New metrics on tangent bundles

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Tangent bundle and Sasaki metric

$\pi : TM \longrightarrow M$ the tangent vector bundle of a manifold M; T_M is the total space.
Then $V = \ker \mathrm{d}\pi \subset T(T_M)$, vertical subvector bundle, giving tangent bundle of the fibres. We find

$$V \simeq \pi^\star TM.$$

We have a **tautological vector field**; ξ is vertical, defined over T_M by

$$\xi_u = u \ \in V, \quad \forall u \in T_M.$$

(M,g) Riemannian of dim m. Then canonical splitting:

$$T(T_M) = H^{\nabla} \oplus V \simeq \pi^* TM \oplus \pi^{\star} TM.$$

$\mathrm{d}\pi_{|H^\nabla}$ vector bundle isometry $H^{\nabla} \simeq \pi^* TM$ over the manifold T_M.
T_M is an oriented Riemannian $2m$-manifold with the **Sasaki metric**

$$g^S = \pi^* g \oplus \pi^{\star} g.$$

$\exists$ a mirror endomorphism, indeed a tensor:

$$B : T\,T_M \longrightarrow T\,T_M \qquad \begin{cases} Bw^h = w^v \\ Bw^v = 0 \end{cases}$$

isometry $H^{\nabla} \longrightarrow V$.

The structure group of manifold T_M reduces to $\mathrm{O}(m)$ as the diagonal subgroup of $\mathrm{O}(m) \times \mathrm{O}(m)$.
Clearly $\mathrm{O}(m) \subset \mathrm{GL}_+(2m, \mathbb{R})$.

Remark: we may take orthonormal frames $T_u(T_M \backslash 0)$ of the form $(m = n + 1)$

$$e_0, e_1, \ldots, e_n, \quad \frac{\xi}{\|\xi\|}, e_{n+1}, \ldots, e_{2n} \qquad \text{with} \quad \begin{cases} Be_i = e_{i+n} \\ e_0 : \; Be_0 = \frac{\xi}{\|\xi\|} \end{cases}$$

These are the **adapted** frames.
($B^t\xi \in H^\nabla$ is known as the geodesic spray.)

The structure group of $T_M \backslash 0$ is reducible, from the principal $\mathrm{O}(2n+2)$-bundle of orthonormal frames to the principal $\mathrm{O}(n)$-bundle over $T_M \backslash 0$ of adapted frames.

Many have worked on the geometry of T_M and that of the tangent sphere bundles: M. T. Abbassi, G. Calvaruso, O. Kowalski, E. Musso, V. Oproiu, N. Papaghiuc, M. Sekizawa, S. Sasaki, L. Vanhecke ... and many collaborators.

Example: Let $M = M_R^{\pm}$ denote the $n+1$-dimensional space-form with metric g of constant sectional curvature $\pm 1/R^2$ where $R > 0$.

Consider the radius $s > 0$ sphere bundle $S_{s,M} \longrightarrow M$.
The scalar curvature of the manifold $S_{s,M}$ is

$$\mathrm{Scal}_{(S_{s,M},g^S)} = \pm\frac{n(n+1)}{R^2} - \frac{s^2 n}{2R^4} + \frac{(n-1)n}{s^2} \ .$$

This is a constant, positive (negative) for small (large) s, though the metric is **not** Einstein.

For M orientable:
We have the exterior differential system on S_M defined by the contact 1-form $\theta = e^0$ (due to Y. Tashiro) and the invariant n-forms

$$\alpha_0, \ldots, \alpha_n$$

where

$$\alpha_i = \frac{1}{i!(n-i)!} \sum_{\sigma \in S_n} \mathrm{sg}(\sigma)\, e^{\sigma_1} \wedge \cdots \wedge e^{\sigma_{n-i}} \wedge e^{(n+\sigma_{n-i+1})} \wedge \cdots \wedge e^{(n+\sigma_n)}.$$

α_i correspond with the generators of 1-dimensional $\mathrm{SO}(n)$ representations in $\Lambda^n(\mathbb{R} \oplus \mathbb{R}^n \oplus \mathbb{R}^n)$.
These yield global forms over S_M.
cf. arXiv:1112.3213 [math.DG] for the applications.

This slide is an add from the white-board written during exposition:
For $n = 1$, we recognize Cartan's structural equations:
$\theta = e^0,\ \mathrm{d}\theta = e^{21},\ \alpha_0 = e^1,\ \alpha_1 = e^2$ where $e_0, e_1 \in H^\nabla$ and $e_2 \in V$, satisfying (K =Gauss curvature):

$$\mathrm{d}\alpha_0 = \theta \wedge \alpha_1, \qquad \mathrm{d}\alpha_1 = K\alpha_0 \wedge \theta.$$

For $n = 2$, i.e. $\dim M = 3$, we have new equations:
$\theta = e^0,\ \mathrm{d}\theta = e^{31} + e^{42},\ \alpha_0 = e^{12},\ \alpha_1 = e^{14} - e^{23},\ \alpha_2 = e^{34}$
where $e_0, e_1, e_2 \in H^\nabla$ and $e_3, e_4 \in V$, such that:

$$\mathrm{d}\alpha_0 = \theta \wedge \alpha_1, \qquad \mathrm{d}\alpha_1 = 2\theta \wedge \alpha_2 - r\,\theta \wedge \alpha_0, \qquad \mathrm{d}\alpha_2 = \mathcal{R}^\xi \alpha_2$$

where $r = r(u) = \mathrm{ric}(u, u),\ \forall u \in S_M$, and $\mathcal{R}^\xi \alpha_2$ is a 3-form, interesting to decompose under $\mathrm{SO}(2)$, cf. `http://arxiv.org/abs/1504.04659`, with further applications in `http://arxiv.org/abs/1604.05390`.

Generalised Sasaki metric

Sasaki also introduced an **almost complex structure**

$$I = I_{1,1} = B - B^t.$$

Integrable if and only if the Levi-Civita connection is flat.

More generally, given any linear connection ∇, we have I integrable if and only if

$$T^\nabla = 0 \quad \text{and} \quad R^\nabla = 0$$

(P. Dombrovski).

For any $f, h \in \mathrm{C}^{\infty}_{T_M}(\mathbb{R}^+)$, we may define a metric and almost complex structure, a Hermitian structure by

$$g_{f,h} = f\pi^* g \oplus h\pi^\star g$$

$$I_{f,h} = \sqrt{\frac{f}{h}}B - \sqrt{\frac{h}{f}}B^t.$$

Then

$$\omega_{f,h} \;\; = \;\; g_{f,h}(I_{f,h} \;\; , \;\;)$$

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Then

$$\begin{aligned} \omega_{f,h} &= g_{f,h}(I_{f,h}\ ,\ ) \\ &= \sqrt{fh}\,\omega_{1,1} \end{aligned}$$

One class of weight functions comes with pullback or constant along the fibre functions

$$f, h \in \mathrm{C}^{\infty}_{M,\pi}(\mathbb{R}^+).$$

Theorem (2011)

For the class of pullback weight functions:

1. $\omega_{f,h}$ *is closed* $\iff T^\nabla = 1 \wedge \mathrm{d}\overline{\psi}$, *where* $\overline{\psi} = \frac{1}{2}\log(hf)$
2. $I_{f,h}$ *is integrable* $\iff R^\nabla = 0$ *and* $T^\nabla = 1 \wedge \mathrm{d}\psi$, *where* $\psi = \frac{1}{2}\log(\frac{h}{f})$
3. $g_{f,h}$ *is Kähler* $\iff R^\nabla = 0$ *and* $T^\nabla = 1 \wedge \mathrm{d}\psi = 1 \wedge \mathrm{d}\overline{\psi}$, *this is,* f *is constant.*

Note the case when ∇ is torsion-free.

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Another important class of functions on T_M:

$$f, h \text{ functions of } r = \|u\|_M.$$

Now let ∇ be the Levi-Civita connection.

Theorem (—, 2015; V. Oproiu and N. Papaghiuc, 2009)

For the class of functions of r:

1. $\omega_{f,h}$ *is closed* $\iff$ *fh is a constant.*
2. $I_{f,h}$ *is integrable* $\iff$ *M has constant sectional curvature K and $\frac{f}{h} = c + Kr^2$, where $c > 0$ constant.*
3. *The metric is Kähler* $\iff$

$$g_{f,h} = \sqrt{c + Kr^2}\,\pi^* g + \frac{1}{\sqrt{c + Kr^2}}\,\pi^\star g.$$

For $K \geq 0$ the metric is complete: $\int_0^{+\infty} \frac{1}{\sqrt[4]{c+Kr^2}}\mathrm{d}r = +\infty$.

For $K < 0$ we must restrict to the disk-bundle

$$D_M = \{u \in T_M : \ \|u\|^2 < -\frac{c}{K}\}.$$

But then

$$\int_0^{\sqrt{-\frac{c}{K}}} \frac{1}{\sqrt[4]{c+Kr^2}}\mathrm{d}r < +\infty.$$

The metric is not complete.

Notice H^∇ and V are Lagrangian. Not complex...

For $K = \pm 1$, let us take isoperimetric coordinates:

$$g = \frac{2}{(1 \pm |z|^2)^2} \mathrm{d}z \odot \mathrm{d}\overline{z}.$$

The germ of T_M is given by (z, w). Then, how do we integrate, i.e. what are the holomorphic charts of the new metric $g_{f,\frac{1}{f}}$ on T_M? ($f(r) = \sqrt{c + Kr^2}$)

We may indeed prove $N = 0$ just using the coordinates z, w.

The metric g_{T_M} has $\mathrm{SU}(2)$ holonomy in case $\dim M = m = 2$.
Non flat, Kähler Ricci-flat metric.

The metric is just $\mathrm{U}(m)$ holonomy in higher dimensions.

Should this be the celebrated Stenzel metric on S^2 ?
But how...
Besides, it is not Ricci-flat in higher dimensions.

Ciconia metric

Let M denote the real Riemannian manifold.
What other metrics may be defined invariantly on T_M as

$$g^S(A\ \ ,\ \ )\ \ ?$$

What other metrics reduce to the principal $\mathrm{O}(m)$-bundle of adapted diagonal frames?

The equation is that the self-adjoint map $A : T(T_M) \longrightarrow T(T_M)$ must satisfy

$$A \begin{bmatrix} o & \\ & o \end{bmatrix} = \begin{bmatrix} o & \\ & o \end{bmatrix} A \,, \qquad \forall o \in \mathrm{O}(m),$$

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respecting canonical decomposition $T(T_M) = H^\nabla \oplus V$.

We have seen

$$A = \begin{bmatrix} f & & & & & \\ & \ddots & & & & \\ & & f & & & \\ & & & h & & \\ & & & & \ddots & \\ & & & & & h \end{bmatrix}.$$

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respecting canonical decomposition $T(T_M) = H^{\nabla} \oplus V$.

We also have

$$A = \begin{bmatrix} f & & & \color{red}{b} & & \\ & \ddots & & & \ddots & \\ & & f & & & \color{red}{b} \\ \color{red}{b} & & & h & & \\ & \ddots & & & \ddots & \\ & & \color{red}{b} & & & h \end{bmatrix}.$$

For $m = 2$ and M **orientable** we have also

$$A = \begin{bmatrix} f & & b & c \\ & f & -c & b \\ b & -c & h & \\ c & b & & h \end{bmatrix}.$$

For m even > 2 and M almost-Hermitian we may consider reduction to unitary group and proceed...

This is called ciconia metric, arXiv:1612.07596.

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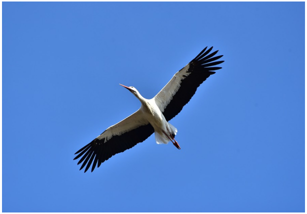

Figure: *Ciconia ciconia*

By construction, ciconia metric is compatible with the almost complex structure $\pi^* J \oplus \pi^\star J$ over T_M.

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We concentrate in dim 2.

A function a will correspond to

$$a = \begin{bmatrix} b & c \\ -c & b \end{bmatrix} = b + ic$$

with b, c new $\mathbb{R}$ functions on T_M. Note: $i = \sqrt{-1}$).
We have positive definite metric iff

$$f > 0, \quad fh - |a|^2 > 0.$$

Take isothermal coordinates on $\mathcal{U} \subset M$, $\quad g = \lambda\, \mathrm{d}z \mathrm{d}\bar{z}$.

Let $\nabla_z \mathrm{d}z = -\Gamma \mathrm{d}z$. Then

$$\nabla_z \mathrm{d}\bar{z} = 0, \qquad \Gamma = \frac{1}{\lambda}\frac{\partial \lambda}{\partial z}.$$

The sectional curvature of M is given by,

$$R^{\nabla}(\partial_z, \partial_{\bar{z}})\partial_z = \nabla_z \nabla_{\bar{z}} \partial_z - \nabla_{\bar{z}} \nabla_z \partial_z = -\frac{\partial \Gamma}{\partial \bar{z}} \partial_z = -\frac{\partial^2 \log \lambda}{\partial z \partial \bar{z}} \partial_z,$$

$$K = \frac{g(R^{\nabla}(\partial_z, \partial_{\bar{z}})\partial_z, \partial_{\bar{z}})}{g(\partial_z, \partial_{\bar{z}})^2} = -\frac{2}{\lambda}\frac{\partial \Gamma}{\partial \bar{z}} = -\frac{2}{\lambda}\frac{\partial^2 \log \lambda}{\partial z \partial \bar{z}}.$$

Next we consider the open subset $T_{\mathcal{U}} = \pi^{-1}(\mathcal{U}) \subset T_M$. We have trivialising coordinates

$$T_{\mathcal{U}} = \{(z, w) : \; z \in \mathcal{U}, \; w \in \mathbb{C}\}.$$

The horizontal lift $X = \pi^* \partial_z = \partial_z - w\Gamma \partial_w$ gives place to the $(1, 0)$-form

$$\eta = w\Gamma \mathrm{d}z + \mathrm{d}w$$

such that $\eta(X) = 0, \; \eta(\partial_w) = 1$.

$$\pi^* g = \lambda \, \mathrm{d}z\mathrm{d}\overline{z}, \qquad \pi^\star g = \lambda \, \eta\overline{\eta}$$

$$a\lambda \, \mathrm{d}z\overline{\eta} + \overline{a}\lambda \, \eta\mathrm{d}\overline{z}$$

are well-defined, where $a : T_M \longrightarrow \mathbb{C}$ global.

Finally, we have ciconia metric

$$g_{f,a,h} = \lambda(f\, \mathrm{d}z\mathrm{d}\overline{z} \; + \; a\, \mathrm{d}z\overline{\eta} + \overline{a}\, \eta\mathrm{d}\overline{z} \; + \; h\, \eta\overline{\eta}),$$

with symplectic 2-form

$$\omega_{f,a,h} = \frac{i\lambda}{2}(f\, \mathrm{d}z \wedge \mathrm{d}\overline{z} \; + \; a\, \mathrm{d}z \wedge \overline{\eta} + \overline{a}\, \eta \wedge \mathrm{d}\overline{z} \; + \; h\, \eta \wedge \overline{\eta}).$$

and Hermitian structure

$$H_{f,a,h} = \lambda(f\, \mathrm{d}z \otimes \mathrm{d}\overline{z} \; + \; a\, \mathrm{d}z \otimes \overline{\eta} + \overline{a}\, \eta \otimes \mathrm{d}\overline{z} \; + \; h\, \eta \otimes \overline{\eta}).$$

Recall $\mathrm{C}^\infty_{\mathcal{U},\pi}(\mathbb{C})$, the set of functions which are the pullback by π of functions on M, i.e. functions which depend only of z.

A second set, $\mathrm{C}^\infty_{r^2} = \mathrm{C}^\infty_{[0,+\infty[}(\mathbb{C})$, where $r^2 = r^2(u) = g(u,u) = \lambda|w|^2,\ u \in T_M$, is the set of functions φ on $\pi^{-1}(\mathcal{U})$ which depend only of r^2 and have derivatives $\varphi', \varphi'', \ldots$ at 0 (n.b.: we let $\varphi' = \mathrm{d}\varphi/\mathrm{d}r^2$).

Next we assume f, h take real values, $f > 0$ and $fh - |a|^2 > 0$.

Theorem

Suppose a given ciconia metric $\mathrm{g}_{f,a,h}$ is Kähler with weight functions of any of the two types above. We have that:
(i) if $f, a, h \in \mathrm{C}^\infty_{M,\pi}$, then $K = 0$, a is holomorphic and h is constant;
(ii) if $f, h \in \mathrm{C}^\infty_{M,\pi}$ and $a \in \mathrm{C}^\infty_{r^2}$, then $K = 0$ and a, h are constant;
(iii) if $f, a \in \mathrm{C}^\infty_{M,\pi}$ and $h \in \mathrm{C}^\infty_{r^2}$, then $K = 0$ and a is holomorphic;
(iv) if $a, h \in \mathrm{C}^\infty_{M,\pi}$ and $f \in \mathrm{C}^\infty_{r^2}$, then $f(r^2) = f_1 r^2 + f_0$, $K = -\frac{2f_1}{h}$, h, f_0, f_1 are constant and a is holomorphic;
(v) if $f \in \mathrm{C}^\infty_{M,\pi}$ and $a, h \in \mathrm{C}^\infty_{r^2}$, then $K = 0$ and a is constant;
(vi) if $a \in \mathrm{C}^\infty_{M,\pi}$ and $f, h \in \mathrm{C}^\infty_{r^2}$, then $K = -\frac{2f'}{h}$ and a is holomorphic;
(vii) if $h \in \mathrm{C}^\infty_{M,\pi}$ and $f, a \in \mathrm{C}^\infty_{r^2}$, then $f(r^2) = f_1 r^2 + f_0$, $K = -\frac{2f_1}{h}$, a, h, f_0, f_1 are constant;
(viii) if $f, a, h \in \mathrm{C}^\infty_{r^2}$, then $K = -\frac{2f'}{h}$ and a is constant.
Reciprocally, any of the conditions above imply the metric is Kähler.

Example:
For any open set $\mathcal{U} \subset \mathbb{C}$ and any $a \in \mathrm{C}^{\infty}_{\mathcal{U}}$ holomorphic in z, the Hermitian metric on $\mathcal{U} \times \mathbb{C}$

$$H = (1 + |a|^2)\mathrm{d}z \otimes \mathrm{d}\overline{z} + a\mathrm{d}z \otimes \mathrm{d}\overline{w} + \overline{a}\mathrm{d}w \otimes \mathrm{d}\overline{z} + \mathrm{d}w \otimes \mathrm{d}\overline{w}$$

is Kähler and flat.
However, it is biholomorphic to the same $\mathcal{U} \times \mathbb{C}$ with canonical metric. We take
$F(z, w) = (z, w + \int a(z)\,\mathrm{d}z)$.

Calabi-Yau type ciconia metric

$H = H_{f,a,h}$. The Ricci-form is the closed $(1,1)$-form

$$\rho = i\bar{\partial}\partial \log \det H.$$

Proposition

We have

$$\rho = 2K\pi^*\omega + i\bar{\partial}\partial \log(fh - |a|^2).$$

Proof.

By invariance of the unitary structure we may use the type-$(1,0)$ frame field $\pi^*\partial_z, \partial_w$, and therefore deduce that $\det H = \lambda^2(fh - |a|^2)$. Combining with $K = -\frac{2}{\lambda}\frac{\partial^2 \log \lambda}{\partial z \partial \bar{z}}$ and $\pi^*\omega = \frac{i\lambda}{2}\mathrm{d}z \wedge \mathrm{d}\bar{z}$, the result follows. □

Notice that

$$H = \lambda\big(f + h|w|^2|\Gamma|^2 + 2\Re(\overline{a}w\Gamma)\big)\,\mathrm{d}z\otimes\mathrm{d}\overline{z} + \lambda(a + hw\Gamma)\,\mathrm{d}z\otimes\mathrm{d}\overline{w} + \lambda(\overline{a} + h\overline{w}\overline{\Gamma})\,\mathrm{d}w\otimes\mathrm{d}\overline{z} + \lambda h\,\mathrm{d}w\otimes\mathrm{d}\overline{w}$$

so $\det H = \lambda^2(fh - |a|^2)$ is also confirmed from the matrix of H on the holomorphic frame ∂_z, ∂_w.

Let $T_M \backslash M$ denote the complement of the zero-section.

Theorem

For any Riemann surface (M, g) and every set of smooth functions f, a, h on T_M such that $f > 0$ and $fh - |a|^2 = \frac{1}{r^4}$, the ciconia metric $\mathrm{g}_{f,a,h}$ on the open manifold $T_M \backslash M$ is Ricci-flat.

Theorem

Let (M, g) be an oriented compact Riemann surface of constant Gauss curvature $K = -1, 0$ or 1. For any $\epsilon_2 > \epsilon_1 \geq 0$, let us denote by $Z = Z_{\epsilon_1,\epsilon_2}$ the open manifold $Z = \{u \in T_M : \ \epsilon_1 < r^2 < \epsilon_2\} \subset T_M$ where $r^2 = g(u, u)$. Given the following conditions on a constant $c_0 \in \mathbb{R}$ and on $\epsilon_1, \epsilon_2, f, a, h$, the respective ciconia metrics $\mathrm{g}_{f,a,h}$ on the manifold Z are Kähler and Ricci-flat:

(i) if $K = 0$, we consider $Z_{0,+\infty}$ with

$$f(r^2) = f > 0 \text{ constant}, \quad a \neq 0 \text{ constant}, \quad h(r^2) = \frac{|a|^2}{f} + \frac{1}{fr^4}$$

and then the metric is complete.

Theorem (cont.)

(ii) if $K = 1$*, we let* $Z = Z_{0,\frac{1}{c_0}}$*, for any* $c_0 > 0$*, and take*

$$f(r^2) = \frac{\sqrt{1 - c_0 r^2}}{r}, \quad a = 0, \quad h(r^2) = \frac{1}{r^3\sqrt{1 - c_0 r^2}};$$

and then the associated metric space may be completed to $\overline{Z}\backslash M$*.*
(iii) also if $K = 1$*, we let* $Z = Z_{0,\beta_+}$*, where* $\forall c_0 \in \mathbb{R}$

$$\beta_+ = \frac{-c_0 + \sqrt{c_0^2 + 4|a|^2}}{2|a|^2},$$

and let

$$f = \frac{1}{r}\sqrt{-|a|^2 r^4 - c_0 r^2 + 1}, \quad a \neq 0 \text{ constant}, \quad h = \frac{|a|^2 r^4 + 1}{r^3\sqrt{-|a|^2 r^4 - c_0 r^2}}$$

so that the associated metric space structure on $\overline{Z}\backslash M$ *is complete.*

Theorem (cont.)

(iv) if $K = -1$*, we let* $Z = Z_{\beta_+,+\infty}$ *with* β_+ *as above and take*

$$f = \frac{1}{r}\sqrt{|a|^2 r^4 + c_0 r^2 - 1}, \quad a \neq 0 \text{ constant}, \quad h = \frac{|a|^2 r^4 + 1}{r^3\sqrt{|a|^2 r^4 + c_0 r^2 - 1}}$$

implying the associated metric space structure on $\overline{Z}$ *is complete. Reciprocally, the above are all the Cauchy-complete solutions.*