Simple proofs of strong cut elimination

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Goal

1. logic: intuitionistic implication $\supset$

2. $SCE = SN + \epsilon$

3. More precisely:

   $$\text{SN for } \lambda^{\mathsf{G}} \quad = \quad (\text{properties of } \lambda) + \epsilon$$

   where

   $$\begin{array}{rcl} \lambda^{\mathsf{G}} & = & \text{neat } \lambda\text{-calculus for sequent calculus} \\ \epsilon & = & \text{small combinatorial overhead} \end{array}$$

4. Plan:

   (a) indicate properties of λ

   (b) advertise λ^{G}

   (c) do ϵ

Properties of λ

1. Strong normalisation

   M typable $\Rightarrow M \in SN_\beta$.

2. Fundamental lemma of perpetuality

$$\left.\begin{array}{r}(\lambda x.M)N \notin SN_\beta \\ x \notin FV(M) \Rightarrow N \in SN_\beta\end{array}\right\} \Rightarrow [N/x]M \notin SN_\beta$$

Stronger version of 2.

$$x \in FV(M) \;\Rightarrow\; ||(\lambda x.M)N||_\beta \leq ||[N/x]M||_\beta + 1$$

$$x \notin FV(M) \;\Rightarrow\; ||(\lambda x.M)N||_\beta \leq ||M||_\beta + ||N||_\beta + 1$$

Sequent calculus/λ^{G}-calculus (1)

Expressions

$$
\begin{array}{lrcl}
\text{(Terms)} & t,u,v & ::= & x \,|\, \lambda x.t \,|\, tk \\
\text{(Contexts)} & k & ::= & (x)v \,|\, u :: k
\end{array}
$$

Sequents $\quad \Gamma \vdash t : A \quad$ and $\quad \Gamma; A \vdash k : B$

Typing / Inference rules

$$
\frac{}{\Gamma, x : A \vdash x : A}\,Axiom \qquad \frac{\Gamma, x : A \vdash v : B}{\Gamma; A \vdash (x)v : B}\,Selection
$$

$$
\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \supset B}\,Right
$$

$$
\frac{\Gamma \vdash u : A \qquad \Gamma; B \vdash k : C}{\Gamma; A \supset B \vdash u :: k : C}\,Left
$$

$$
\frac{\Gamma \vdash t : A \qquad \Gamma; A \vdash k : B}{\Gamma \vdash tk : B}\,Cut
$$

Sequent calculus/λ^{G}-calculus (2)

<u>Cuts:</u> $tk = t(u_1 :: ... :: (u_m :: (x)v))$, $m \geq 0$

$m = 1$	$t(u :: (x)v)$	generalized application
$m = 0$	$t(x)v$	"explicit" substitution

N.B. $t(x)v$ is written $\langle t/x \rangle v$

<u>Reduction rules</u>

$$\begin{array}{lrcl} (\sigma) & \langle t/x \rangle v & \rightarrow & subst(t, x, v) \\ (\pi) & (tk)k' & \rightarrow & t\, append(k, k') \\ (\beta) & (\lambda x.t)(u :: k) & \rightarrow & \langle u/x \rangle (tk) \end{array}$$

<u>Logical reading (cut elimination)</u>

- (σ) right permutation
- (π) left permutation
- (β) key step

Doing ϵ (1)

First step: reduce SN for λ^{G} to SN for a version of $\lambda\mathtt{x}$

1. Recall the terms of $\lambda\mathtt{x}$:

   $$M, N, P ::= x \mid \lambda x.M \mid MN \mid \langle N/x \rangle M$$

2. Equip this set with

   $$\begin{array}{rrcl} (\beta) & (\lambda x.M)N & \to & \langle N/x\rangle M \\ (\sigma) & \langle N/x\rangle M & \to & [N/x]M \\ (\pi_1) & (\langle P/x\rangle M)N & \to & \langle P/x\rangle (MN) \\ (\pi_2) & \langle\langle P/y\rangle N/x\rangle M & \to & \langle P/y\rangle\langle N/x\rangle M \end{array}$$

   and put $\pi = \pi_1 \cup \pi_2$

3. Define $Q : \lambda^{\mathsf{G}} \to \lambda\mathtt{x}$ as follows:

   $$t(u_1 :: ... :: u_m(x)v) \mapsto \langle MN_1...N_m/x\rangle P$$

4. SN for $\lambda\mathtt{x}$ $\Rightarrow$ SN for λ^{G}

Doing ϵ (2)

Second step: enrich λ

1. Equip the set of λ-terms with

$$\begin{array}{lrcl} (\pi_1) & (\lambda x.M)NP & \to & (\lambda x.MP)N \\ (\pi_2) & M((\lambda y.P)N) & \to & (\lambda y.MP)N \end{array}$$

   and put $\pi = \pi_1 \cup \pi_2$

2. In λ, $\to_\pi$ terminates

3. In λ, $\to_\pi$ does not increase $||\_||_\beta$

   (fund. lemma perpetuality used here)

Doing ϵ (3)

Third step: conclude

1. Define $(\_)^\bullet : \lambda\mathtt{x} \to \lambda$ by

   $((\langle N/x \rangle M)^\bullet = (\lambda x.M^\bullet)N^\bullet$

2. In $\lambda\mathtt{x}$, $\to_{\beta\pi}$ terminates

3. For all $M \in \lambda\mathtt{x}$, $M^\bullet \in SN_\beta \Rightarrow M \in SN_{\beta\pi\sigma}$

4. Corollary (SN for $\lambda\mathtt{x}$):

   For all $M \in \lambda\mathtt{x}$, M typable $\Rightarrow M \in SN_{\beta\pi\sigma}$

   (SN for λ used here)

5. Theorem (SN for λ^{G}):

   For all $t \in \lambda^{\mathsf{G}}$, t typable $\Rightarrow t \in SN_{\beta\pi\sigma}$

Conclusions

1. Consequences:

   (a) SN for variants of λ with generalised application

   (b) SN for LJ equipped with various "protocols"

2. Simple proof:

   (a) reuse a previous result instead of repeating a previous proof

   (b) substitution is suspended, but not explicitly executed

   (c) simple $\Rightarrow$ stronger