomorphism holds between two intermediate extension of simply-typed λ-calculus. One extension is based on the idea of having a separate syntactic class of "applicative contexts", as in [5]. The other extension, instead, adds a separate syntactic class of "functions", *i.e.* expressions that only occur in the function position of applications. There is an appealing duality here: an applicative context specifies an immediate and linear use (to the expression that goes to the "hole" of the context); a "function" requires such kind of use.

References

[1] J. Espírito Santo, *Conservative extensions of the λ-calculus for the computational interpretation of sequent calculus*, *PhD thesis*, University of Edinburgh, 2002.

[2] J. Espírito Santo, *An isomorphism between a fragment of sequent calculus and an extension of natural deduction*, *in: M. Baaz and A. Voronkov (Eds.), Proc. of LPAR'02*, 2002, Springer-Verlag, Lecture Notes in Artificial Intelligence, vol. 2514, 354–366.

[3] J. Gallier,*Constructive Logics. Part I: A tutorial on proof systems and typed λ-calculi*, *Theoretical Computer Science*, vol. 110 (1993) 248–339.

[4] G. Gentzen, *Investigations into logical deduction*, *in: M. E. Szabo(ed.), The collected papers of Gerhard Gentzen*, North Holland, 1969.

[5] H. Herbelin, *A λ-calculus structure isomorphic to a Gentzen-style sequent calculus structure*, *in: L. Pacholski and J. Tiuryn (Eds.), Proceedings of CSL'94*, 1995, Springer-Verlag, Lecture Notes in Computer Science, vol. 933, 61–75.

[6] J. von Plato, *Natural deduction with general elimination rules*, *Annals of Mathematical Logic*, 40 (2001) 541–567.