Issues in a calculus of multiary sequent terms

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Plan

1. The system $\lambda\mathbf{Jm}$

2. Permutative conversions

3. Combining reduction and permutation

4. Redundancies, overlap between multiarity and generality

5. Refinements

$\lambda\mathbf{Jm}$: the generalised multiary λ-calculus

Expressions

$$
\begin{array}{rcl}
t, u, v & ::= & x \mid \lambda x.t \mid \underbrace{t(u, l, (x)v)}_{\text{gm-application}} \\
l & ::= & [] \mid u :: l
\end{array}
$$

Sequents $\qquad \Gamma \vdash t : A \qquad \Gamma ; B \vdash l : C$

Typing rules

$$
\frac{}{x : A, \Gamma \vdash x : A}\, Axiom \qquad \frac{x : A, \Gamma \vdash t : B}{\Gamma \vdash \lambda x.t : A \supset B}\, Right
$$

$$
\frac{\Gamma \vdash t : A \supset B \qquad \Gamma \vdash u : A \quad \Gamma ; B \vdash l : C \quad x : C, \Gamma \vdash v : D}{\Gamma \vdash t(u, l, (x)v) : D}\, gm - Elim
$$

$$
\frac{}{\Gamma ; C \vdash [] : C}\, Ax \qquad \frac{\Gamma \vdash u : A \quad \Gamma ; B \vdash l : C}{\Gamma ; A \supset B \vdash u :: l : C}\, Lft
$$

Remark: In $\Gamma ; B \vdash l : C$

1. B is "main and linear"
2. $B = B_1 \supset ... \supset B_k \supset C$, for some $k \geq 0$.

Reduction rules

$$
\begin{array}{rcl}
(\lambda x.t)(u,[],(y)v) & \rightarrow_{\beta_1} & \mathbf{s}(\mathbf{s}(u,x,t),y,v) \\
(\lambda x.t)(u,v::l,(y)v) & \rightarrow_{\beta_2} & \mathbf{s}(u,x,t)(v,l,(y)v) \\
t(u,l,(x)v)(u',l',(y)v') & \rightarrow_{\pi} & t(u,l,(x)v(u',l',(y)v'))
\end{array}
$$

(s stands for *gm-substitution*)

$\beta = \beta_1 \cup \beta_2$

$\beta\pi$-normal forms:

$$
\begin{array}{rcl}
t,u,v & ::= & x \mid \lambda x.t \mid x(u,l,(y)v) \\
l & ::= & u::l \mid []
\end{array}
$$

Other reduction rules

$$
\begin{array}{rcl}
t(u,l,(x)x(u',l',(y)v')) & \rightarrow_{\mu} & t(u,\mathbf{append}(l,u'::l'),(y)v') \\
t(u,l,(x)x)(u',l',(y)v') & \rightarrow_{h} & t(u,\mathbf{append}(l,u'::l'),(y)v')
\end{array}
$$

Proviso for μ: $x \notin u',l',v'$
(i.e., $v = x(u',l',(y)v')$ introduces x in a linear fashion)

Remark: $h \subseteq \pi;\mu$

Results:

1. Each combination of β, π and μ is confluent.

2. $\rightarrow_{\beta\pi\mu}$ is SN for typable terms.

Elimination rules for subsystems of $\lambda\mathbf{Jm}$

$$\frac{\Gamma \vdash t : A \supset B \qquad \Gamma \vdash u : A \qquad \overbrace{\Gamma ; B \vdash l : C}^{(2)} \qquad \overbrace{x : C, \Gamma \vdash v : D}^{(1)}}{\Gamma \vdash t(u, l, (x)v) : D} \; gm - Elim$$

Subsystem $\lambda\mathbf{J}$

(2) must be an instance of Ax, $l = []$ and $B = C$. $t(u{\cdot}(x)v)$ abbreviates $t(u, [], (x)v)$

$$\frac{\Gamma \vdash t : A \supset B \qquad \Gamma \vdash u : A \qquad x : B, \Gamma \vdash v : D}{\Gamma \vdash t(u{\cdot}(x)v) : D} \; g - Elim$$

Subsystem $\lambda\mathbf{m}$ ($=\lambda\mathcal{P}\mathbf{h}$)

(1) must be an instance of $Axiom$, $v = x$ and $C = D$. $t(u{\cdot}l)$ abbreviates $t(u, l, (x)x)$

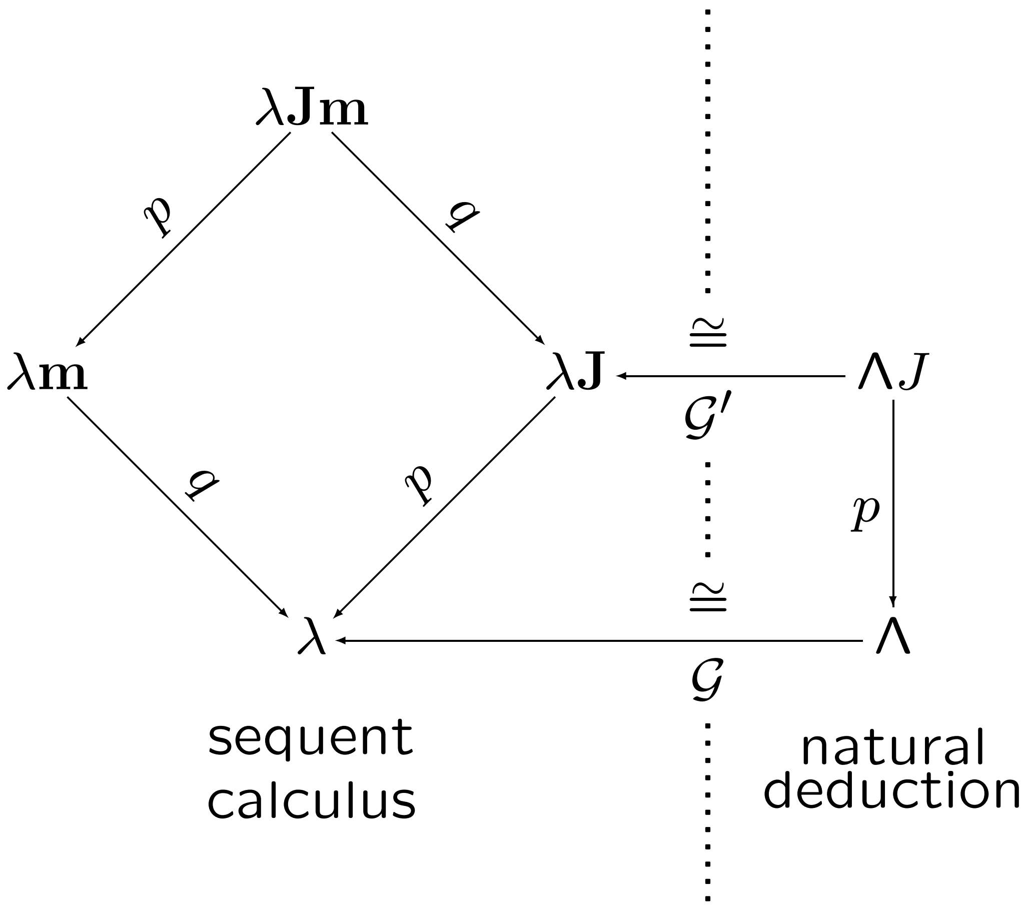
$$\frac{\Gamma \vdash t : A \supset B \qquad \Gamma \vdash u : A \qquad \Gamma ; B \vdash l : C}{\Gamma \vdash t(u{\cdot}l) : C} \; m - Elim$$

Subsystem λ ($=\lambda\mathcal{G}$)

Both restrictions defining $\lambda\mathbf{J}$ and $\lambda\mathbf{m}$ apply.
$t[u]$ abbreviates $t(u, [], (x)x)$

$$\frac{\Gamma \vdash t : A \supset B \qquad \Gamma \vdash u : A}{\Gamma \vdash t[u] : B} \; Elim$$

$\lambda\mathbf{Jm}$, subsystems and natural fragments



q	$=$	express multiary application with iterated application
p	$=$	execute explicit substitution
$\mathcal{G}$	$=$	Gentzen's mapping
$\mathcal{G}'$	$=$	Extension of $\mathcal{G}$ to ΛJ

The 2 features added to λ-calculus

$$
\begin{array}{rcll}
t, u, v & ::= & x \mid \lambda x.t \mid t(u, l, (x)x) & \\
& \mid & t(u, l, (x)v) \quad \& x \neq v & (1) \\
l & ::= & [] & \\
& \mid & u :: l & (2)
\end{array}
$$

	added feature	abbreviation	map/perm.
(1)	generality	$\mathbf{J}$	p
(2)	multiarity	$\mathbf{m}$	q

Permutation rules

Rules $(p_1), (p_2), (p_3)$:

(p_1) $t(u, l, (x)y) \to y$ if $x \neq y$

(p_2) $t(u, l, (x)\lambda y.v) \to \lambda y.t(u, l, (x)v)$

(p_3) $t_1(u_1, l_1, (x)t_2(u_2, l_2, (y)v)) \to$
$t_1(u_1, l_1, (x)t_2)(t_1(u_1, l_1, (x)u_2), t_1(u_1, l_1, (x)l_2), (y)v)$
if $x \notin v$

$p = p_1 \cup p_2 \cup p_3$

Rule (q):

$$(q) \quad t(u, v :: l, (x)v') \quad \to \quad t[u](v, l, (x)v')$$

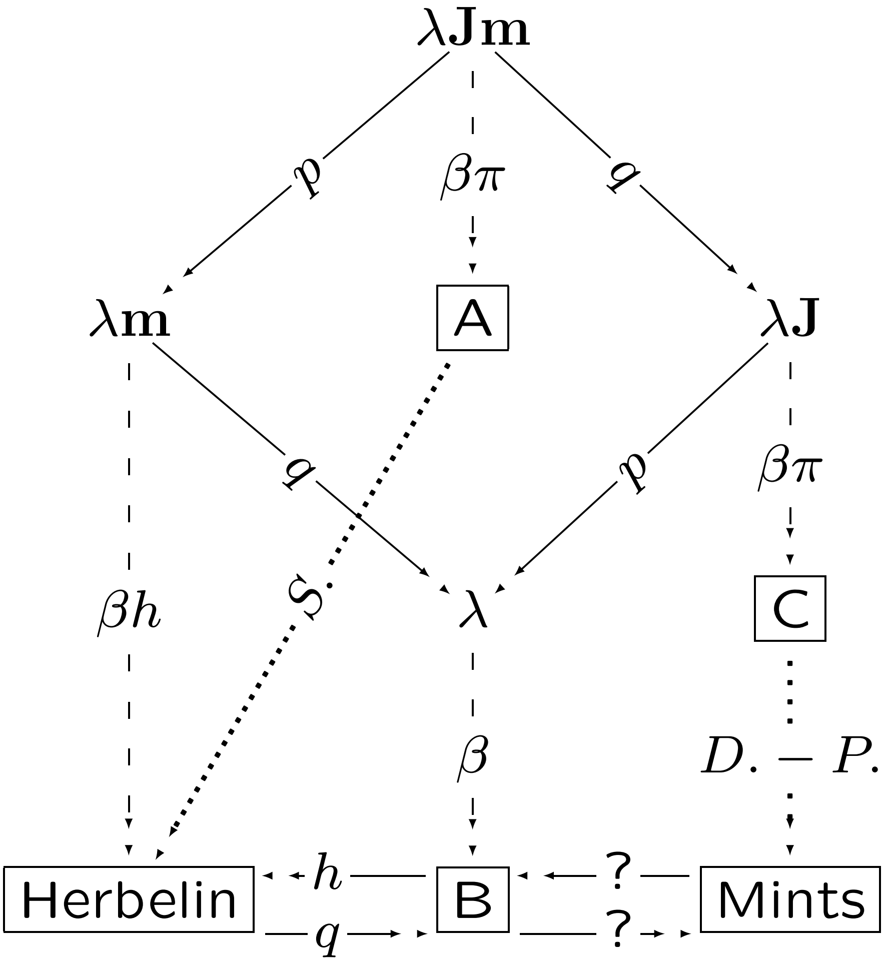
Results:

1. The rewriting system induced by p (resp. q, pq) is confluent and SN.

2. The p (resp. q, pq) normal forms are the set of terms of $\lambda\mathbf{m}$ (resp. $\lambda\mathbf{J}$, λ).

3. Hence p eliminate generality and q eliminates multiarity.

Combining reduction and permutation



A	cut-free multiary sequent terms (S.)
C	cut-free sequent terms (≅ von Plato's fully-normal natural deductions)
Herbelin	"π-normal" multiary sequent terms (S.) (= Herbelin's cut-free terms)
Mints	Mint's "normal" cut-free sequent derivations
B	β-normal λ-terms

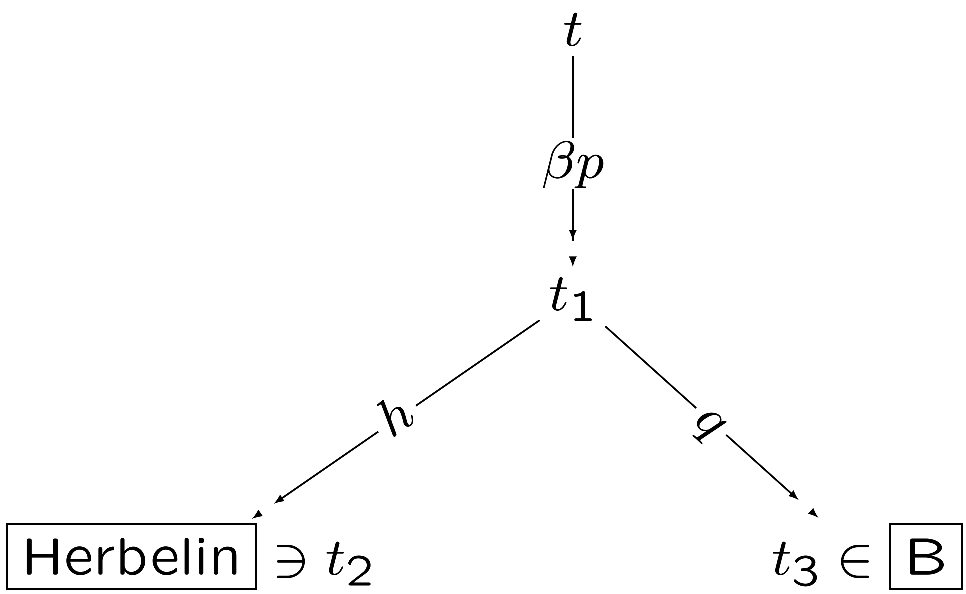
Combining reduction and permutation (2)

1. $p \# \pi$ and $h \# q$

2. $t \in$ Herbelin iff t is βph-nf
   $t \in$ B iff t is βpq-nf

3. Any combination of β, p and q (resp. h) is confluent.

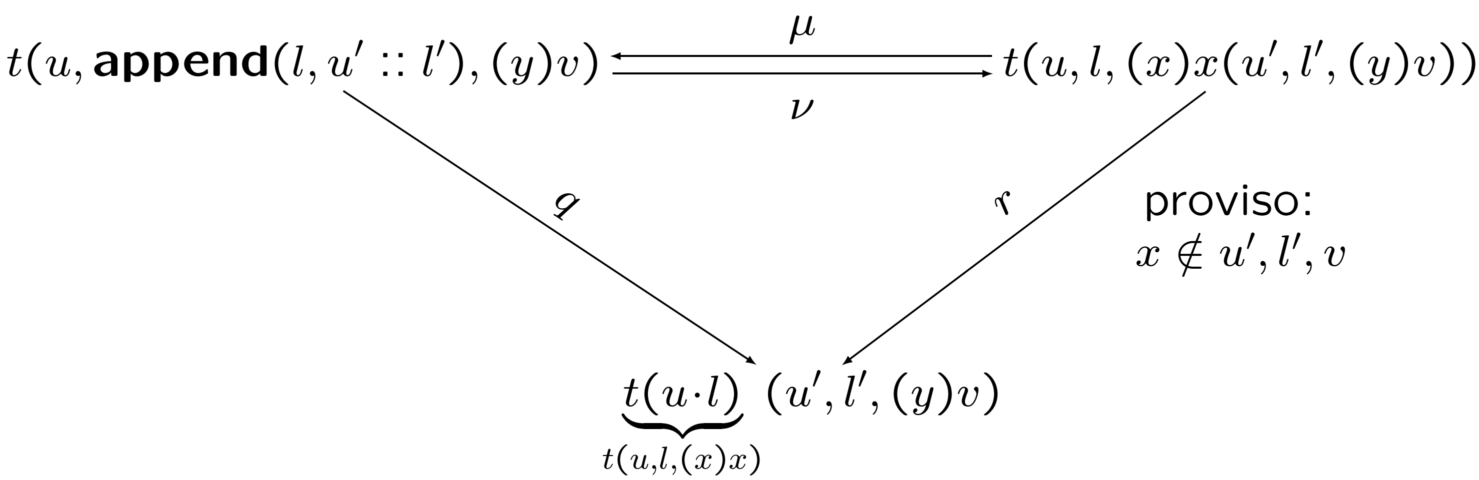
4. Reduction to βpq-nf and βph-nf can always be split into **two stages**



5. There is a typed $t \in \lambda\mathbf{J}$ s. t. t is not βp-SN. Hence $\beta \# p$.

A systematic analysis of overlaps in $\lambda\mathbf{Jm}$

Three ways of expressing multiple application: (1) multiary application. (2) **normal** generality. (3) iterated application.



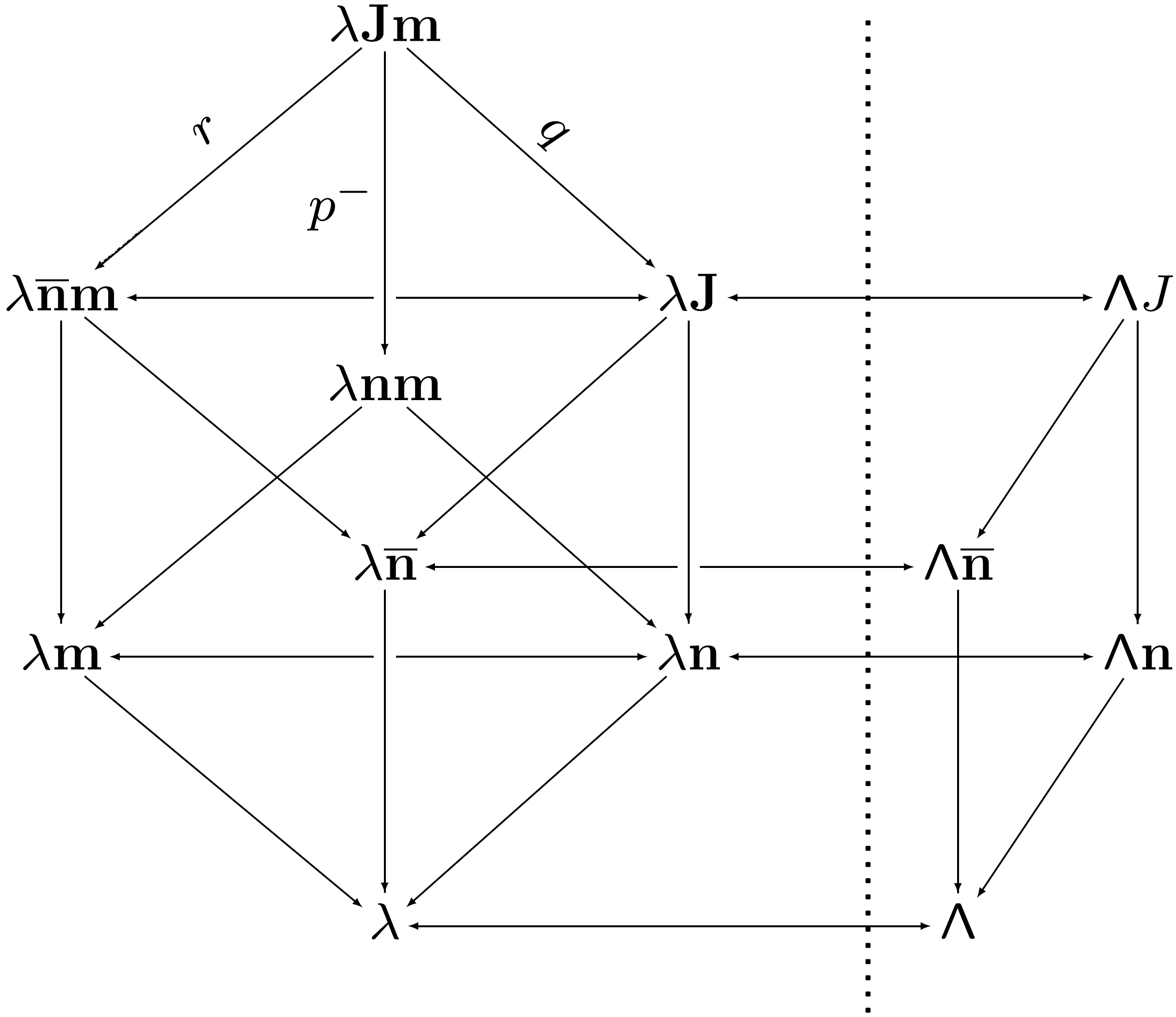
- ν: express multiarity (=non-empty lists) with generalised application
- p is split into
  1. r: eliminates normal generality
  2. p^-: eliminates non-normal generality
- q is slightly extended

The 3 features added to λ-calculus

$$
\begin{array}{rcll}
t, u, v & ::= & x \mid \lambda x.t \mid t(u, l, (x)x) & \\
& \mid & t(u, l, (x)v) \quad \& mla(x, v) & (1) \\
& \mid & t(u, l, (x)v) \quad \& \neg mla(x, v) \& x \neq v & (2) \\
l & ::= & [] & \\
& \mid & u :: l & (3)
\end{array}
$$

	feature	abbrev.	map/perm.
(1)	normal generality	$\mathbf{n}$	r
(2)	non-normal generality	$\overline{\mathbf{n}}$	p^-
(3)	multiarity	$\mathbf{m}$	q

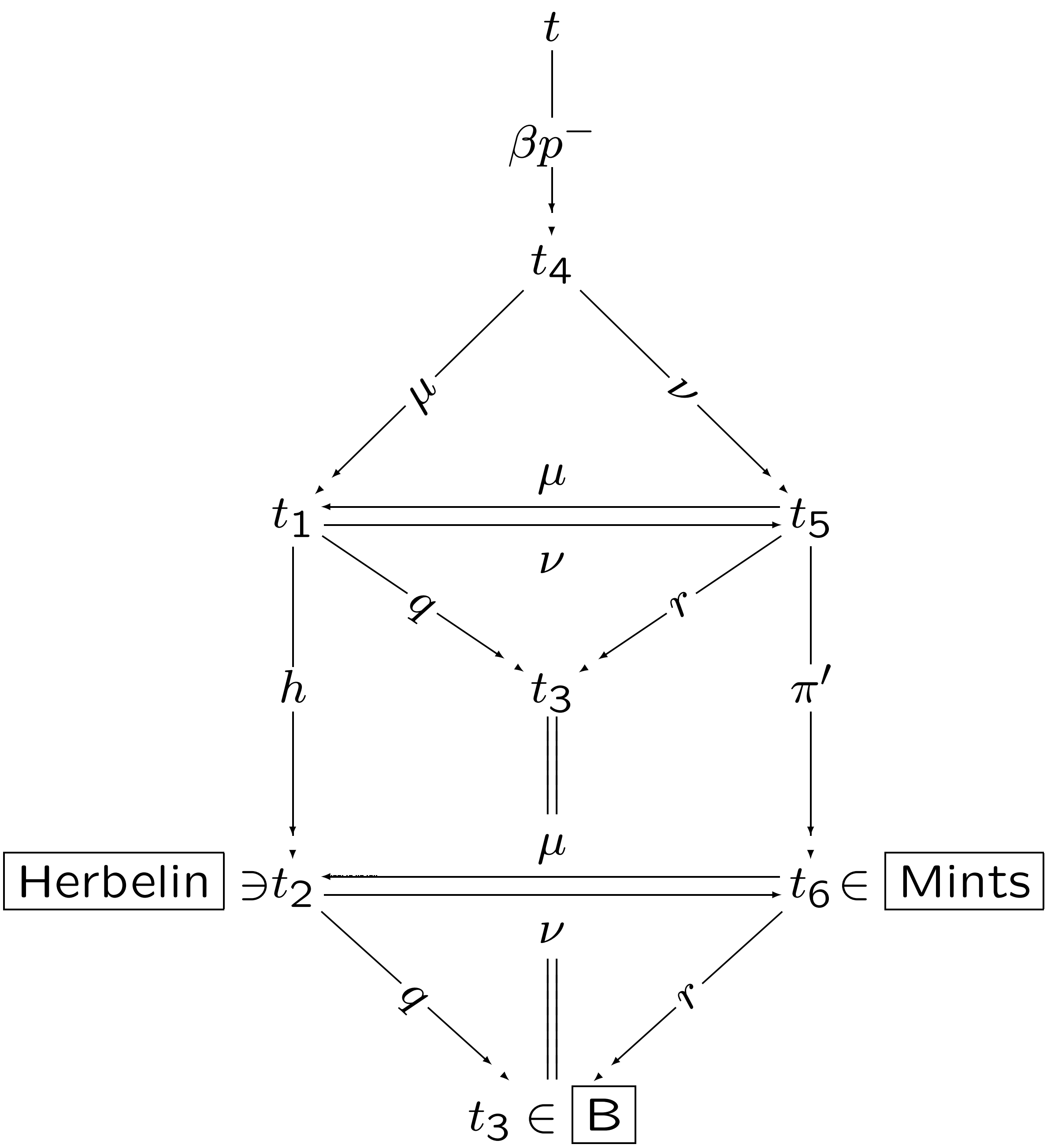
Refinements (1)
$\lambda\mathbf{Jm}$, subsystems and natural fragments



Nomenclature: each system is named as λ [1] [2]:

- [1] is either empty or $\mathbf{n}$ or $\overline{\mathbf{n}}$ or $\mathbf{J}(=\mathbf{n}\overline{\mathbf{n}})$
- [2] is either empty or $\mathbf{m}$

Refinements (2)
Combining reduction and permutation



Current goals

1. Goal: to prove the conjectured relationships between the 3 features, subsystems of $\lambda\mathbf{Jm}$ and permutations

2. Goal: to prove the conjectured results about the combination of reduction and permutation. In particular, to prove termination of β together with weakened form of p (p^{-}?)