Deformations of symplectic groupoids

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Outline

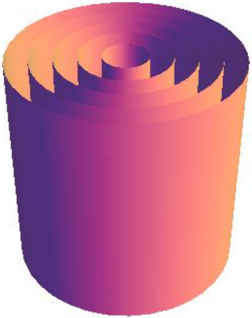
- **Deformations**
- **Lie groupoids**
- **Deformations of Lie groupoids**
(with M. Crainic and I. Struchiner)
- **Deformations of symplectic groupoids**
(with C. Cardenas and I. Struchiner)

Deformations

- Deformations of (algebraic or geometric) structures.
- Examples:
  - Lie algebras
  - associative algebras
  - Lie groups
  - Lie group actions
  - foliations
  - principal bundles
  - complex manifolds

Deformations

Deformations



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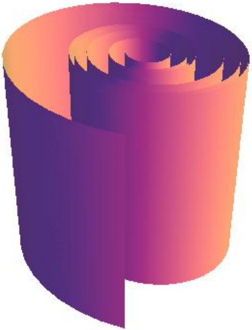

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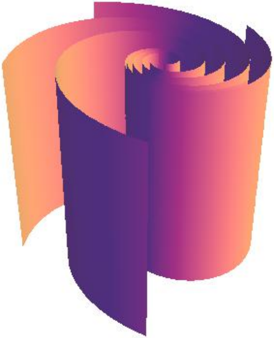

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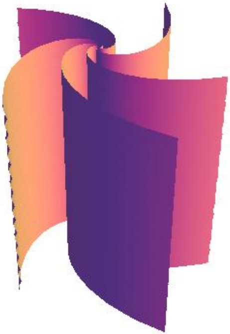

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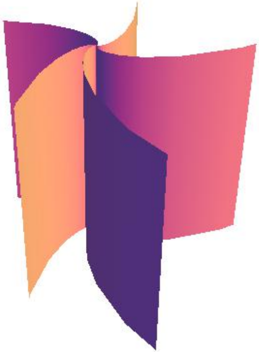

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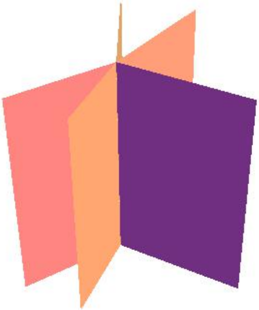

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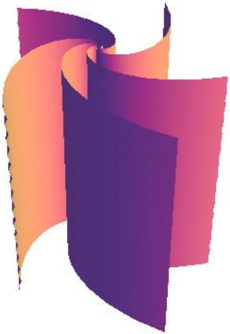

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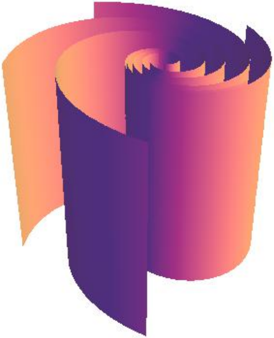

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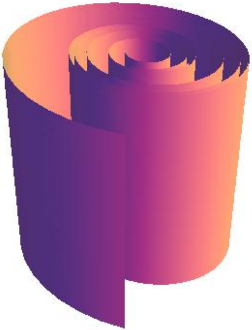

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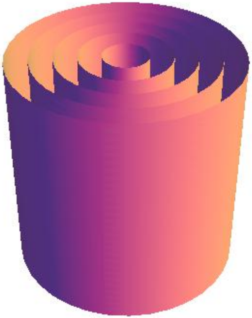

Deformations and rigidity

- Goal: find moduli space = $\{\text{deformations}\}/\{\text{isomorphisms}\}$.
- Rigid structures = isolated point in moduli space.

Cohomology and infinitesimal rigidity

- Structures $\to$ deformation theory $\to$ controlled by cohomology.
- Deformation cohomology $= T_{\mathrm{structure}}$(moduli space).
- Vanishing of cohomology $=$ infinitesimal rigidity.
- The guiding principle:

Infinitesimal rigidity $\Longrightarrow$ rigidity.

Lie groupoids

- A groupoid is a (small) category with all arrows invertible.
- A Lie groupoid is a smooth groupoid.
- Examples:
  - fundamental groupoids
  - holonomy groupoids

Examples of Lie groupoids

- Lie groups
- Principal bundles
- Submersions
- Foliations
- Lie group actions

Why Lie groupoids?

• This framework lets us treat all these geometrical objects on equal footing.

• It permits a unified study of their deformation theories.

• Example of an application: simultaneous deformations of a Lie group and of its action on a manifold.

Deformations of Lie groupoids

• Deformation of a Lie groupoid = family $\{\mathcal{G}_\varepsilon\}_{\varepsilon \in I}$ of Lie groupoids smoothly parametrized by a real parameter ε.

• $H^*_{\mathrm{def}}(\mathcal{G})$ = Deformation cohomology for a Lie groupoid $\mathcal{G}$;

• deformations of a groupoid $\mathcal{G} \to$ elements in $H^2_{\mathrm{def}}(\mathcal{G})$, given by variation of the structure maps: $(\frac{d}{d\varepsilon} m_\varepsilon$, etc);

• vanishing of cohomology $\to$ vector fields whose flows are used to prove rigidity results.

Some of the results

- Relations between $H^*_{\mathrm{def}}(\mathcal{G})$ and the classical examples.

- Relation with an infinitesimal counterpart (deformation theory of Lie algebroids).

- Infinitesimal rigidity for *proper* Lie groupoids

- Rigidity for *compact* groupoids.

- Application: normal forms

Deformations of Symplectic groupoids

- Symplectic groupoid = Lie groupoid + multiplicative symplectic form

- Deformation complex - constructed out of the Bott-Shulman-Stasheff double complex.

- Infinitesimal rigidity for *proper* Symplectic Lie groupoids

- Rigidity for *compact* symplectic groupoids (up to gauge equivalences (also called B-fields)).

References

- **Main reference**

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- **Some earlier work for groupoids and algebroids**

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References

- **Some further developments**

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- **Pictures by Ioan Marcut**

Thank you!