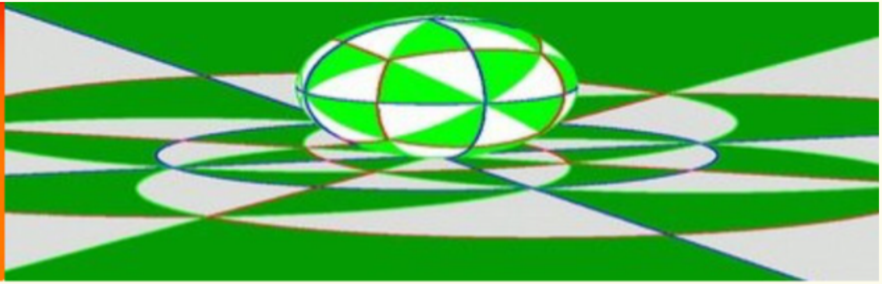
XXVI International Fall Workshop on Geometry and Physics

Braga, 4-7 September 2017



ROUTH REDUCTION AS AN EXAMPLE OF LAGRANGIAN REDUCTION

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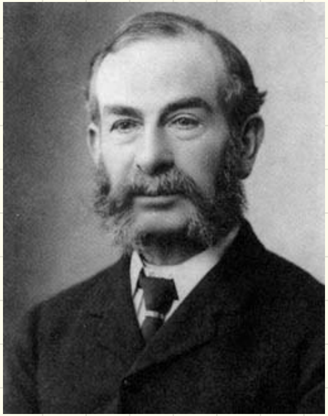


Edward John Routh
1831 - 1907

CALCULATIONS IN COORDINATES

$$\mathbb{R}^{n+m} \ni (q, x, \dot{q}, \dot{x}) \longmapsto L(q, x, \dot{q}, \dot{x}) \in \mathbb{R}$$

cyclic variables : $\frac{\partial L}{\partial q} = 0$

$$R(q, x, p, \dot{x}) = p\dot{q} - L(q, x, \dot{q}, \dot{x}) \qquad \frac{\partial R}{\partial q} = 0$$

$$\dot{p} = -\frac{\partial R}{\partial q} = 0 \qquad \dot{q} = \frac{\partial R}{\partial p} \qquad \frac{d}{dt}\left(\frac{\partial R}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\Downarrow$$

$$p = a \qquad R_a(x, \dot{x}) = a\dot{q}(a) - L(x, \dot{q}(a), \dot{x})$$

Routhian depends only on $(x, \dot{x})$

WHAT IS GEOMETRIC PICTURE BEHIND THESE CALCULATIONS ?

J.E. Marsden, T.S. Ratiu, J. Scheurle, F. Cantrijn, B. Langerock, T. Mestdag, E. García-Toraño Andrés, J. Vankerschaver, M. Crampin ...

WHY DO IT AGAIN ?

$$
\begin{tikzcd}
 & D \arrow[d, squiggly] & \\
T^*T^*Q \arrow[d] & TT^*Q \arrow[l, "\beta_Q"'] \arrow[r, "\alpha_Q"] & T^*TQ \arrow[d] \\
T^*Q \arrow[u, bend left, "dH"] & & TQ \arrow[u, bend right, "dL"']
\end{tikzcd}
$$

$$\beta_Q^{-1}(dH(T^*Q)) = D = \alpha_Q^{-1}(dL(TQ))$$

EASY $\rightsquigarrow$

INTERESTING $\rightsquigarrow$ ROUTH REDUCTION RELATION

$$
\begin{tikzcd}
 & ??? \arrow[d] & ??? \arrow[l] \arrow[r] & ??? \arrow[d] \\
 & ??? \arrow[u, bend left] & & ??? \arrow[u, bend right, "dR???"']
\end{tikzcd}
$$

TOOLS:

→ SYMPLECTIC GEOMETRY: SYMPLECTIC REDUCTIONS, LAGRANGIAN SUBMANIFOLDS, GENERATING OBJECTS:

GENERATING FUNCTION

$$f : Q \longrightarrow \mathbb{R}$$

$$df(Q) \subset T^*Q$$

LAGRANGIAN SUBMANIFOLD

FUNCTION ON A SUBMANIFOLD

$$Q \hookleftarrow C \xrightarrow{f} \mathbb{R}$$

$$\{\varphi \in T^*Q : \forall v \in TC \ \langle df, v\rangle = \langle \varphi, v\rangle\} \subset T^*Q$$

GENERATING FAMILY

$$\begin{array}{ccc} M & \xrightarrow{f} & \mathbb{R} \\ {\scriptstyle \rho}\downarrow & & \\ Q & & \end{array}$$

$$C = \{m \in M : d^v f(m) = 0\}$$

$$\{\varphi \in T^*Q : \rho^*\varphi(m) = df(m)\}$$

SYMPLECTIC RELATIONS ARE LAGRANGIAN SUBMANIFOLDS

$$\rho : P_1 \longrightarrow P_2$$

$$\mathrm{graph}(\rho) \subset (P_1 \times P_2 \, , \; \omega_1 - \omega_2)$$

IF P_1, P_2 ARE COTANGENT BUNDLES WE CAN GENERATE THE GRAPH

COMPOSING SYMPLECTIC RELATIONS MEANS ADDING GENERATING OBJECTS

EXAMPLE

$$L : TQ \longrightarrow \mathbb{R}$$

$$dL(TQ) \subset T^*TQ$$

LAGRANGIAN SUBMANIFOLD

$$T^*TQ \xrightarrow{\gamma_Q} T^*T^*Q$$

SYMPLECTOMORPHISM GENERATED BY

$$TQ \times T^*Q \supset TQ \times_Q T^*Q \ni (v, p) \longmapsto -\langle p, v \rangle \in \mathbb{R}$$

WHAT GENERATES $\gamma_Q(dL(TQ))$?

$$\begin{array}{ccc} TQ \times_Q T^*Q & \ni (v, p) \longmapsto & \underline{L(v) - \langle p, v \rangle \in \mathbb{R}} \\ \downarrow & & \\ T^*Q & & \end{array}$$

TOOLS:

→ GEOMETRY OF AFFINE VALUES: AV-BUNDLES, AFFINE PHASE SPACES

$$\begin{tikzcd} Q \times \mathbb{R} \arrow[d, shift right] \\ Q \arrow[u, shift right, "f"'] \end{tikzcd} \rightsquigarrow \begin{tikzcd} Z \circlearrowleft (\mathbb{R},+) \arrow[d, shift right] \\ Q \arrow[u, shift right, "\sigma"'] \end{tikzcd}$$

AFFINE PHASE SPACE:

$$(\sigma_1, q) \sim (\sigma_2, q) \iff d(\sigma_1 - \sigma_2)(q) = 0$$

$$d\sigma(q) := [(\sigma, q)]$$

$$PZ = \{ d\sigma(q) \}$$ ← AFFINE BUNDLE OVER Q

SYMPLECTIC MANIFOLD

MODELED ON T^*Q

$$\omega_Z = \Sigma^* \omega_Q$$

$$\Sigma : PZ \longrightarrow T^*Q$$

$\tilde{T}Z \longrightarrow TQ$ IS A BUNDLE OF AFFINE VALUES

ATIYAH ALGEBROID $\tilde{T}Z = TZ/\mathbb{R}$

VECTOR BUNDLE OVER Q

ROUTH REDUCTION RELATION

Q - CONFIGURATION MANIFOLD

X - VECTOR FIELD ON Q - SYMMETRY

$$L: TQ \longrightarrow \mathbb{R} \qquad \underbrace{d_T X \cdot L = 0}$$

GEOMETRIC VERSION OF A LAGRANGIAN WITH CYCLIC VARIABLE

IN FACT, WHAT IS IMPORTANT IS THE DISTRIBUTION

$$\Delta_X = \langle d_T X \rangle$$

IT IS ONE DIMENSIONAL DISTRIBUTION, DOUBLE VECTOR SUBBUNDLE OF

$$\begin{tikzcd} & TTQ \arrow[dl, "T\tau_Q"'] \arrow[dr, "\tau_{TQ}"] & \\ TQ & & TQ \end{tikzcd}$$

ROUTH REDUCTION RELATION VIA HAMILTONIAN MECHANICS

$$\begin{tikzcd}
T^*T^*Q & TT^*Q \arrow[l, "\beta_Q"'] \arrow[r, "\alpha_Q"] & T^*TQ \arrow[ll, bend right, "\gamma_Q"'] \arrow[d] \\
 & & TQ \arrow[u, bend right, "dL"']
\end{tikzcd}$$

$$\begin{tikzcd}
T^*Q \times_Q TQ \arrow[r] \arrow[d] & \mathbb{R} \\
T^*Q &
\end{tikzcd} \qquad (p,v) \longmapsto L(v) - \langle p, v \rangle$$

→ IN HAMILTONIAN MECHANICS SYMMETRY IS GIVEN BY

$d_{T^*}X$ ← COTANGENT LIFT OF X

↑ HAMILTONIAN VECTOR FIELD FOR $T^*Q \ni p \longmapsto i_X(p) = \langle p, X(\pi_Q(p)) \rangle$

→ PHASE TRAJECTORIES LIE IN LEVEL SETS OF i_X

$C_\alpha = \{ p \in T^*Q : \ i_X(p) = \alpha \}$ ← COISOTROPIC SUBMANIFOLD, REDUCTION WITH RESPECT TO C_α = DIVIDING BY LIFTED $(\mathbb{R},+)$ ACTION

↑ AFFINE SUBBUNDLE OF $T^*Q \to Q$ WITH MODEL $C_0 \to Q$

ROUTH REDUCTION RELATION VIA HAMILTONIAN MECHANICS

$$T^*Q \supset C_\alpha \xrightarrow{\rho} P_\alpha \qquad Q \longrightarrow M = Q/\mathbb{R}$$

MANIFOLD OF TRAJECTORIES OF X

AFFINE BUNDLE OVER M MODELED ON T^*M

SYMPLECTIC MANIFOLD

REDUCED AFFINE PHASE SPACE

$$\begin{tikzcd}
T^*T^*Q \arrow[d, "T^*\rho"] & TT^*Q \arrow[l, "\beta_Q"'] \arrow[r, "\alpha_Q"] & T^*TQ \arrow[d] \\
T^*P_\alpha & TP_\alpha \arrow[l, "\beta"'] \arrow[r] & ??? \quad TQ \arrow[u, bend right, "dL"']
\end{tikzcd}$$

SYMPLECTIC RELATION GENERATED BY A FUNCTION EQUAL ZERO ON

$$T^*Q \times P_\alpha \supset C_\alpha \times P_\alpha$$

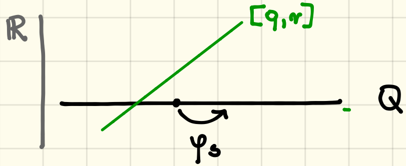
$$\begin{tikzcd}
C_\alpha \times_Q TQ \arrow[r] \arrow[d] & \mathbb{R} \\
P_\alpha &
\end{tikzcd}$$

$$(p, v) \longmapsto L(v) - \langle p, v \rangle$$

ROUTH REDUCTION RELATION VIA HAMILTONIAN MECHANICS

HERE COMES AV-GEOMETRY!

$$Q \times \mathbb{R} \;:\; (q, r) \sim (\underbrace{\varphi_s}_{\text{FLOW OF } X}(q),\; r + s\alpha)$$



$$Z_\alpha \simeq = Q \times \mathbb{R} / \sim \qquad Z_\alpha \longrightarrow M$$

$$[q, r] + t := [q, r + t] = [\varphi_s(q), r + t + s\alpha]$$

$$P_\alpha \simeq PZ_\alpha$$

$$T^*P_\alpha \xleftarrow{\beta} TP_\alpha \longrightarrow P\tilde{T}Z_\alpha$$

WHAT IS IT ??

$\tilde{T}Z_\alpha = TZ_\alpha / (\mathbb{R},+)$ ATIYAH ALGEBROID

$TZ_\alpha \longrightarrow TM$ AV- BUNDLE

$w \in TZ_\alpha$ $\quad [w] + s = [w + s\xi]$

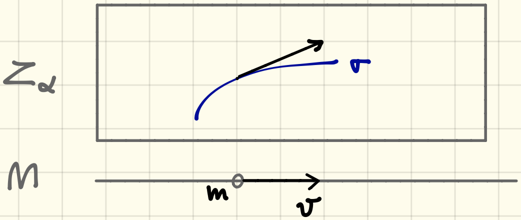
FUNDAMENTAL VECTOR FIELD OF $\mathbb{R}$-ACTION ON Z_α

$P\tilde{T}Z_\alpha$ - AFFINE DIFFERENTIALS OF SECTIONS OF $\tilde{T}Z_\alpha \longrightarrow TM$

ROUTHIAN IS NOT A FUNCTION ON TM BUT A SECTION OF AN AV- BUNDLE

AFFINE ANALOG OF A MAP $T^*Q \times_Q TQ \ni (p,v) \longmapsto \langle p,v \rangle \in \mathbb{R}$

$PZ_\alpha \times_M TM \ni (p,v) \longmapsto \langle p,v \rangle \in \tilde{T}Z_\alpha$



$p = d\sigma(m) \qquad \langle p,v \rangle = [\,T\sigma(v)\,]$

$$\begin{tikzcd} T^*PZ_\alpha & TPZ_\alpha \arrow[l, "\beta"'] \arrow[r, "\alpha"] & P\tilde{T}Z_\alpha \arrow[from=1-1, to=1-3, bend right, "\gamma"'] \end{tikzcd}$$

$T^*PZ_\alpha \simeq P\tilde{T}Z_\alpha$

SYMPLECTOMORPHISM GENERATED BY A SECTION OVER A SUBMANIFOLD

$$PZ_\alpha \times TM \supset PZ_\alpha \times_M TM \longrightarrow \tilde{T}Z_\alpha \qquad (p_\alpha, w) \longmapsto \langle p_\alpha, w\rangle$$

$$\begin{tikzcd} C_\alpha \times_Q TQ \arrow[r] \arrow[d] & \mathbb{R} \\ P_\alpha & \end{tikzcd} \qquad (p, v) \longmapsto L(v) - \langle p, v\rangle$$

$$\begin{tikzcd} TM \times_M C_\alpha \times_Q TQ \arrow[r] \arrow[d] & \tilde{T}Z_\alpha \\ TM & \end{tikzcd} \qquad (w, p, v) \longmapsto L(v) - \langle p, v\rangle + \langle p_\alpha, w\rangle$$

GENERATING FAMILY WITH PARAMETERS p, v

$$
\begin{tikzcd}
T^*T^*Q \arrow[d] & TT^*Q \arrow[l, "\beta_Q"'] \arrow[r, "\alpha_Q"] & T^*TQ \arrow[d, "??"] \\
T^*PZ_\alpha & TPZ_\alpha \arrow[l, "\beta"'] \arrow[r, "\alpha"] & P\tilde{T}Z_\alpha
\end{tikzcd}
$$

QUESTIONS:

1. IS IT POSSIBLE TO SIMPLIFY

$$
\begin{tikzcd}
TM \times_M C_\alpha \times_Q TQ \arrow[r] \arrow[d] & \tilde{T}Z_\alpha \\
TM &
\end{tikzcd}
\qquad (w,p,v) \longmapsto L(v) - \langle p, v\rangle + \langle p_\alpha, w\rangle
$$

YES, BUT NOT ALWAYS TO ONE ROUTHIAN SECTION $\longrightarrow$ EXAMPLE

2. IS IT POSSIBLE TO GO DIRECTLY FROM T^*TQ TO $P\tilde{T}Z_\alpha$?

YES, BUT YO HAVE TO INCLUDE VALUES OF GENERATING OBJECTS

$\hookrightarrow$ LAGRANGIAN REDUCTION

ANSWER TO QUESTION 1 WITH AN EXAMPLE:

$$\begin{array}{ccc} TM \times_M C_\alpha \times_Q TQ & \longrightarrow & \tilde{T} Z_\alpha \\ \downarrow & & \\ TM & & (w, p, v) \longmapsto L(v) - \langle p, v\rangle + \langle p_\alpha, w\rangle \end{array}$$

$\rightsquigarrow$

$$\begin{tikzcd} TQ \arrow[r] \arrow[d] & \tilde{T} Z_\alpha \\ TM & \end{tikzcd}$$

$$v \longmapsto L(v) - \langle p, v\rangle + \langle p_\alpha, w\rangle$$

$$TQ \ni v \longmapsto w \in TM$$

FORMULA DOES NOT DEPEND ON p,
ANY $p \in C_\alpha$ OVER $\tau_Q(v)$ IS OK!

EXAMPLE:

$$Q = \mathbb{R}^2 \ni (x, y) \qquad L : TQ \ni (x, y, \dot{x}, \dot{y}) \longmapsto \dot{x}\dot{y} - y^2 \in \mathbb{R} \qquad M = \mathbb{R} \ni (y)$$

$$X = \frac{\partial}{\partial x}$$

ROUTHIAN FAMILY READS $(y, \dot{x}, \dot{y}) \longmapsto \dot{x}\dot{y} - y^2 - \underset{\uparrow \text{PARAMETER}}{\alpha}\dot{x} \in \mathbb{R}$

$$T^*TM \supset \{ (y, \dot{y}, a, b) : \dot{y} = \alpha, \; a = -2y \}$$

$$TT^*M \supset \{ (y, p, \dot{y}, \dot{p}) : \dot{y} = \alpha, \; \dot{p} = -2y \}$$

$$X_h(y, p) = \alpha \frac{\partial}{\partial y} - 2y \frac{\partial}{\partial p}$$

$$h(y, p) = \alpha p + y^2$$

ANSWER TO QUESTION 2

THE RELATION $T^*TQ \xrightarrow{R} P\tilde{T}Z_\alpha$ IS THE AFFINE PHASE LIFT OF THE REDUCED TANGENT RELATION $\tilde{T}(Q\times\mathbb{R}) \longrightarrow \tilde{T}Z_\alpha$ OF THE PROJECTION

$Q\times\mathbb{R} \longrightarrow Z_\alpha$

$$Q\times\mathbb{R} \xrightarrow{\zeta_\alpha} Z_\alpha \qquad (q,r) \longmapsto [q,r] = \{(\varphi_s(q), r+s\alpha)\}$$ AV-BUNDLE MORPHISM

$$\tilde{T}(Q\times\mathbb{R}) = TQ\times\mathbb{R} \xrightarrow{T\zeta_\alpha} \tilde{T}Z_\alpha \qquad (v,t) \longmapsto [v,t] = \{(T\varphi_s(v) + \dot{s}X(q),\ t+\alpha\dot{s})\}$$

AV-BUNDLE MORPHISM

ROUTH REDUCTION R IS $P\tilde{T}\zeta_\alpha$. IT IS GENERATED BY NONTRIVIAL SECTION OF CERTAIN AV-BUNDLE

$$T^*T^*Q \longrightarrow T^*PZ_\alpha$$

$$T^*Q \supset C_\alpha \longrightarrow PZ_\alpha$$ PHASE LIFT GENERATED BY FUNCTON ZERO ON $C_\alpha \times_M PZ_\alpha$

FINAL COMMENT ON (NOT REALLY) A GENERALISATION

LAGRANGIAN WITH SYMMETRY $d_T X \cdot L = 0$

$d_T X \rightsquigarrow \Delta_X = \langle d_T X \rangle \qquad dL(TQ) \subset \Delta_X^\circ$

ONE DIMENSIONAL LINEAR DISTRIBUTION ON TQ,
DOUBLE VECTOR SUBBUNDLE OF TTQ

ONE CAN SHOW THAT IT DEFINES X UP TO
MULTIPLICATION BY NUMBERS

NO DISTINGUISHED PARAMETERIZATION OF $\{C_\alpha\}$ BY α ...