Interpreting Heegaard Floer homology

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Invariants?

Geometry & Topology:

(1) Construct spaces.

(2) Study structures on those spaces.

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Geometry & Topology:

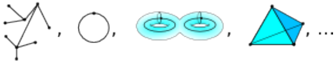
(1) Construct spaces.

$(2-\varepsilon)$ Compute invariants for those spaces.

(2) Study structures on those spaces.

Invariants

Eg: **Cell Complexes & Euler Characteristic (χ)**

(1) Construct spaces. $M \in \{ \cdot,$  $\}$

$(2-\varepsilon)$ Compute invariants. $\chi(M) := \sum_i (-1)^i c_i(M)$.

(2) Study structures:
$\chi(M)$ counts **holes in** M.

Invariants

Invariant: Choose extra data, ask a question, quotient.

Finite Question (e.g. Cell decomposition, Morse singularities)
$\rightsquigarrow$ **Integer Answer** ($\chi(M)$)
$\rightsquigarrow$ **Vector Space Answer** ($H_*(M)$).

Linear Question (e.g. Hodge theory)
$\rightsquigarrow$ **Vector Space Answer** ($H^*(M)$).

Nonlinear Question (e.g. Complex structures)
$\rightsquigarrow$ **Moduli space Answer** $\mathcal{M}$ (Extract invariants from $\mathcal{M}$).

Moduli Spaces from Physics

Physics (Mechanics):

1. Hamiltonian mechanics on a spacetime W
   $\rightsquigarrow$ **symplectic** configuration space $\mathcal{C}$.

2. Localizing to critical points in $\mathcal{C}$ of action functional
   $\rightsquigarrow$ "equations of motion."

3. $\{$Critical points$\}$ corrected by **flows** between critical points
   $\rightsquigarrow$ moduli space $\mathcal{M}$ of solutions.

Gauge Theory

Co-opting moduli spaces from physics.
(Anti-self-dual connections on a principal $SU(2)$-bundle over W?)

$\rightsquigarrow$ Donaldson Theory

$\rightsquigarrow$ Instanton Floer Theory

$\rightsquigarrow$ Seiberg-Witten Theory

$\rightsquigarrow$ Monopole Floer Theory

$\rightsquigarrow$ **Heegaard Floer Theory** (HF).

Gauge Theory $\rightsquigarrow$ Floer theory?

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But often, physics uses a spacetime W with **boundary** $\partial W \neq \emptyset$.
$\rightsquigarrow$ Dimensional reduction.

Gauge Theory $\rightsquigarrow$ Floer theory?

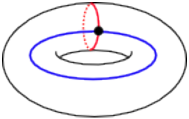
Dimensional Reduction / TQFT (Atiyah, Floer, Gromov, Witten):
Spacetime Cobordism $W : M_1 \to M_2, \quad \partial W = M_1 \amalg -M_2$.

- Imposing **Dirichlet** boundary conditions on configurations
  $\rightsquigarrow$ **Lagrangian** submanifolds $L_i \subset \mathcal{C}$ of configuration space:
  $\omega|_{L_i} = 0$, $\dim L_i = \frac{1}{2} \dim \mathcal{C}$.
- Localizing to critical points of the functional
  $\rightsquigarrow$ **Intersections of Lagrangians** $L_1 \cap L_2$.
- **Flows** between critical points are given by
  "**Whitney discs**" — psuedo-holo discs bounded by Lags.

Heegaard Floer homology: $HF(M)$

Peter Ozsváth, Zoltan Szabó (2000). M closed oriented 3-mfd.

$M^3 = U_\alpha \cup_\Sigma U_\beta$.
Heegaard diag. $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$.



$HF(M) = HF_{\mathsf{Lag}}(\mathbb{T}_\alpha, \mathbb{T}_\beta)$, $\mathbb{T}_\alpha, \mathbb{T}_\beta \subset \mathsf{Sym}^{g(\Sigma)}(\Sigma)$.
—$CF(M)$ generated by points $\mathbb{T}_\alpha \cap \mathbb{T}_\beta \subset \mathsf{Sym}^{g(\Sigma)}(\Sigma)$.
—Differentials: psuedoholomorphic Whitney disks.

$HF(M)$ is mysterious

when M is a *rational homology sphere:* $H^1(M) = 0$.

L-spaces

Definition

An *L-space* is a 3-manifold M with $HF_{\mathrm{red}}(M) = 0$.

Basic properties of L-spaces:

- M closed, oriented, $b_1(M) = 0$. (M $\mathbb{Q}$HS.)
- $\widehat{HF}(M) = \bigoplus_{\mathfrak{s}\in\mathrm{Spin}^c(M)} \widehat{HF}(M,\mathfrak{s}) = \bigoplus_{h\in H_1(M)} \mathbb{Z} = \mathbb{Z}^{|H_1(M)|}$.

Example L-spaces:

- **L**ens spaces.
- Branched double covers of alternating knots.

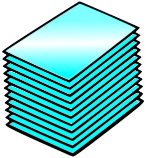
Boyer-Gordon-Watson + Juhász + Némethi

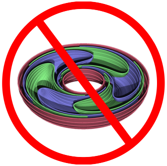
Conjecture: M is **not** an L-space $\iff$ ...

$\pi_1(M)$ has a left order (LO).

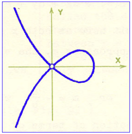
$g_1 > g_2 \iff hg_1 > hg_2$

M admits a co-oriented taut foliation (CTF).





M links a nonrational singularity.



The conjecture also demands M irreducible, because....

Suppose $M = M_1 \# M_2$.

- $\widehat{HF}(M) = \widehat{HF}(M_1) \otimes \widehat{HF}(M_2)$.
- $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2)$.
- No taut foliations on $M_1 \# M_2$.

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- $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2)$.
  LO $\iff$ LO LO
- No taut foliations on $M_1 \# M_2$.
- $\xi = \xi_1 \# \xi_2$.
  tight $\iff$ tight tight

Heegaard Floer homology: HF(M)

HF well behaved under cobordism.

Heegaard Floer homology: HF(M)

HF well behaved under **elementary** cobordism.

$\rightsquigarrow$ Dehn Surgery!

L-space intervals (j/w J. Rasmussen)

Definition

If Y is a compact oriented three-manifold with torus boundary, then the *L-space interval* of Y is the space $\mathcal{L}(Y) \subset \mathbb{P}(H_1(\partial Y))$ of L-space Dehn filling slopes of Y.

Definition

$\mathcal{NL}(Y) := \mathbb{P}(H_1(\partial Y)) \setminus \mathcal{L}(Y)$ of is the set of non-L-space Dehn filling slopes of Y, *i.e.*, slopes for Dehn fillings with nontrivial HF.

Theorem (J. Rasmussen, –R)

If $\mathcal{L}(Y)$ contains more than one element, then $\mathcal{L}(Y)$ is an interval, and its endpoints can be computed from the Turaev torsion of Y.

L-space Classification of Seifert Fibered Spaces

History:
Eisenbud-Hirsch-Neumann, Jankins-Neumann,
Naimi, Calegari-Walker, Eliashberg-Thurston, Ozsváth-Szabó,
Lisca-Matić, Lisca-Stipcisz.

Theorem (J. Rasmussen, –R)

$M = M(\frac{\beta_0}{\alpha_0}; \frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_n}{\alpha_n})$ Seifert fibered over S^2. Write $y_i := \frac{\beta_i}{\alpha_i}$. Then M is not an L-space $\iff$ $0 \in (y_-, y_+) \subset \mathbb{Q} \cup \{\infty\}$, where

$$y_- := \max_{k>0} -\frac{1}{k}\left(1 + \sum \left\lfloor \frac{\beta_i}{\alpha_i} k \right\rfloor\right),$$

$$y_+ := \min_{k>0} -\frac{1}{k}\left(-1 + \sum \left\lceil \frac{\beta_i}{\alpha_i} k \right\rceil\right).$$

Matches (complement of) **Jankins-Neumann-Naimi** classification of Seifert fibered spaces admitting **transverse foliations.**

Gluing Theorem

Theorem (J. Rasmussen, –R; Hanselman, Watson)

Suppose $\varphi : \partial Y_1 \to -\partial Y_2$ *is a gluing map for* Y_i *boundary incompressible and* $\mathcal{L}^\circ(Y_i) \neq \emptyset$. *Then*
$Y_1 \cup_\varphi Y_2$ *not L-space* $\iff \varphi_*^{\mathbb{P}}(\overline{\mathcal{NL}}(Y_1)) \cap \overline{\mathcal{NL}}(Y_2) \neq \emptyset$.

Proof:

- Use mapping cone for zero surgery to obtain formula computing $\mathcal{L}(Y)$ from Turaev torsion $\tau(Y)$.
- Express $Y_1 \cup Y_2$ as Dehn filling of a Floer simple manifold.
- Compare L-space interval boundaries (Euc. alg. argument).

L-space Classification of Graph manifolds

Theorem (–R)

A graph manifold is not an L-space (or equivalently, admits a CTF or an LO fundamental group) if and only if either its L-space surgery interval is empty or 0 *is contained in the appropriate interval with endpoints y_+, y_-, where*

$$y_- := \max_{k>0} -\frac{1}{k}\left(1 + \sum_{i=0}^{n} \lfloor y_i k \rfloor + \sum_{i=0}^{n_{\mathrm{G}}} (\lceil y^{\mathrm{G}}_{i+} k \rceil - 1)\right)$$

$$y_+ := \min_{k>0} -\frac{1}{k}\left(-1 + \sum_{i=0}^{n} \lceil y_i k \rceil + \sum_{i=0}^{n_{\mathrm{G}}} (\lfloor y^{\mathrm{G}}_{i-} k \rfloor + 1)\right).$$

Boyer-Gordon-Watson + Juhász + Némethi

Theorem (Boyer-Clay)

A graph manifold M admits CTFs $\iff$ $\pi_1(M)$ *admits LOs.*

Theorem (Hanselman, J. Rasmussen, –R, Watson)

A graph manifold M is **not** *an L-space* $\iff$ *M admits CTFs.*
$\rightsquigarrow$ *The BGW+J conjecture holds for all graph manifolds.*

Theorem (Némethi)

If $(X, \circ)$ *is the germ of a normal complex surface singularity, then* $(X, \circ)$ *is rational* $\iff$ $M := Link(X, \circ)$ *is an L-space.*

The L-space Conjecture

M irreducible is **not** an L-space $\iff$ ...

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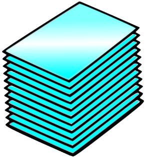

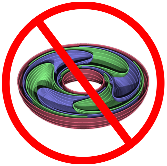

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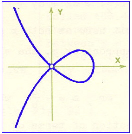

Thanks!