NORMALIZATION BY EVALUATION FOR $\lambda^{\rightarrow,2}$

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Which is the simplest typed λ-calculus without uninterpreted type variables? We suggest that the answer should be $\lambda^{\rightarrow,2}$: simply typed λ-calculus with a type constant 2 for the type of booleans. The $\beta\eta$ equational theory introduces some interesting equalities. E.g., let

$$once = \lambda f : 2 \rightarrow 2.\ \lambda x : 2.\ f\ x$$
$$thrice = \lambda f : 2 \rightarrow 2.\ \lambda x : 2.\ f\ (f\ (f\ x))$$

It turns out (perhaps somewhat surprisingly) that $once =_{\beta\eta} thrice$, and the derivation of this equality is not entirely trivial. But it is easy to see that $once$ and $thrice$ have the same denotation in the standard set-theoretic model of $\lambda^{\rightarrow,2}$: there are only four functions f in $2 \rightarrow 2$ (constant truth, identity, negation and constant falsity) and for each of them we have $f = f^3$.

May we use set-theoretic reasoning to prove equalities up to $\beta\eta$? We show that, for $\lambda^{\rightarrow,2}$, the answer is yes, since for $\lambda^{\rightarrow,2}$ evaluation of typed closed terms into the set-theoretic model is invertible, i.e., for every type σ, there is a function $\mathsf{quote}^\sigma : [\![\sigma]\!] \rightarrow \mathsf{Tm}\ \sigma$ such that for any $t \in \mathsf{Tm}\ \sigma$, $t =_{\beta\eta} \mathsf{quote}^\sigma[\![t]\!]$. Consequently, the set-theoretic model is universal: for any type σ and any $t, t' \in \mathsf{Tm}\ \sigma$, $t = \beta\eta t'$ iff $[\![t]\!] = [\![t']\!]$.

The argument is constructive, so we could use, e.g., Martin-Löf's type theory as the metalanguage and obtain a reduction-free implementation of normalization: $\mathsf{nf}^\sigma t = \mathsf{quote}^\sigma[\![t]\!]$. We use the functional language Haskell as a poor man's type theory and obtain a Haskell program to normalize terms. In our implementation of quotation, we think of higher-order boolean functions concretely as decision trees, which leads to reasonably simple normal forms. Alternatively, one could use truth tables or binary decision diagrams.

Normalization by evaluation and algorithms for deciding $\beta\eta$-equality have been studied by several authors in the literature, but our construction is novel in that it uses the simplest possible model of first-order simply typed λ-calculi—the set-theoretic model—(instead of a model invented specifically to do NBE) and that the construction is very simple. It is also extendible to simply typed λ-calculus with finite products and coproducts. But it remains open to see whether the approach is also combinable with the standard techniques for NBE for systems with uninterpreted type variables.

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