the GAUSS MAP for CMC SURFACES in HOMOGENEOUS SPACES

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Joint work with
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The Gauss map of surfaces in PSL(2,R),
Calc. Var. Partial Differ. Equ. (2015)

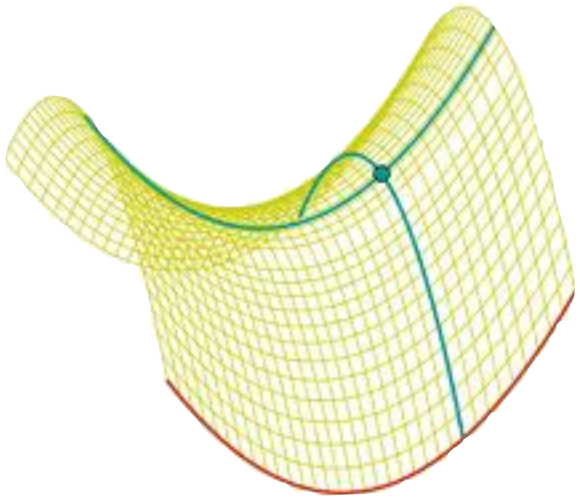
XXVI Fall workshop on Geometry and Physics

Braga, September 2017

Mean curvature of surfaces

Mean curvature of surfaces

CURVATURE:



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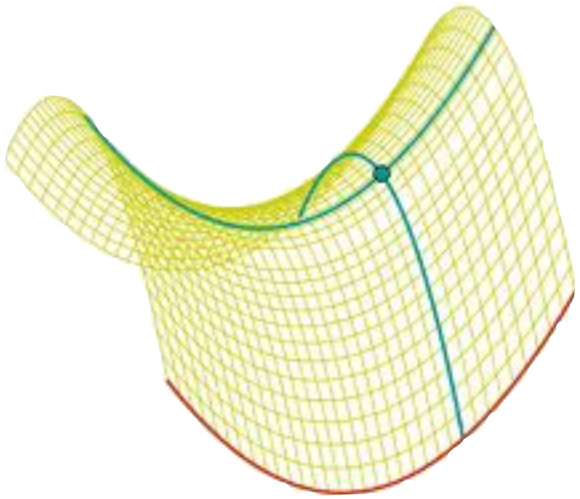

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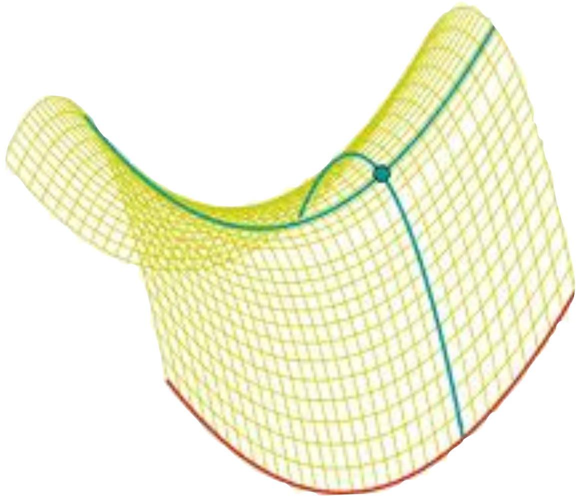

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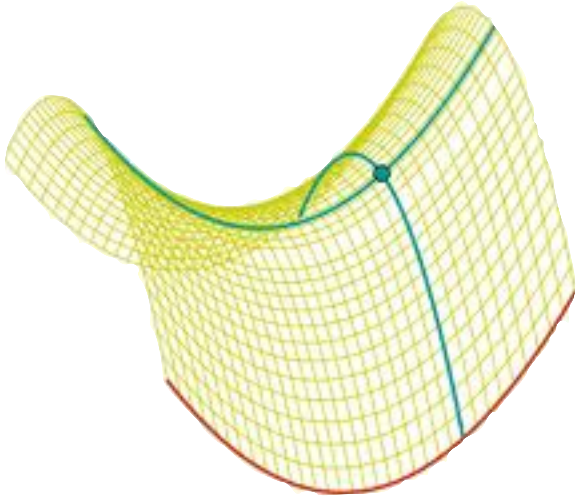

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CMC surface = CONSTANT MEAN CURVATURE surface

Surfaces that minimize the area (I)

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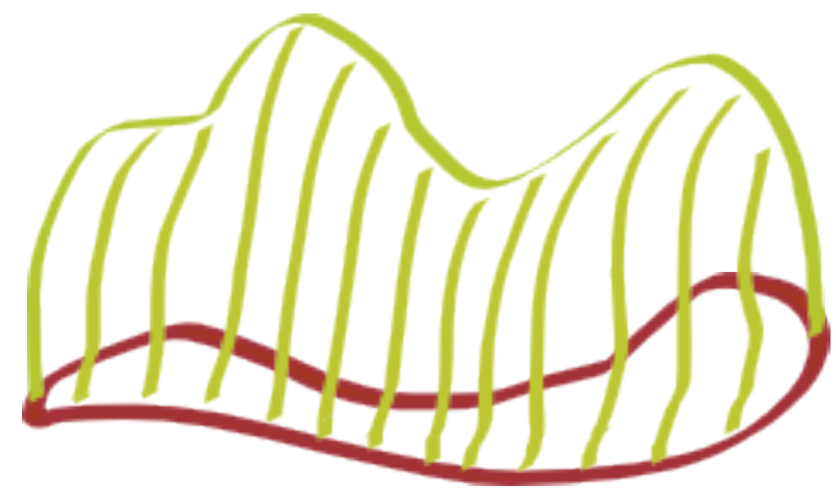
Among all the surfaces
lying on a fixed curve,

Which one has less area ?



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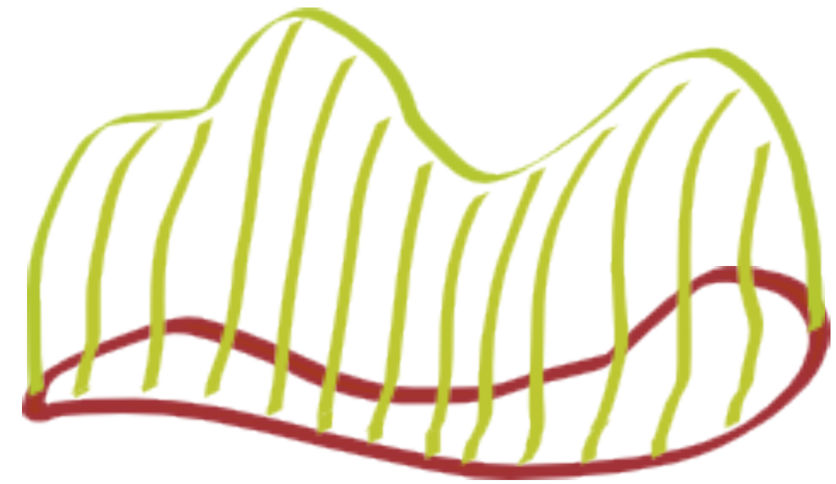

closed curve

A surface is locally
area minimizing

↔

Its mean curvature
vanishes identically

(CMC=0, MINIMAL SURFACES)

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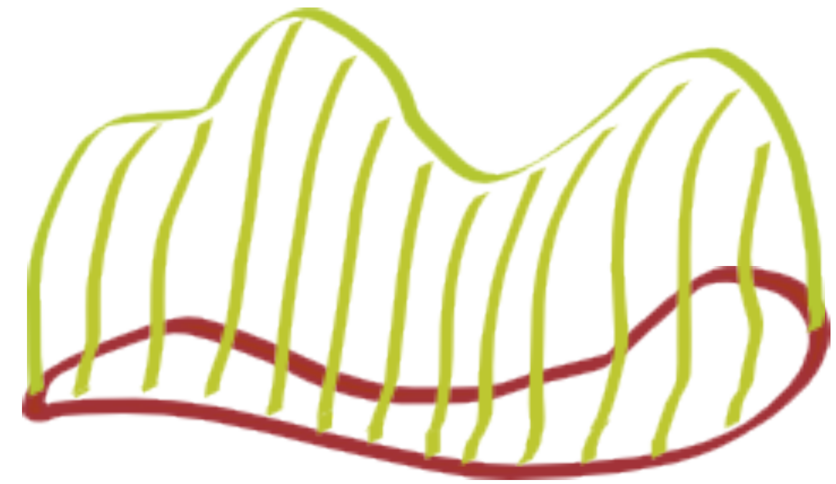

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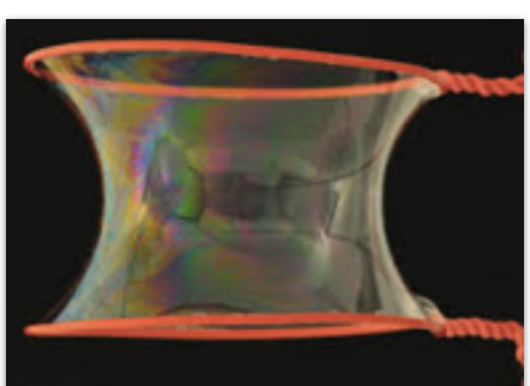

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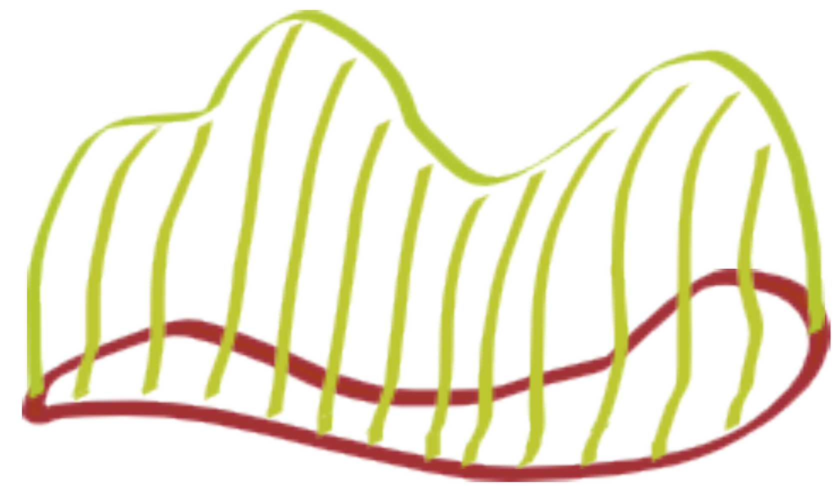

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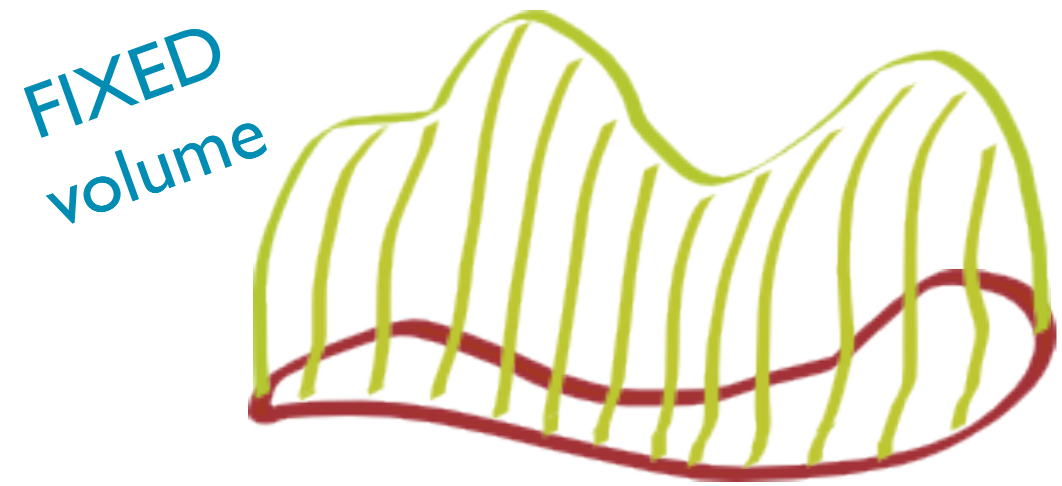

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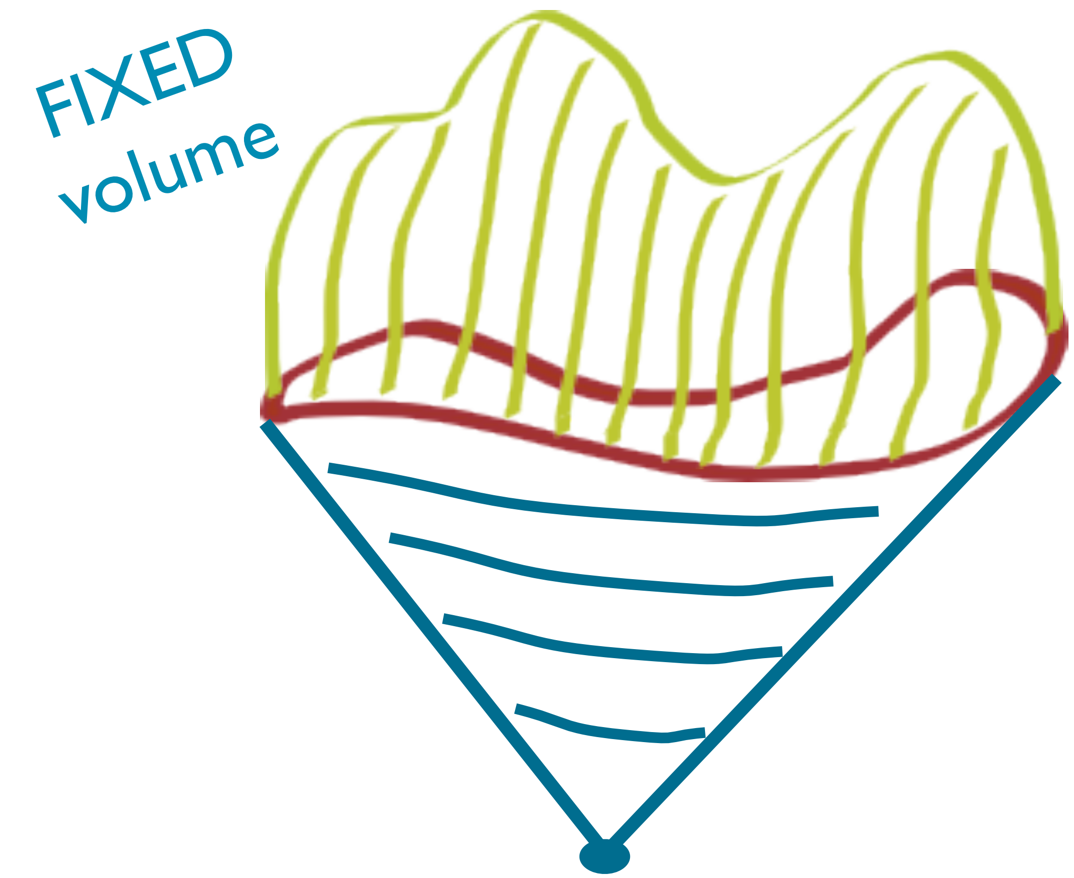

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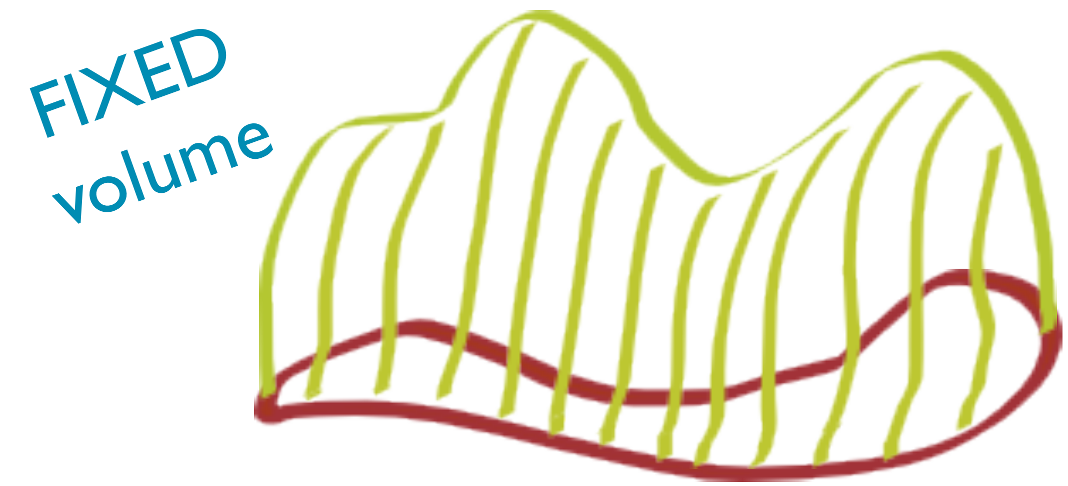

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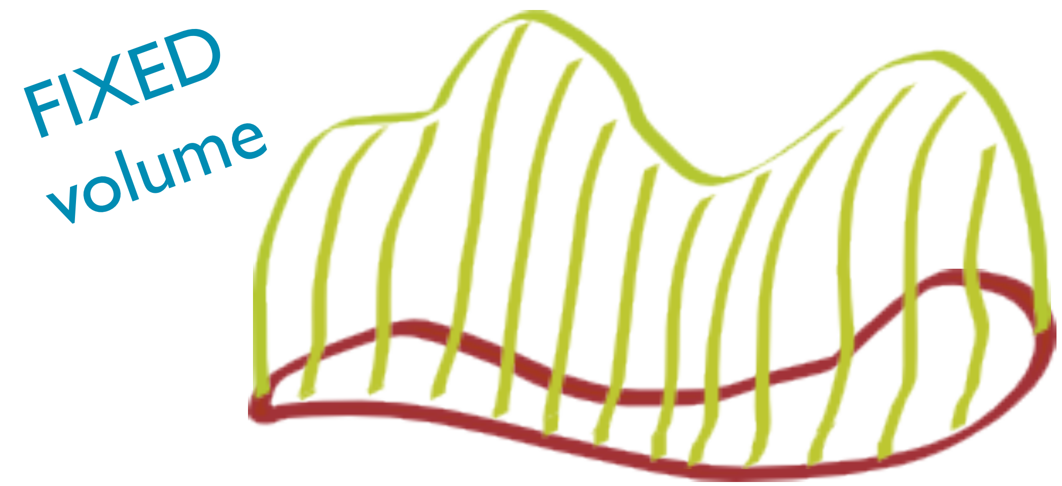

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Its mean curvature is
constant everywhere

(CMC SURFACES)

Some examples in $\mathbb{R}^3$

$H \neq 0$

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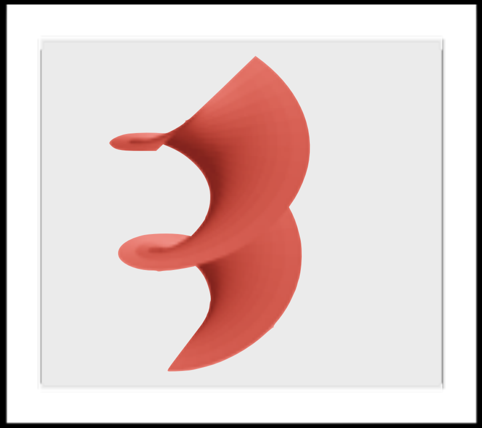
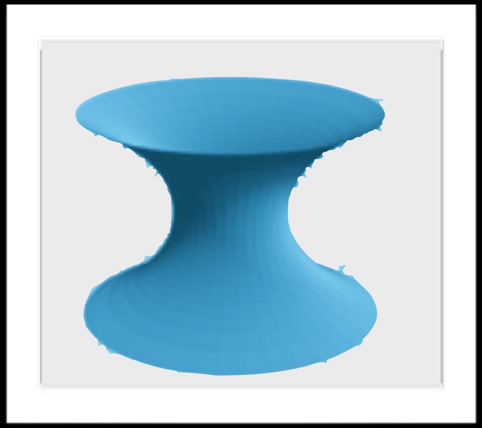
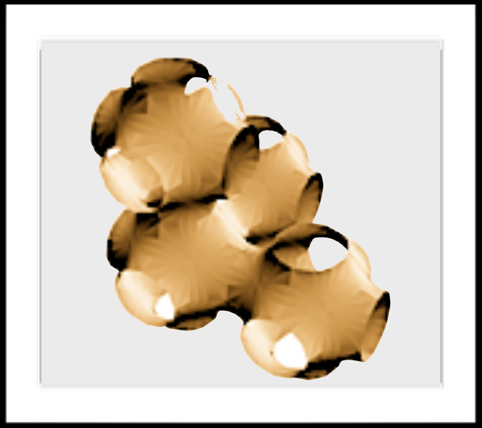
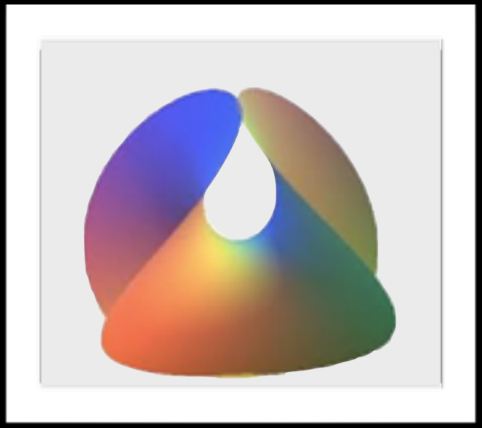



MINIMAL
(H=0)

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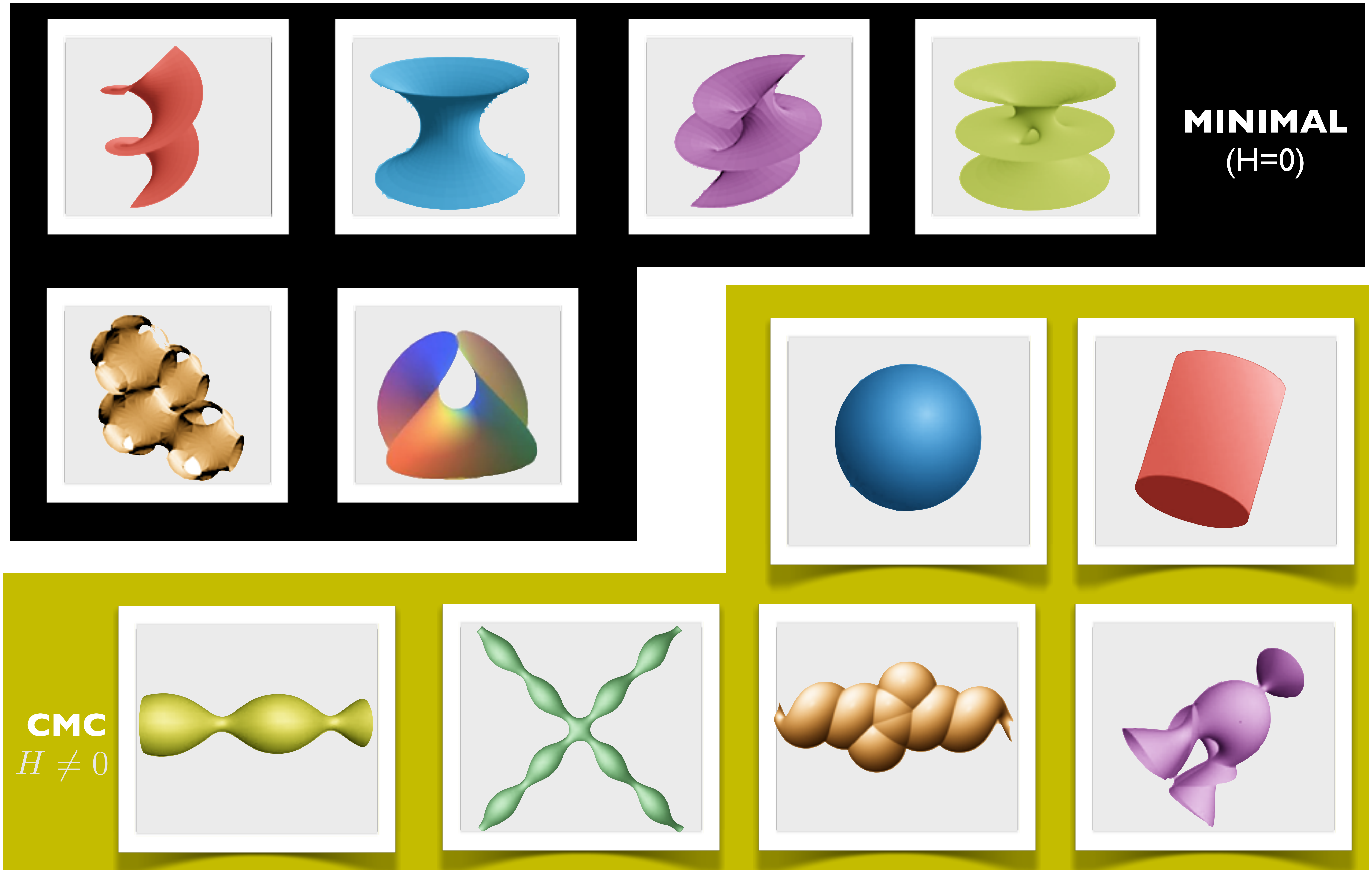

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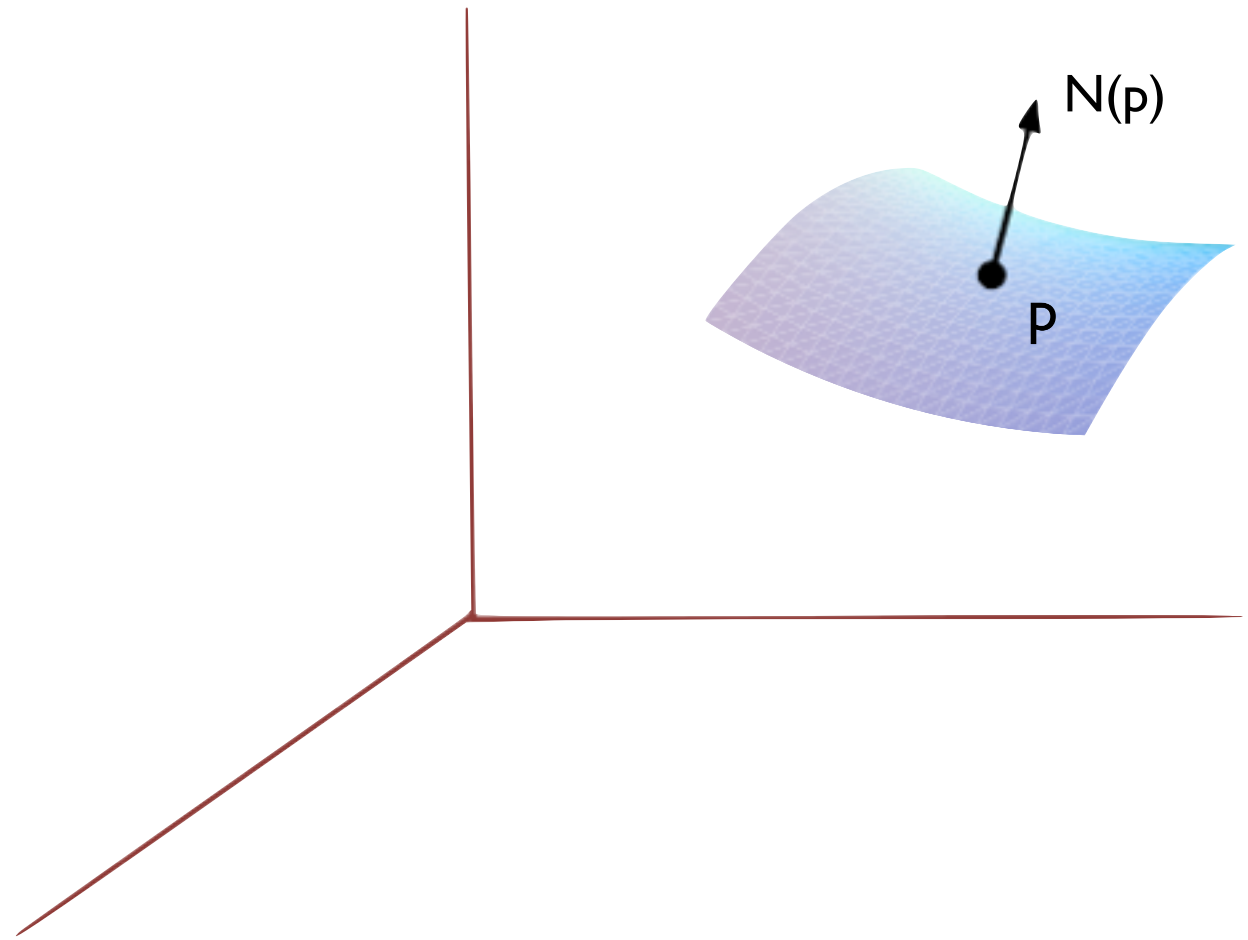
(if the surface is an entire graph, it is a holomorphic map from $\mathbb{C}$ into $\mathbb{S}^2_+$)

The Gauss map in $\mathbb{R}^3$

$\Sigma \subset \mathbb{R}^3$ surface



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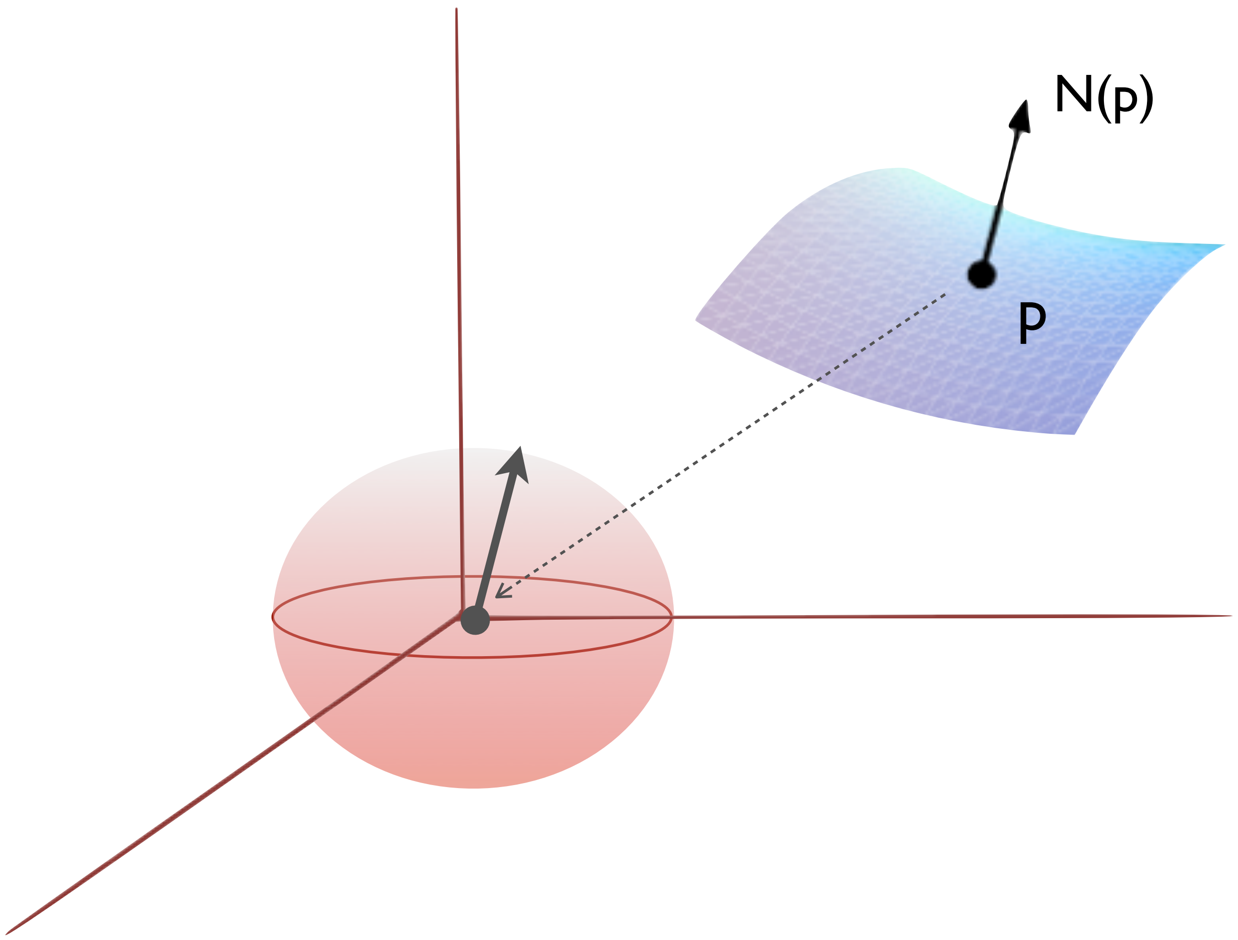

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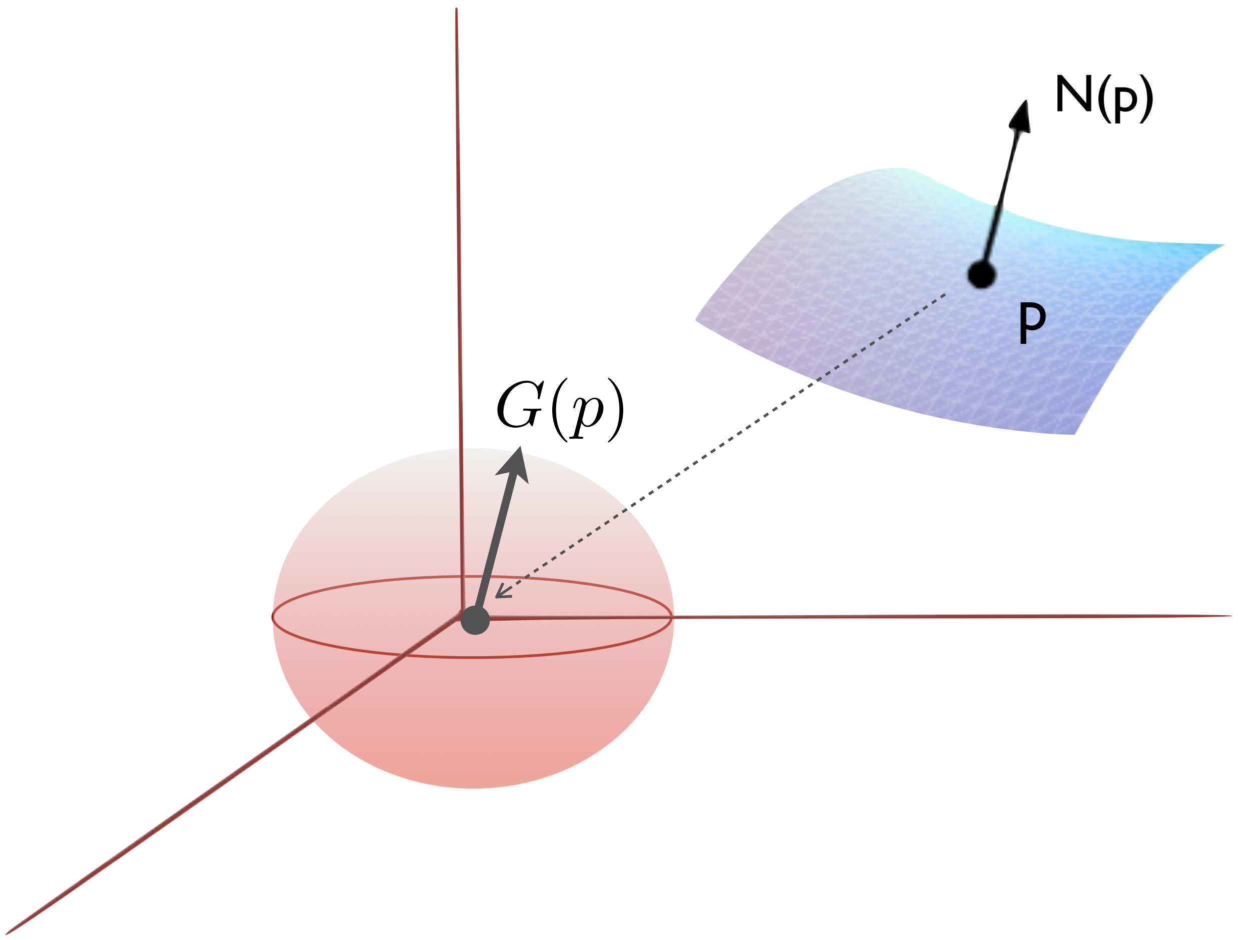
st.proj.



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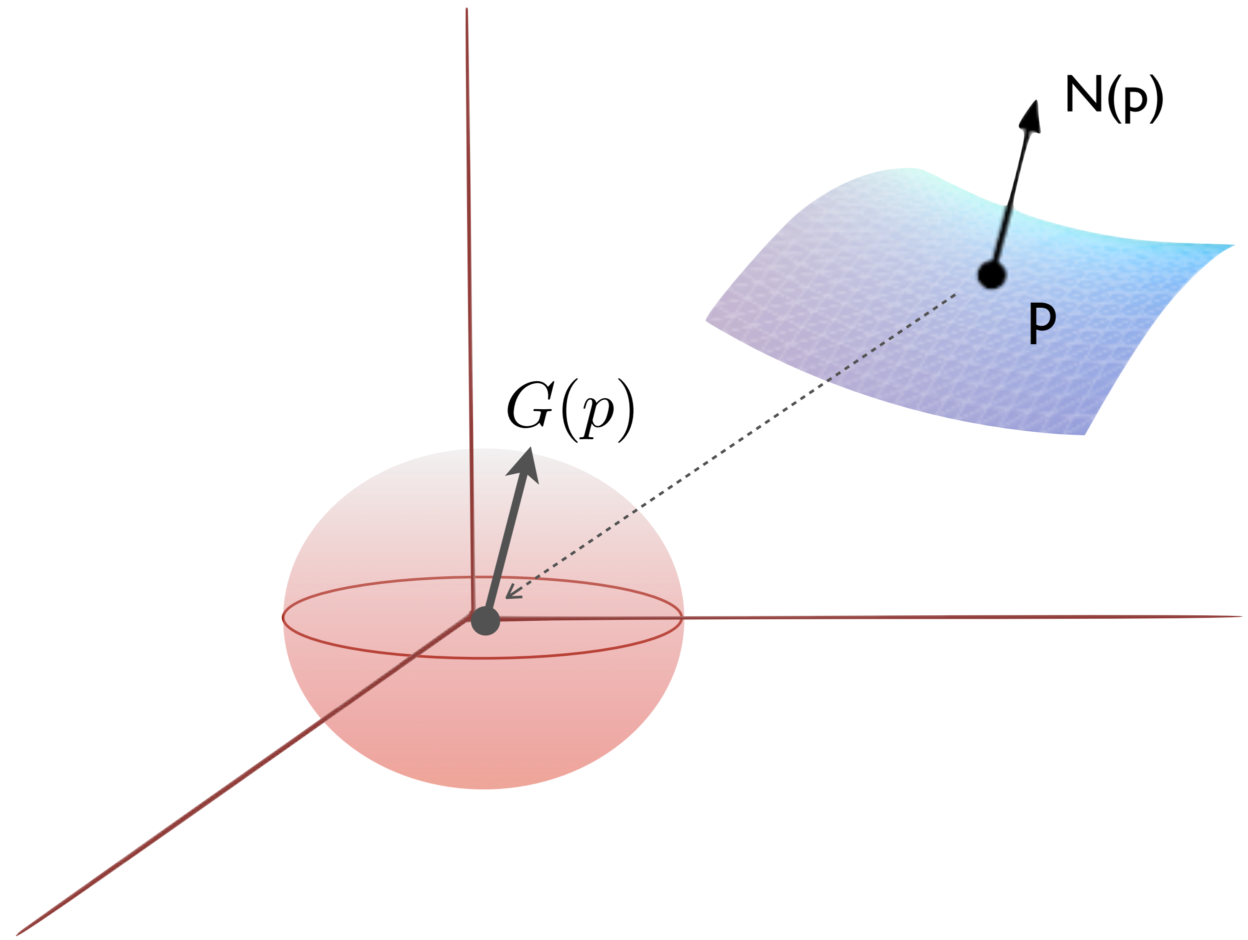

- The Gauss map of CMC surfaces in $\mathbb{R}^3$ is harmonic.
- The Gauss map of minimal surfaces in $\mathbb{R}^3$ is holomorphic.

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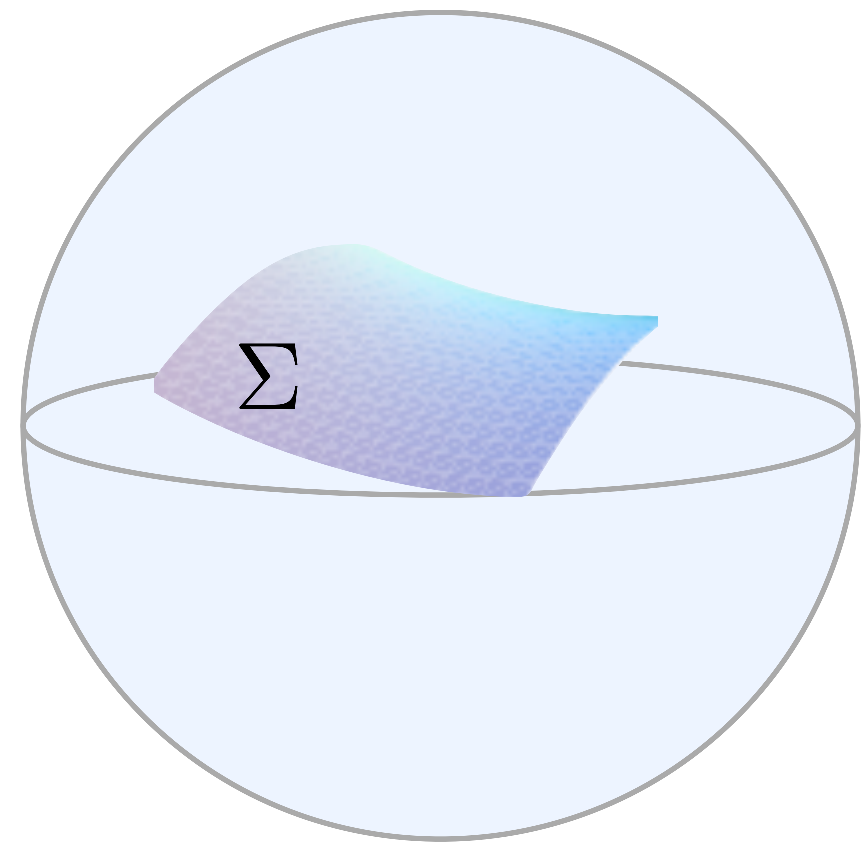
The Gauss map in $\mathbb{H}^3$



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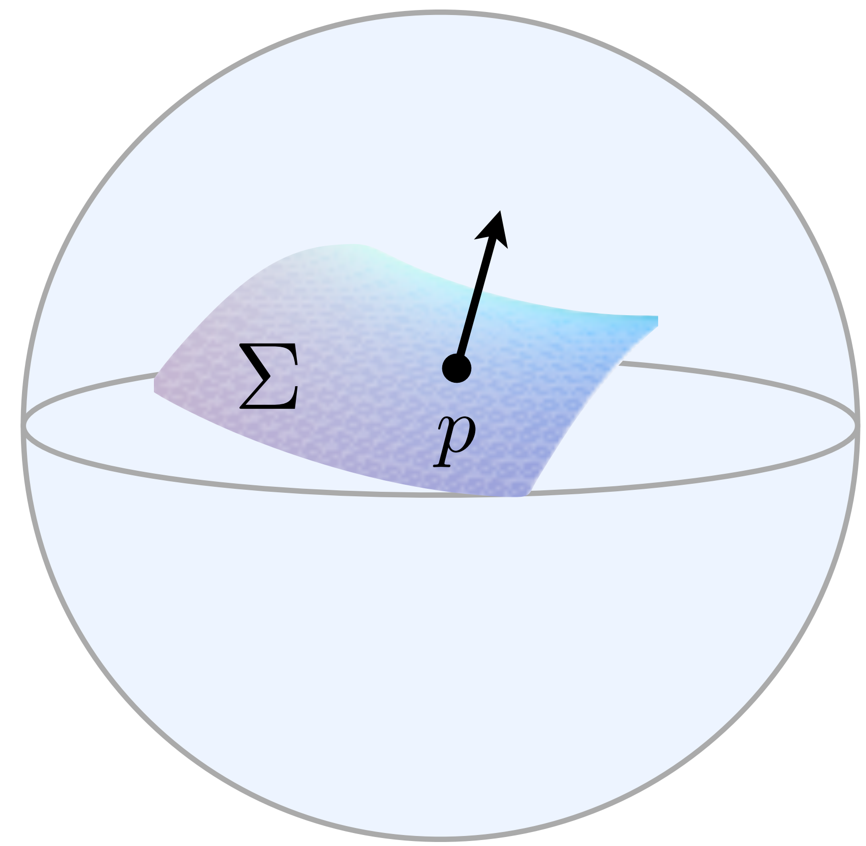

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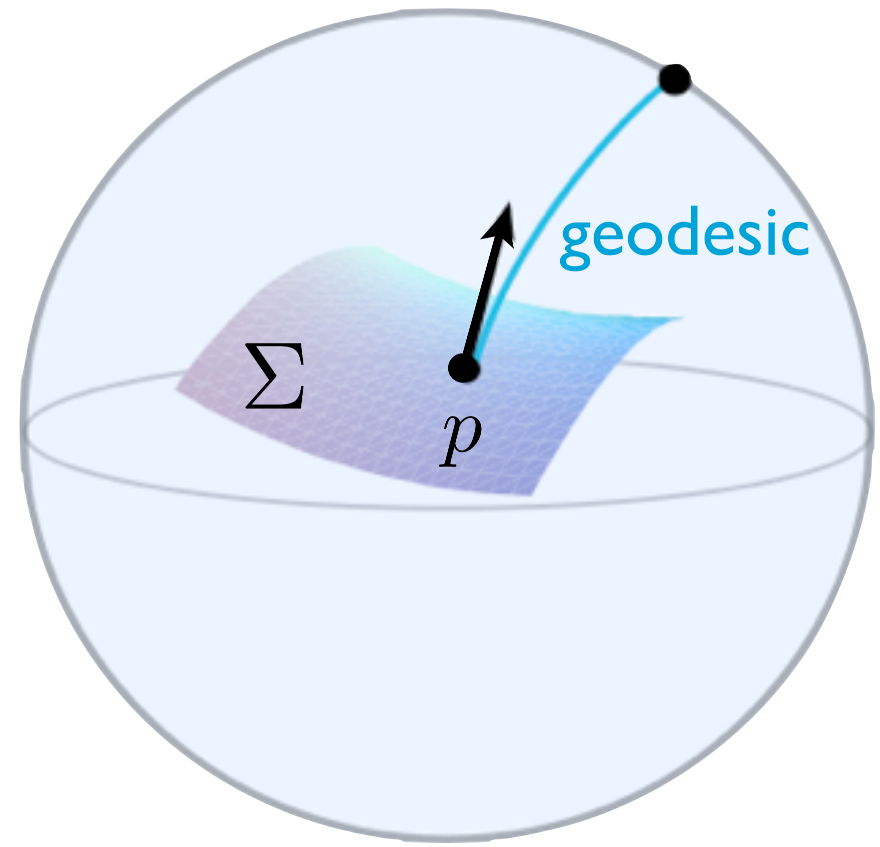

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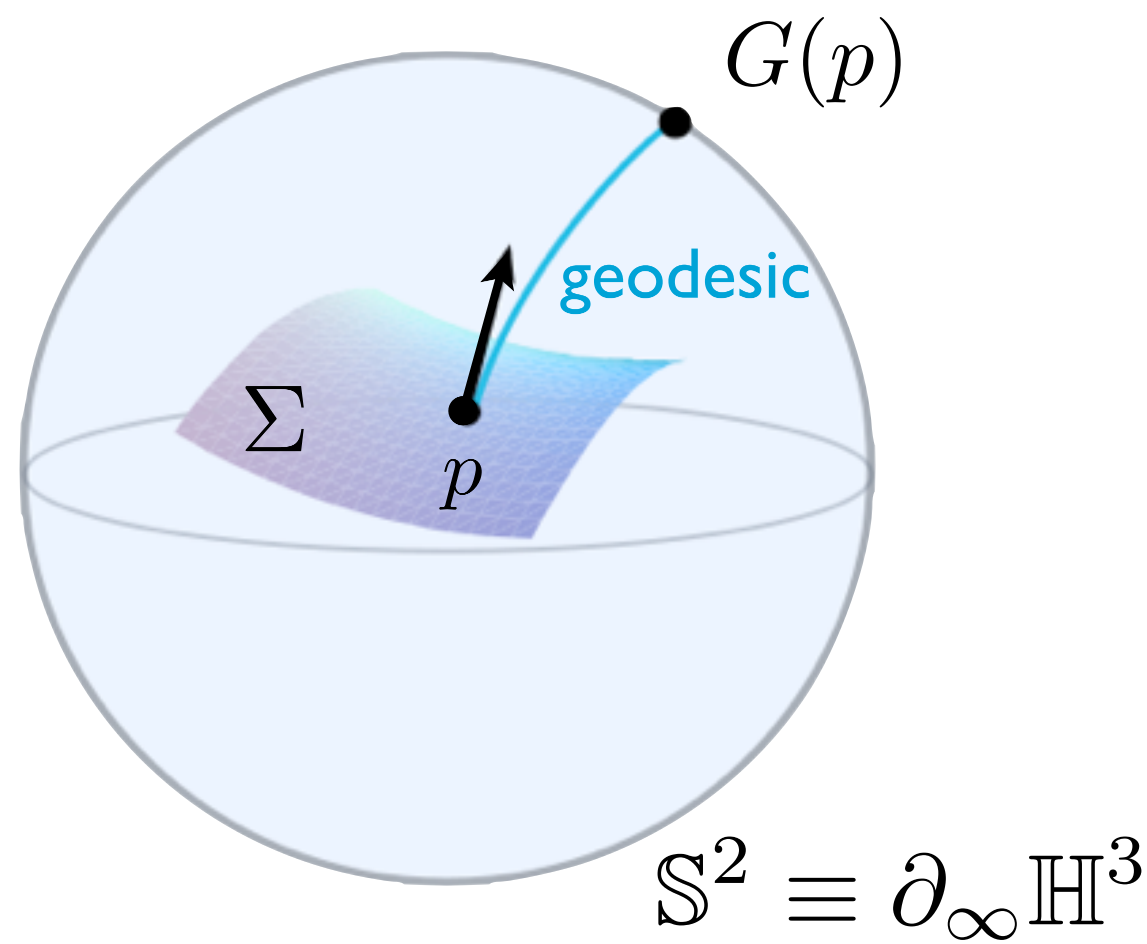

The Gauss map in $\mathbb{H}^3$

"hyperbolic Gauss map"

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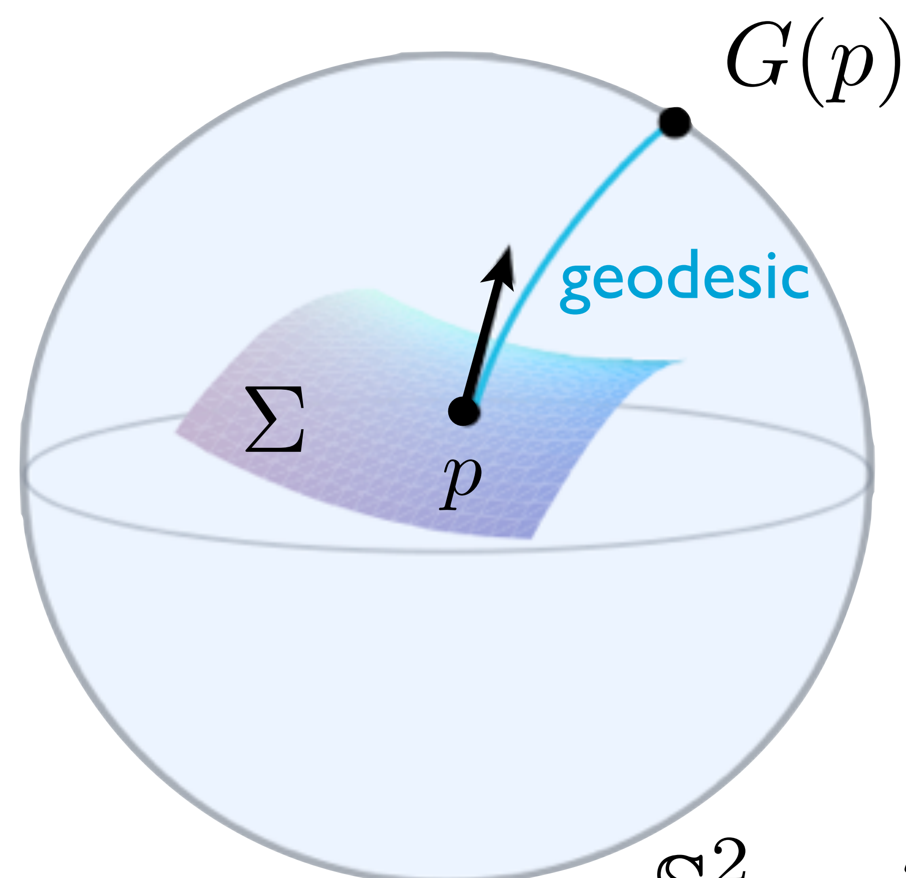
st.proj.



$\mathbb{S}^2 \equiv \partial_\infty \mathbb{H}^3$

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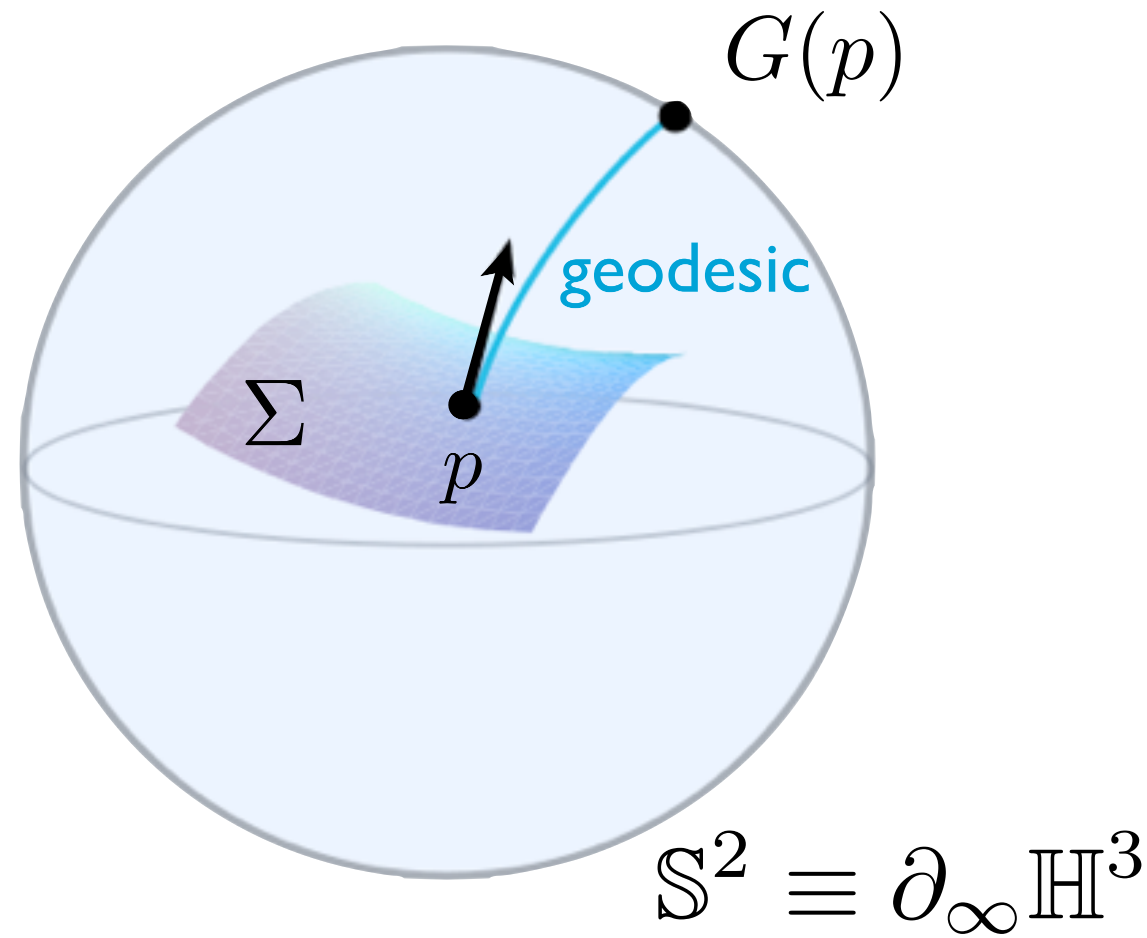

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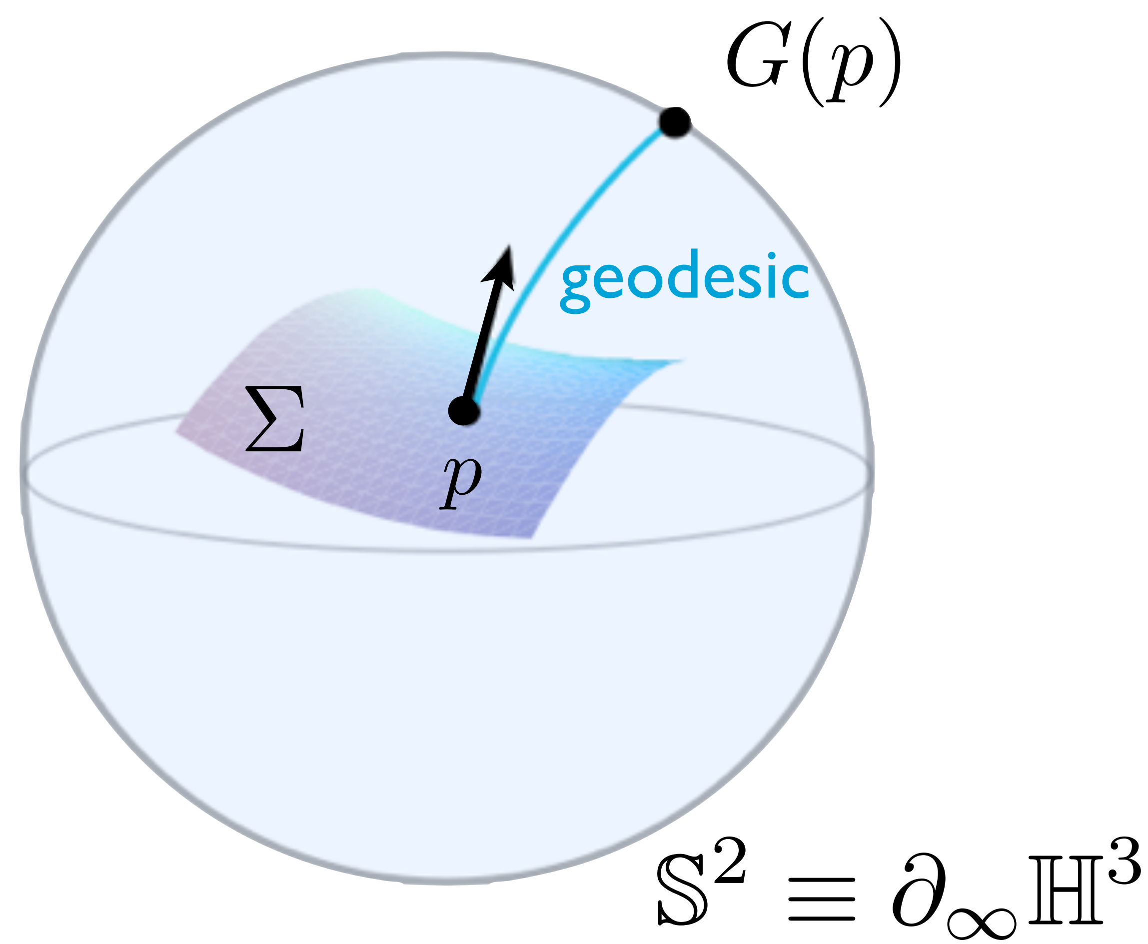

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Bryant surfaces

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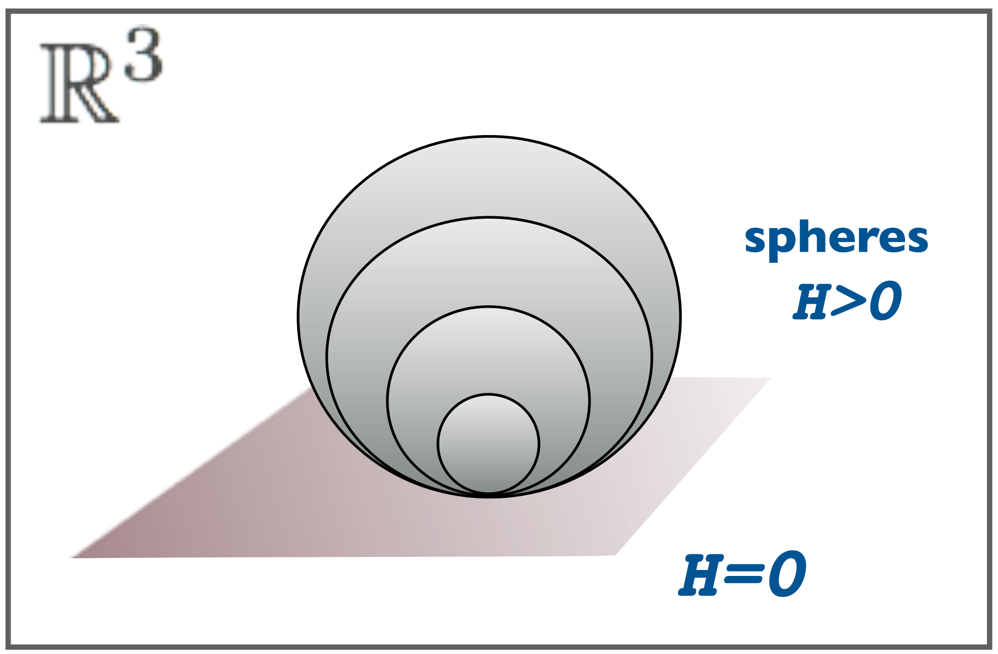

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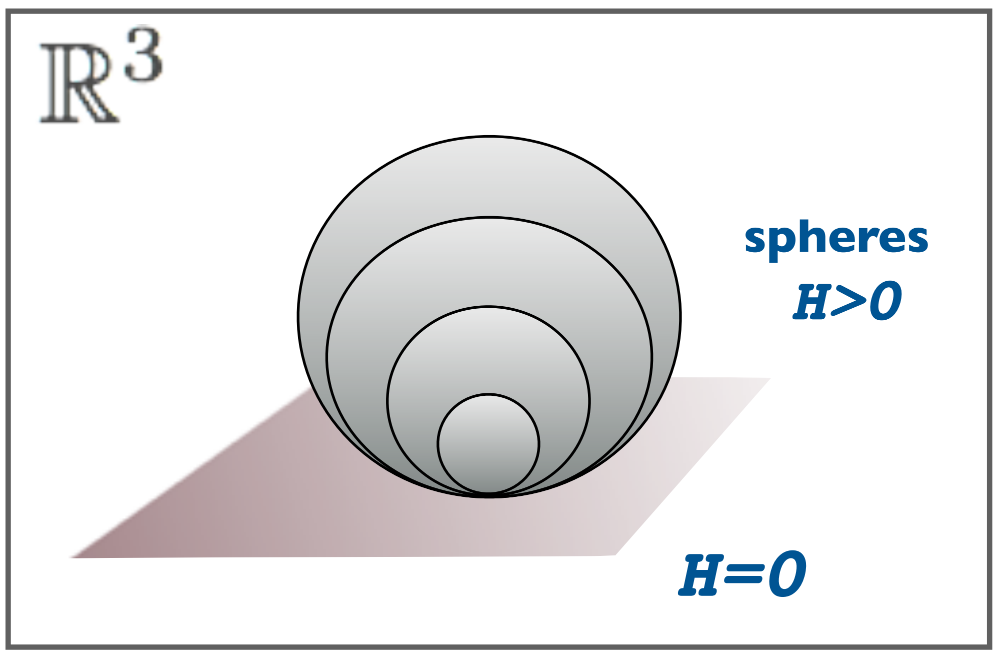

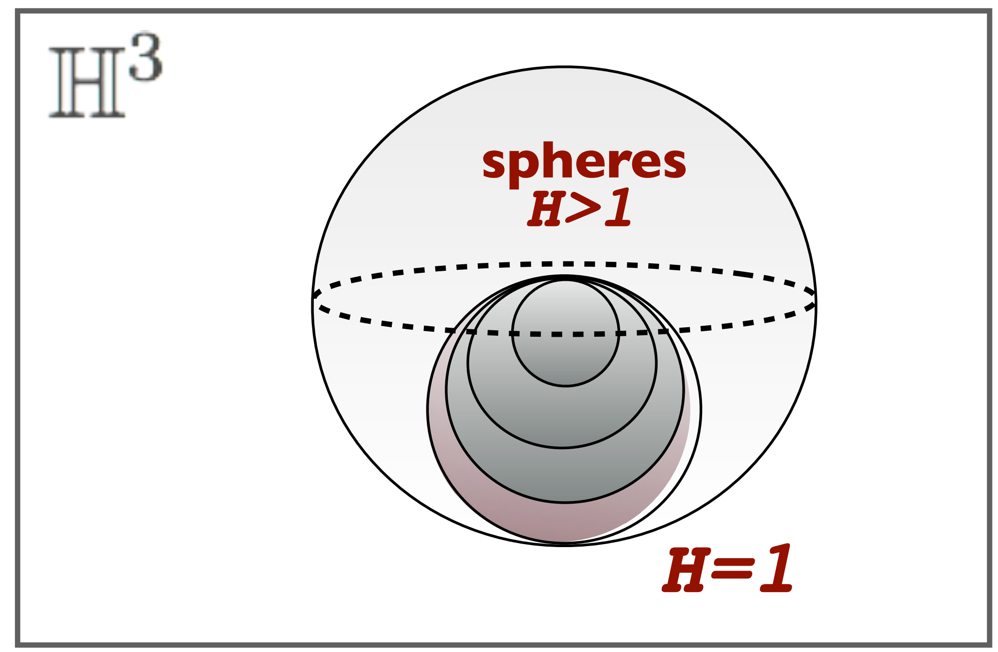

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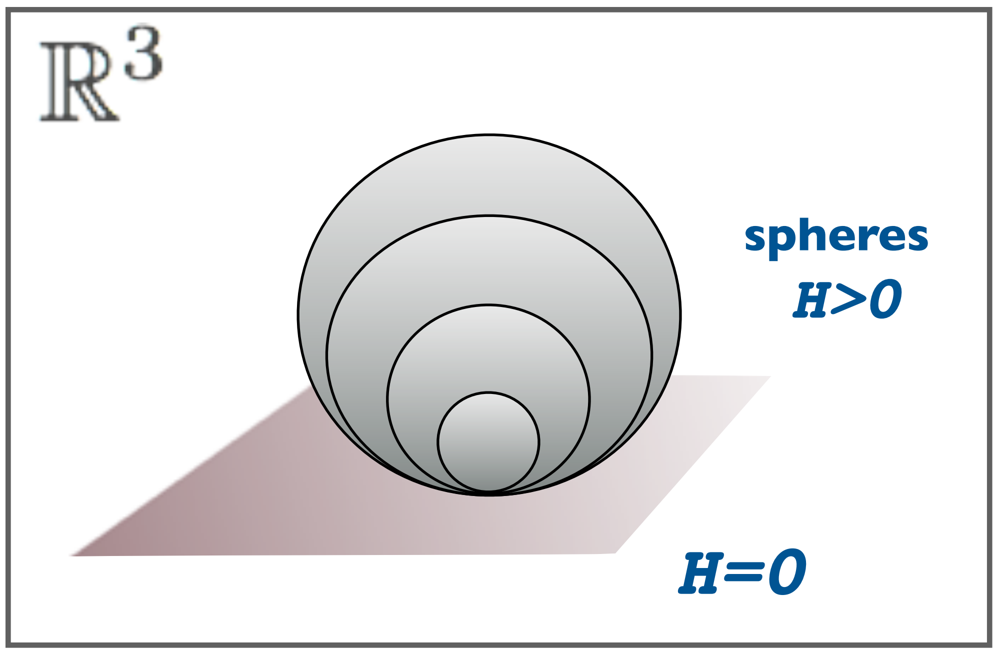

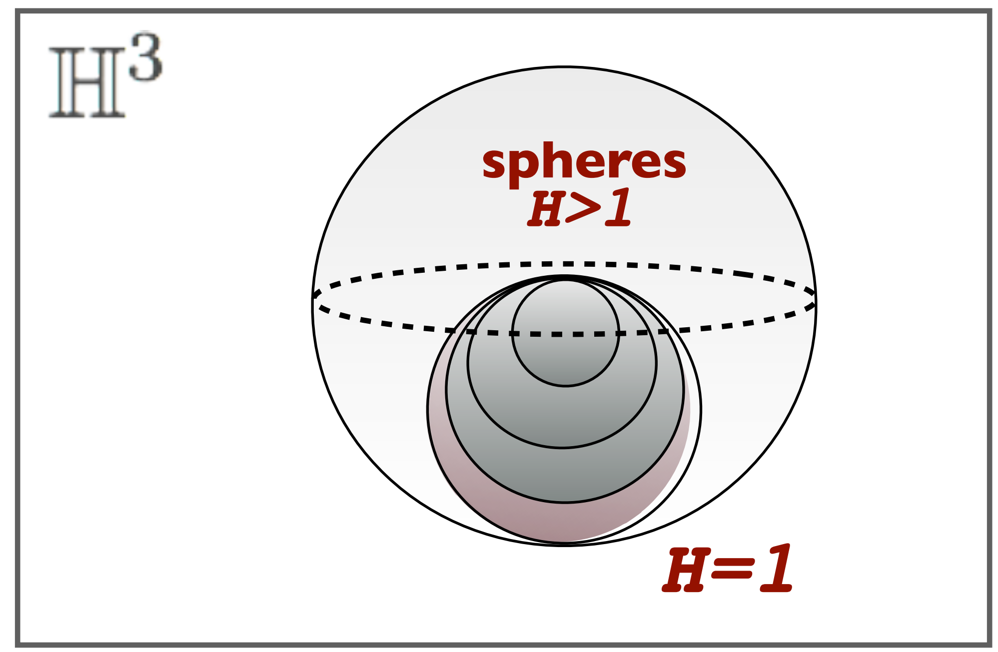

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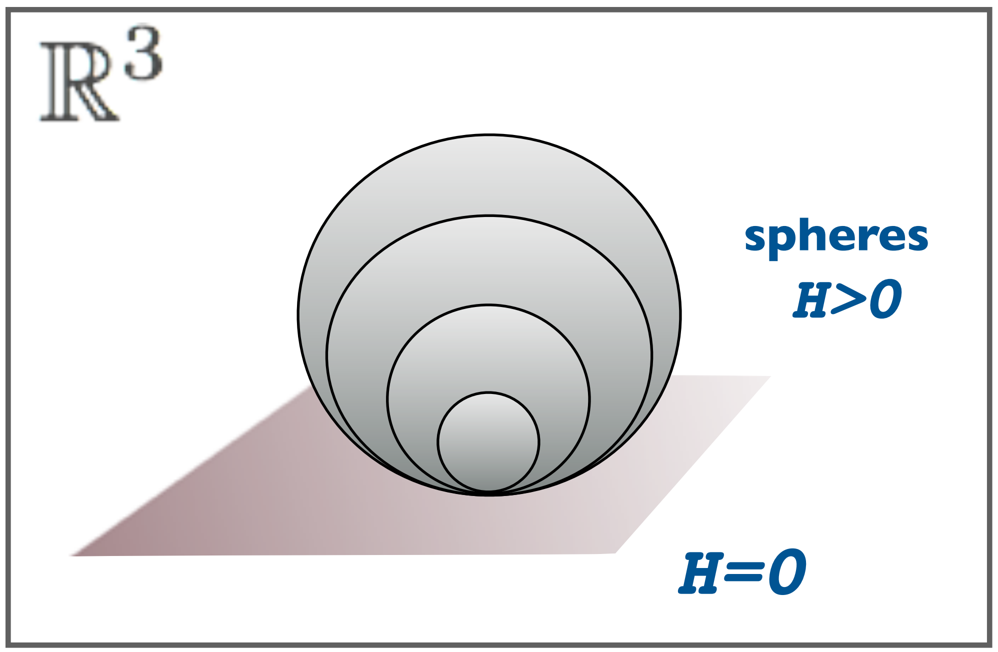

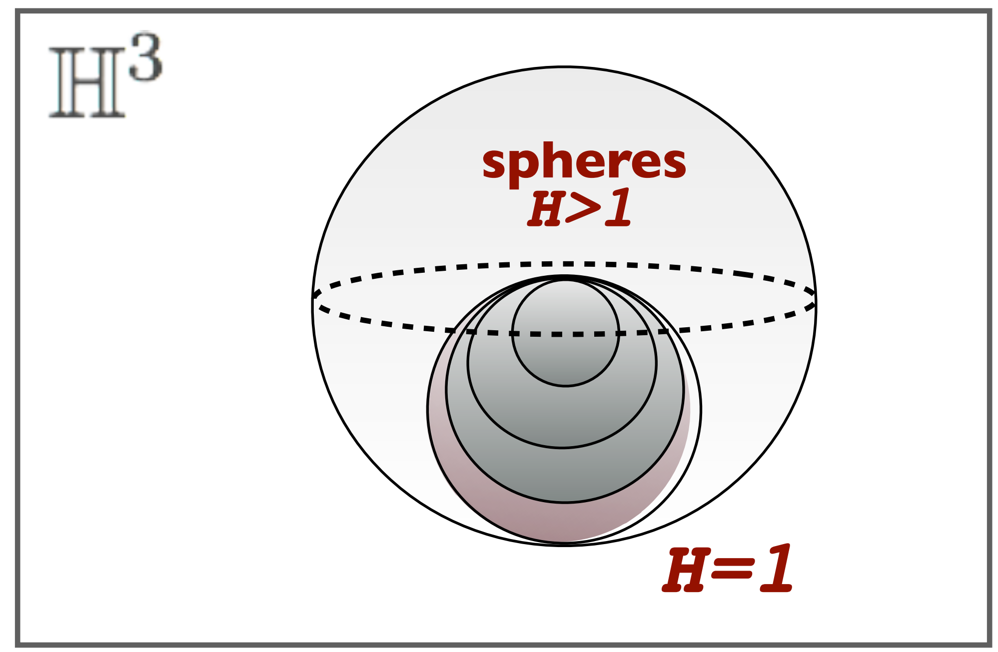

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CRITICAL CMC surfaces in space forms → Minimal (H=0) surfaces in $\mathbb{R}^3$; Bryant (H=1) surfaces in $\mathbb{H}^3$

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$$\mathbb{E}^3(\kappa,\tau) = \mathbb{D}\left(\frac{2}{\sqrt{-\kappa}}\right) \times \mathbb{R} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1^2 + x_2^2 < -4/\kappa\}$$

$$ds^2 = \Lambda^2(\mathrm{d}x_1^2 + \mathrm{d}x_2^2) + (\tau\Lambda(x_2\mathrm{d}x_1 - x_1\mathrm{d}x_2) + \mathrm{d}x_3)^2 \qquad \Lambda = \frac{1}{1 + \frac{\kappa}{4}(x_1^2 + x_2^2)}$$

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base curvat. / bundle curvat.	$\kappa < 0$	$\kappa = 0$	$\kappa > 0$
$\tau = 0$	$\mathbb{H}^2 \times \mathbb{R}$	~~$\mathbb{R}^3$~~	$\mathbb{S}^2 \times \mathbb{R}$
$\tau \neq 0$	$\widetilde{\mathrm{PSL}(2, \mathbb{R})}$	Nil_3	Ber_3

Critical Mean Curvature

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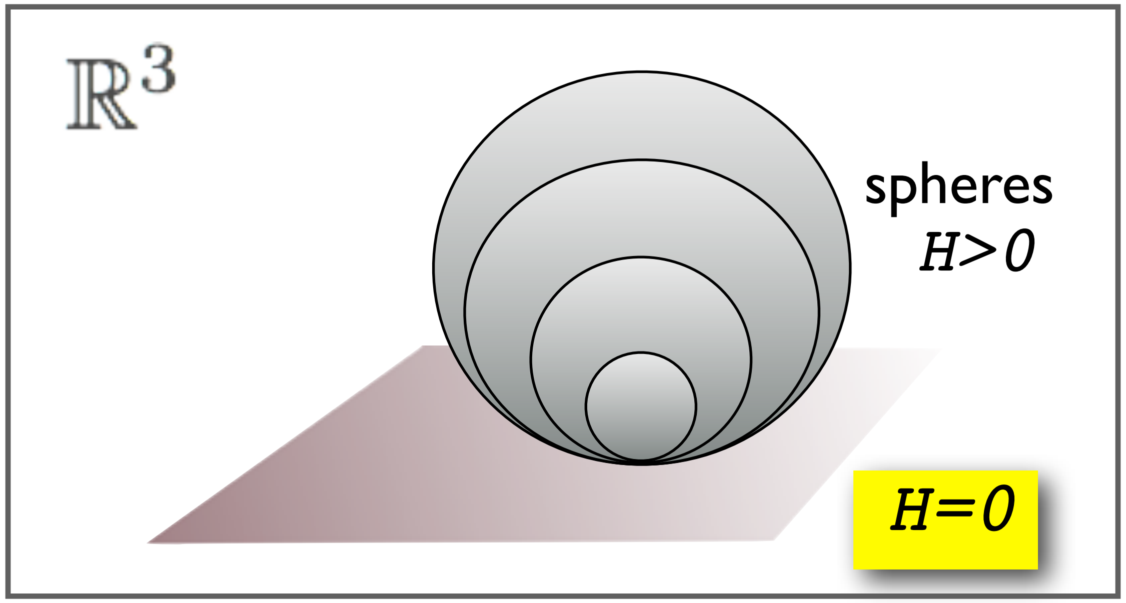

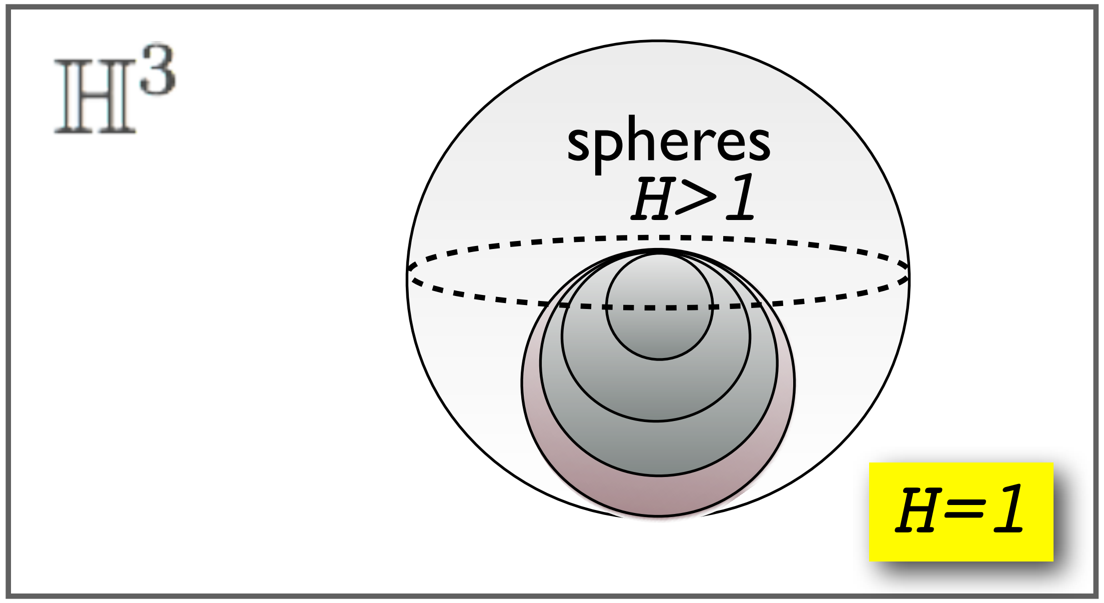

Critical Mean Curvature

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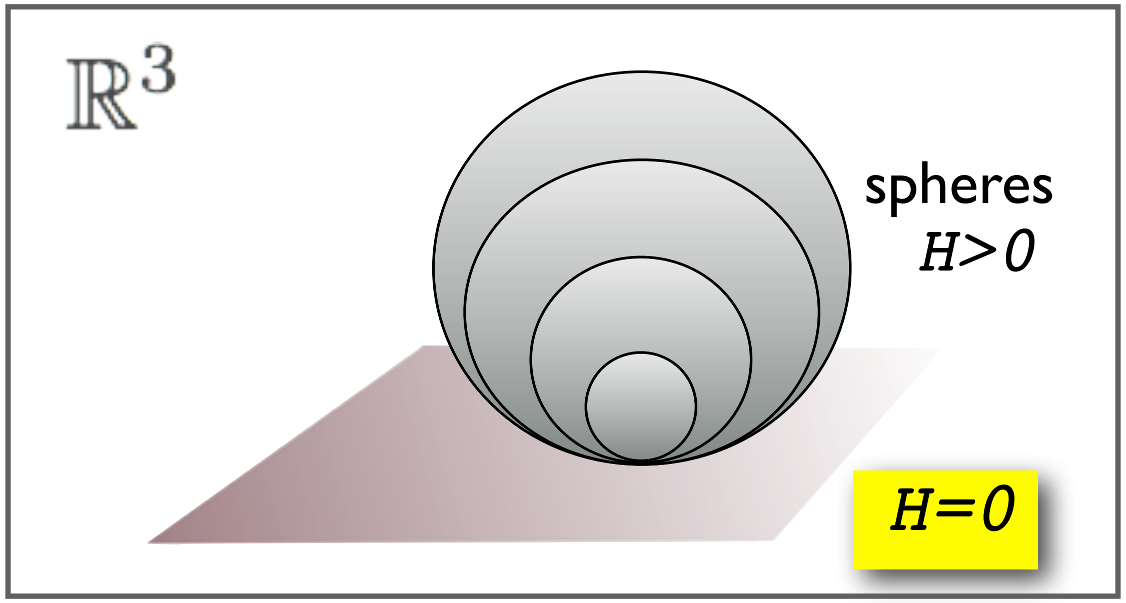

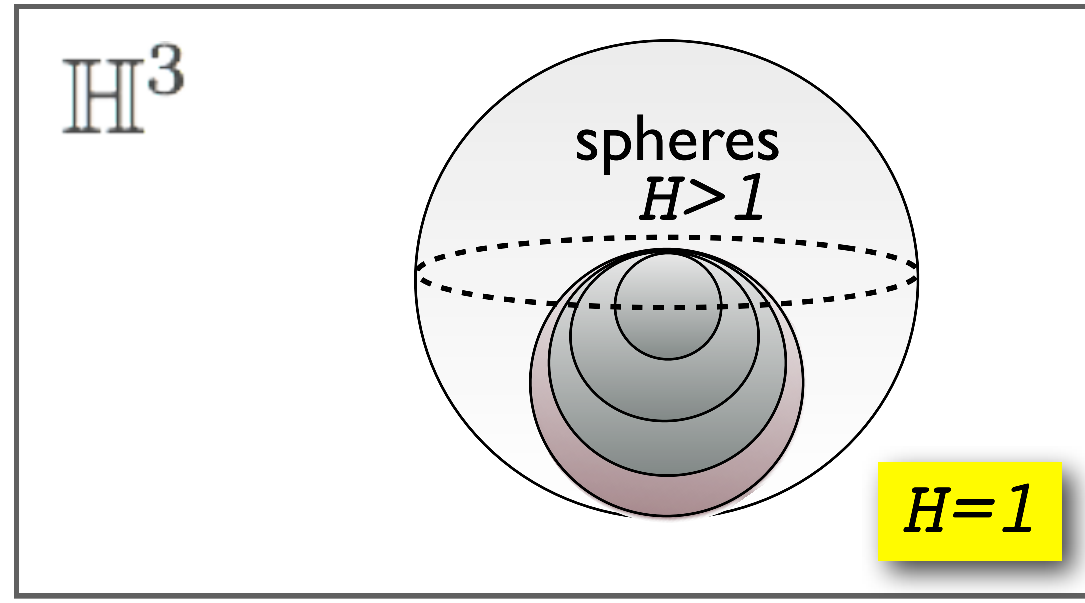

Critical value in $\mathbb{E}^3(\kappa,\tau)$ →

$$H = \frac{\sqrt{-k}}{2}$$

$k \leq 0$

	$\kappa < 0$	$\kappa = 0$	$\kappa > 0$
$\tau = 0$	$\mathbb{H}^2 \times \mathbb{R}$	~~$\mathbb{R}^3$~~	$\mathbb{S}^2 \times \mathbb{R}$
$\tau \neq 0$	$\widetilde{\mathrm{PSL}}(2,\mathbb{R})$	Nil_3	Ber_3

Extension of Hopf thm to $\mathbb{E}^3(\kappa, \tau)$

Hopf (1951): Any topological sphere with CMC in $\mathbb{R}^3$ is a round sphere

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PROOF: There is a modification of the Hopf differential that is holomorphic for CMC surfaces!! (AR differential)

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Classification of entire graphs with zero CMC in $\mathbb{R}^3$: they are all planes.

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SPACE FORMS vs. HOMOGENOUS FIBRATIONS

$\mathbb{R}^3, \mathbb{S}^3, \mathbb{H}^3$

simply connected
homogeneous 3-spaces
with dim(Iso)= **6**

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Σ

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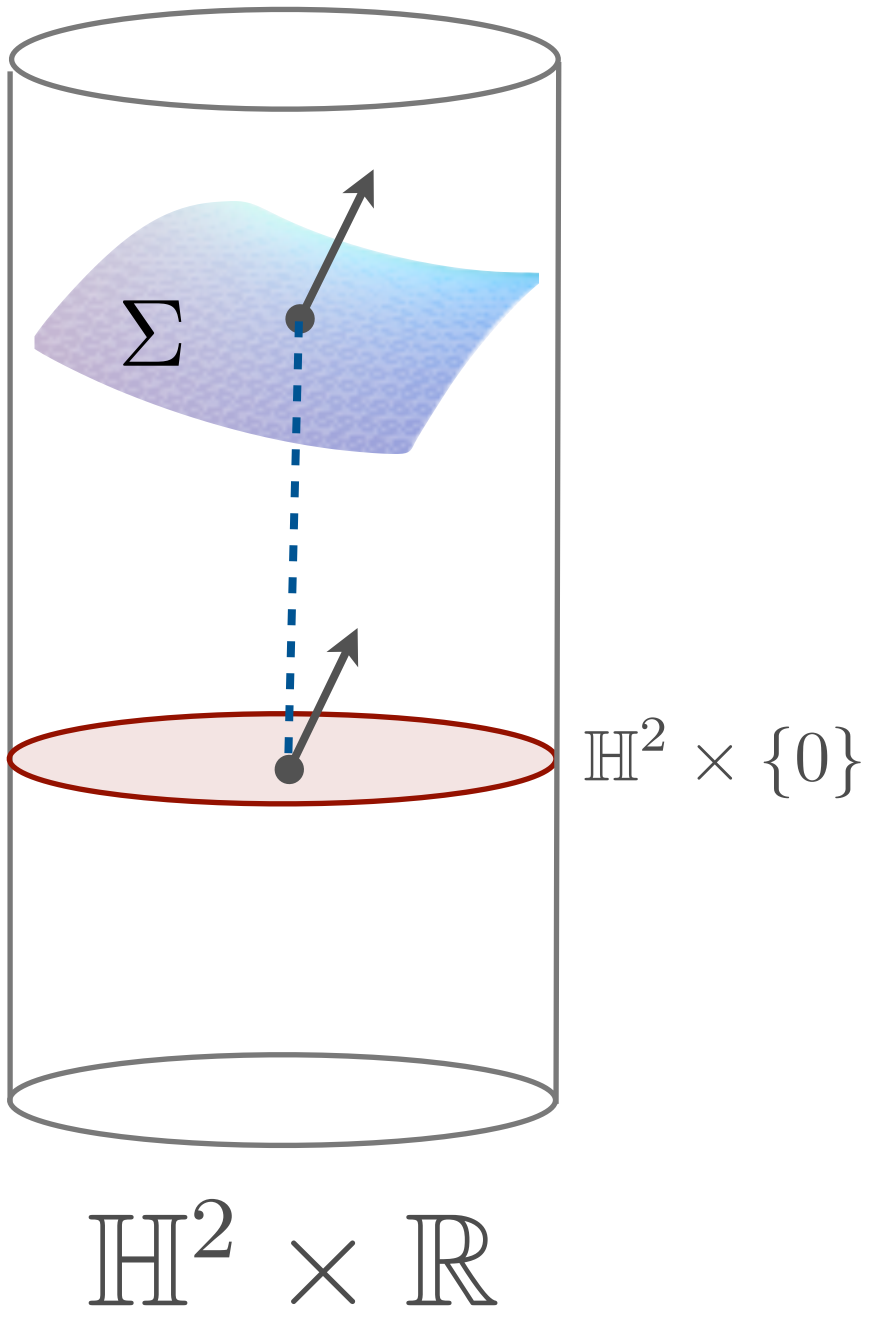

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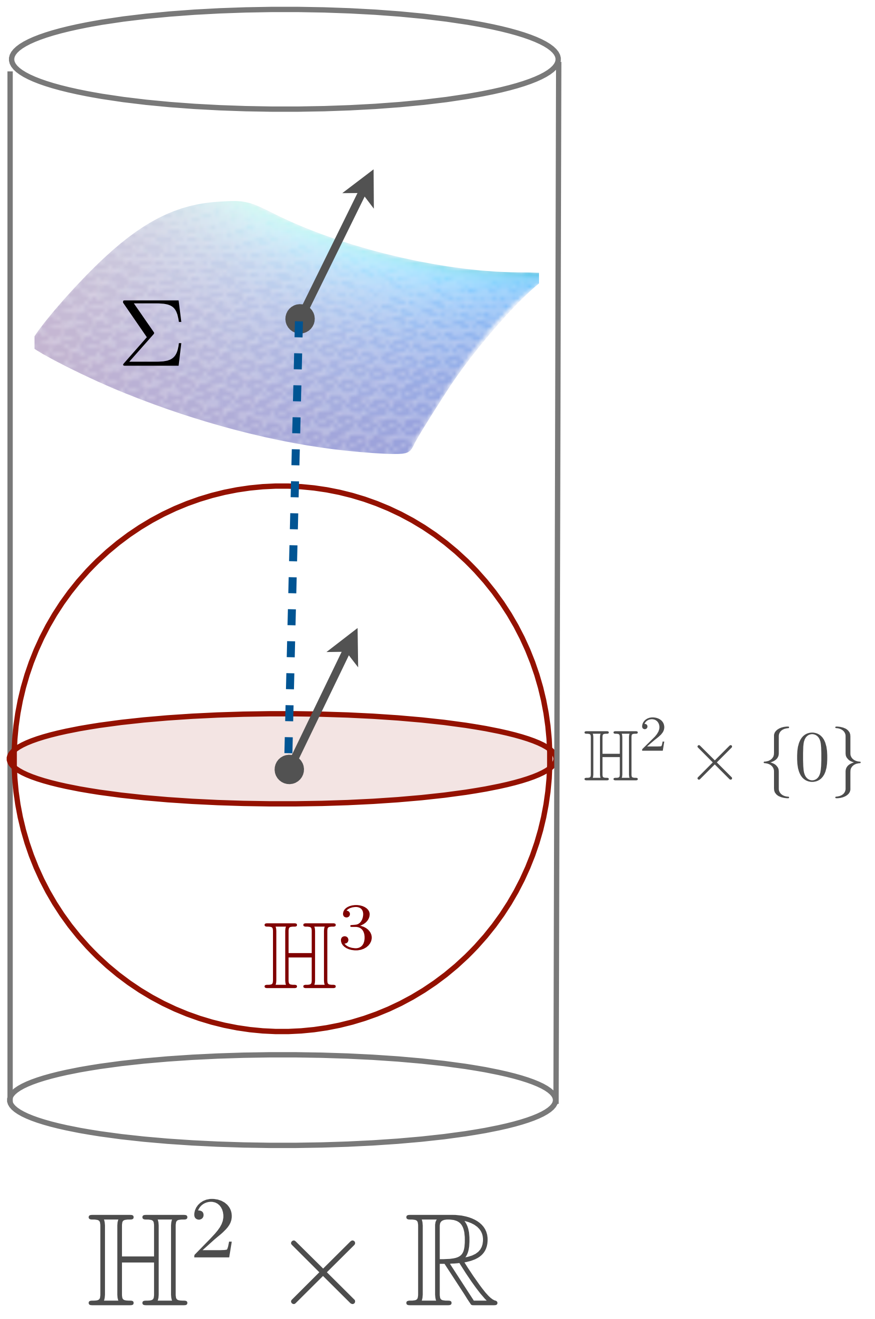

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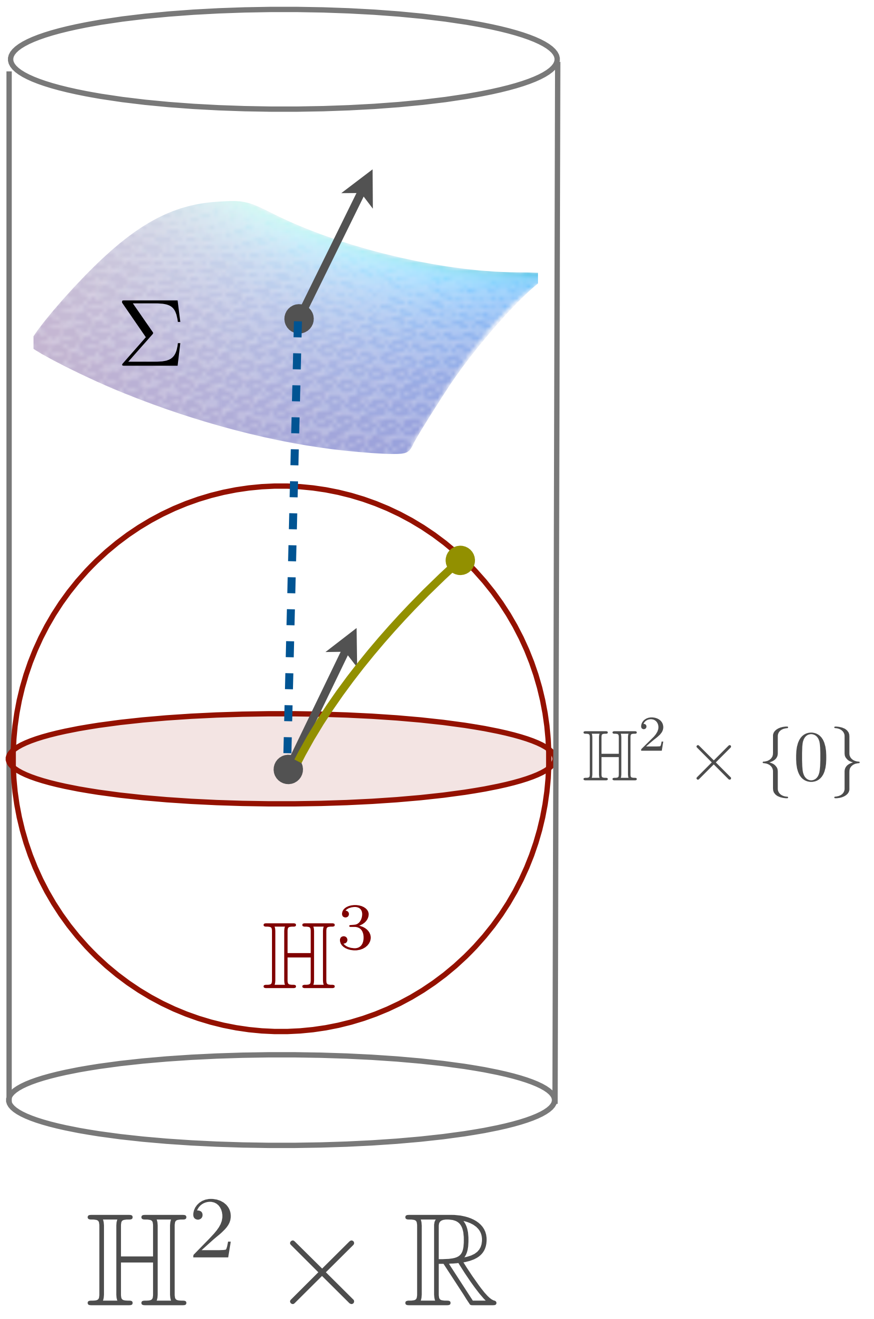

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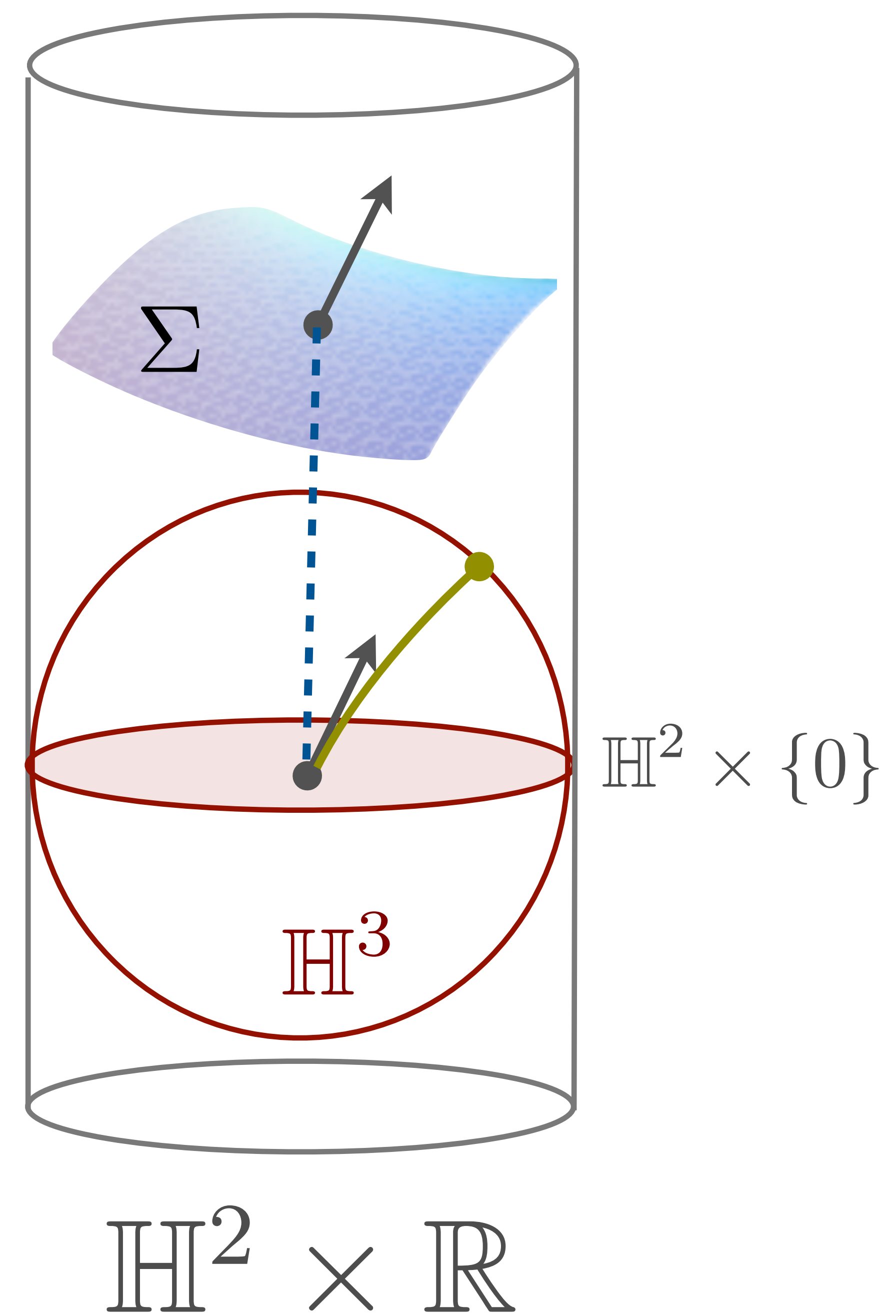

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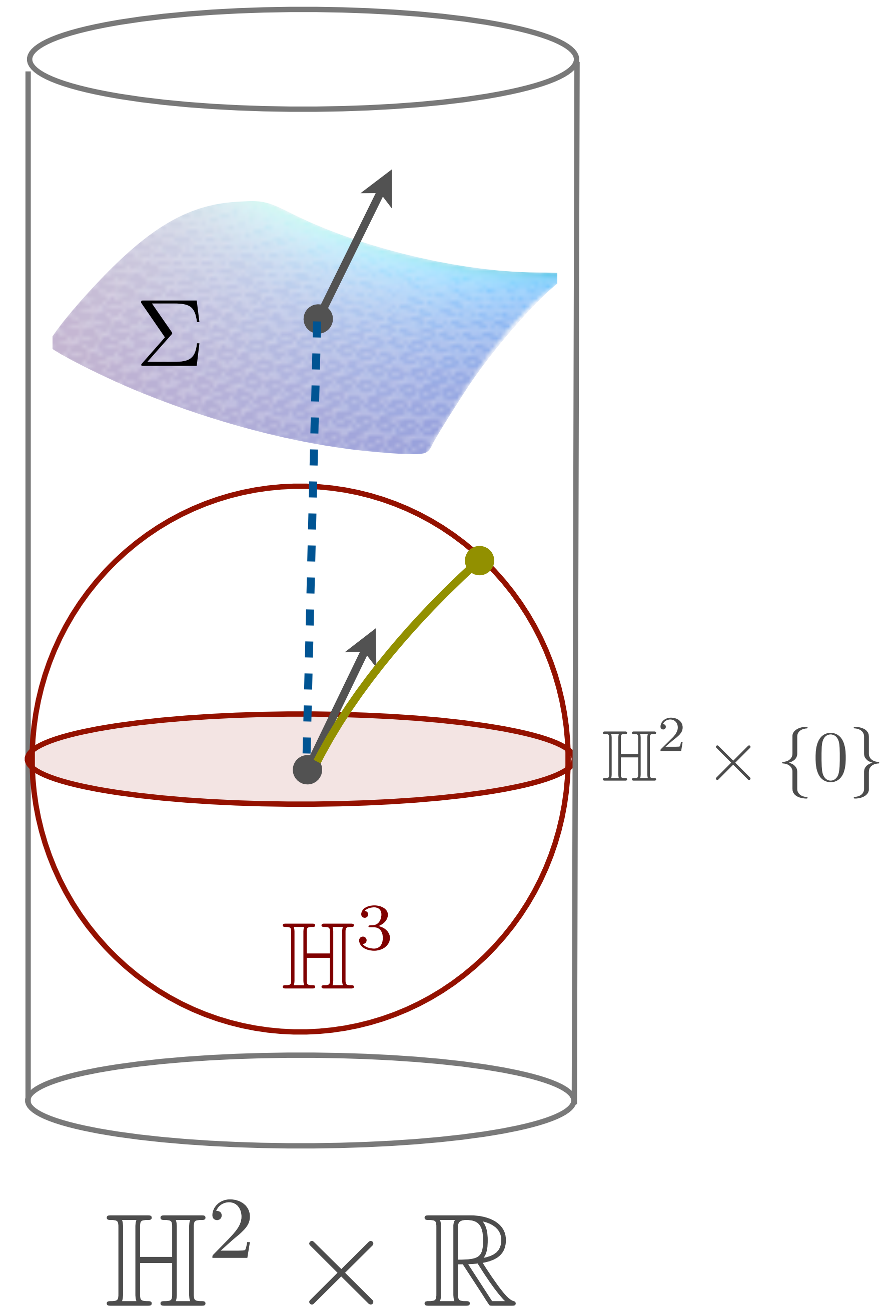

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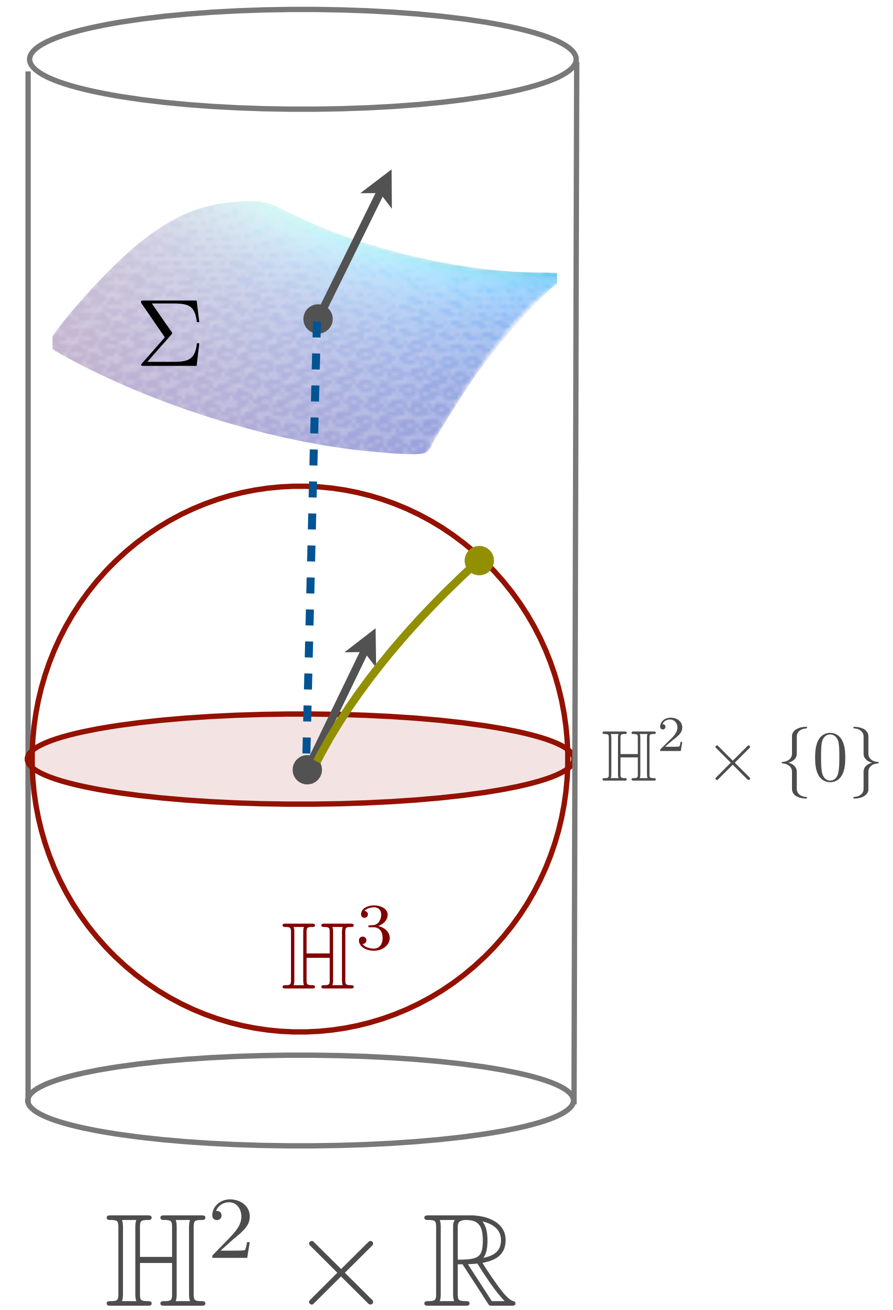

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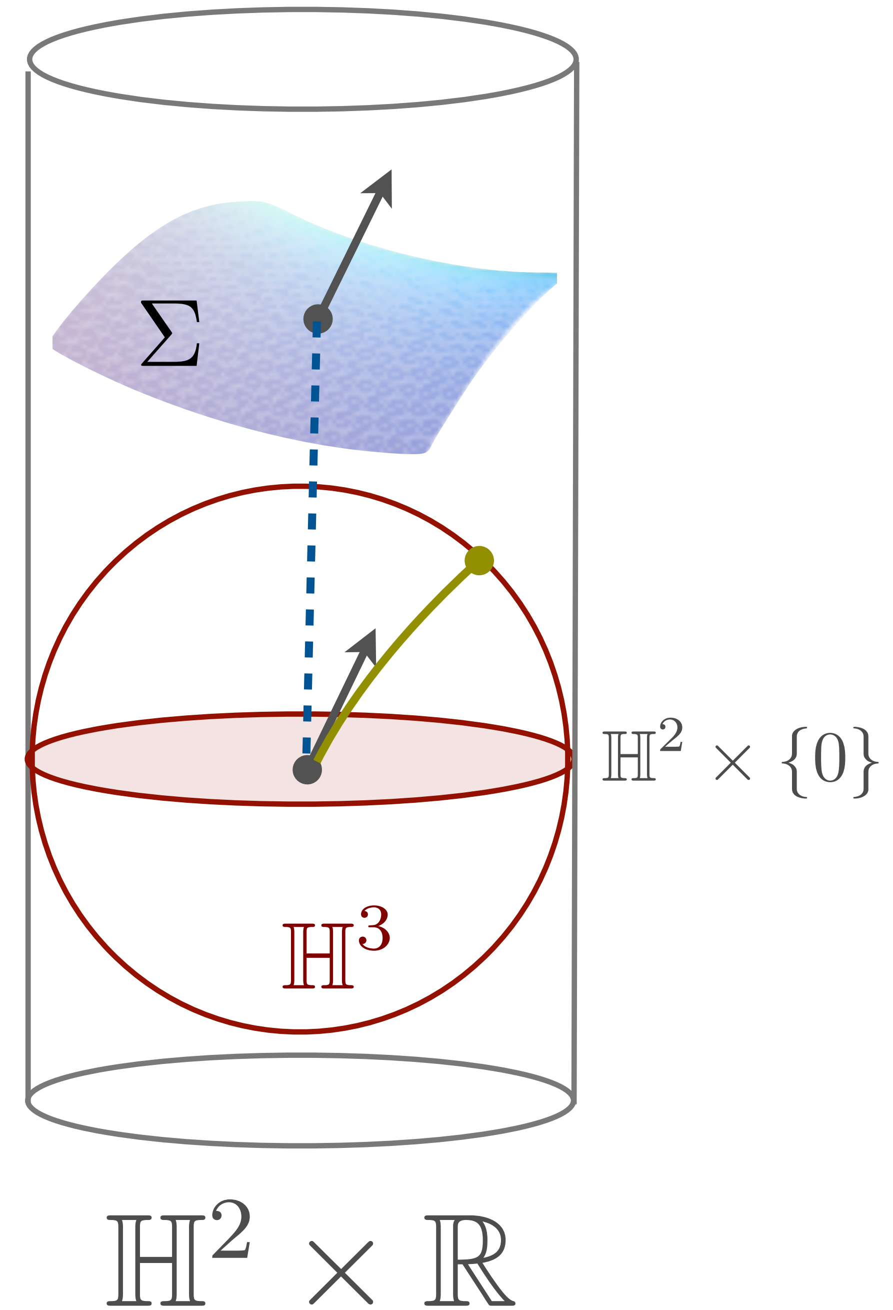

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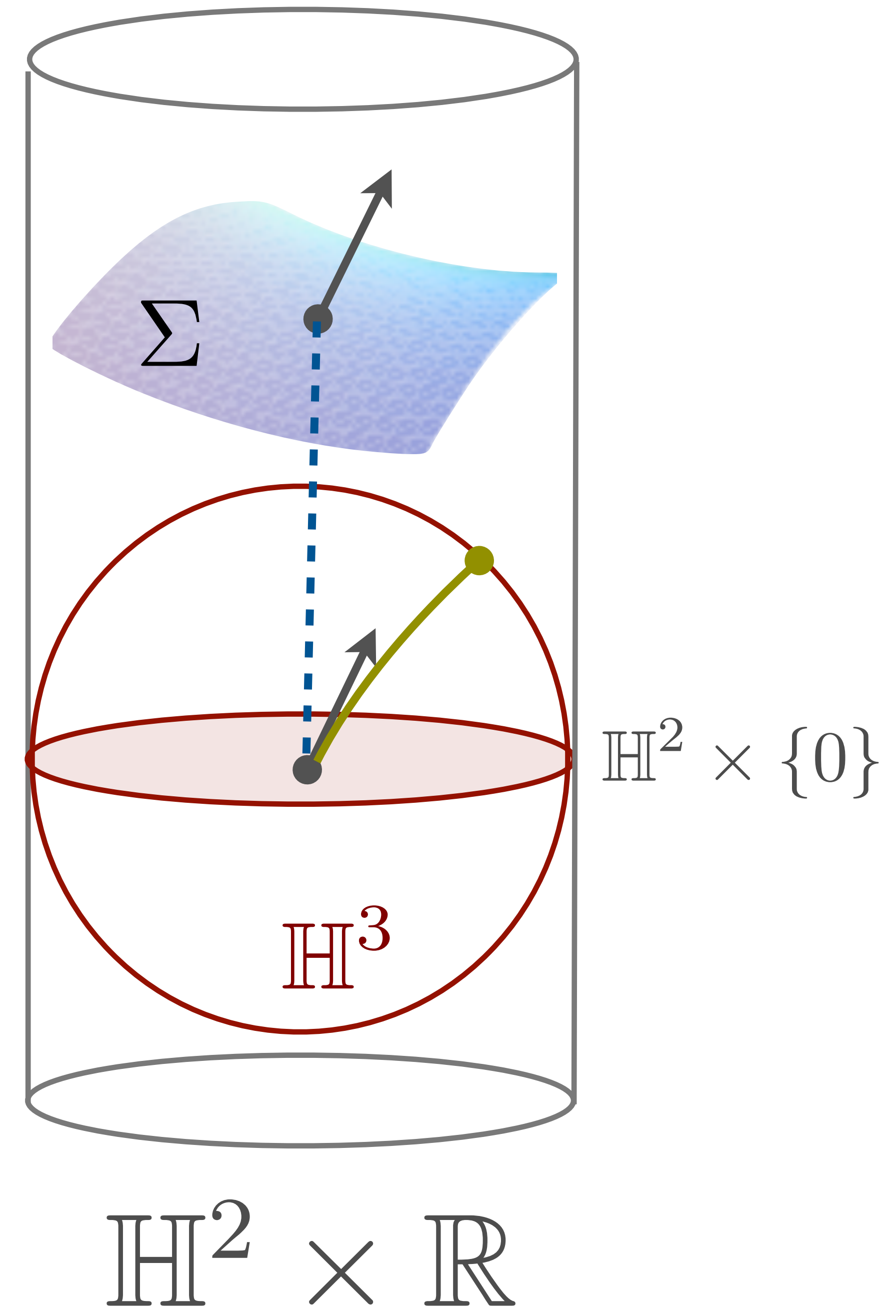

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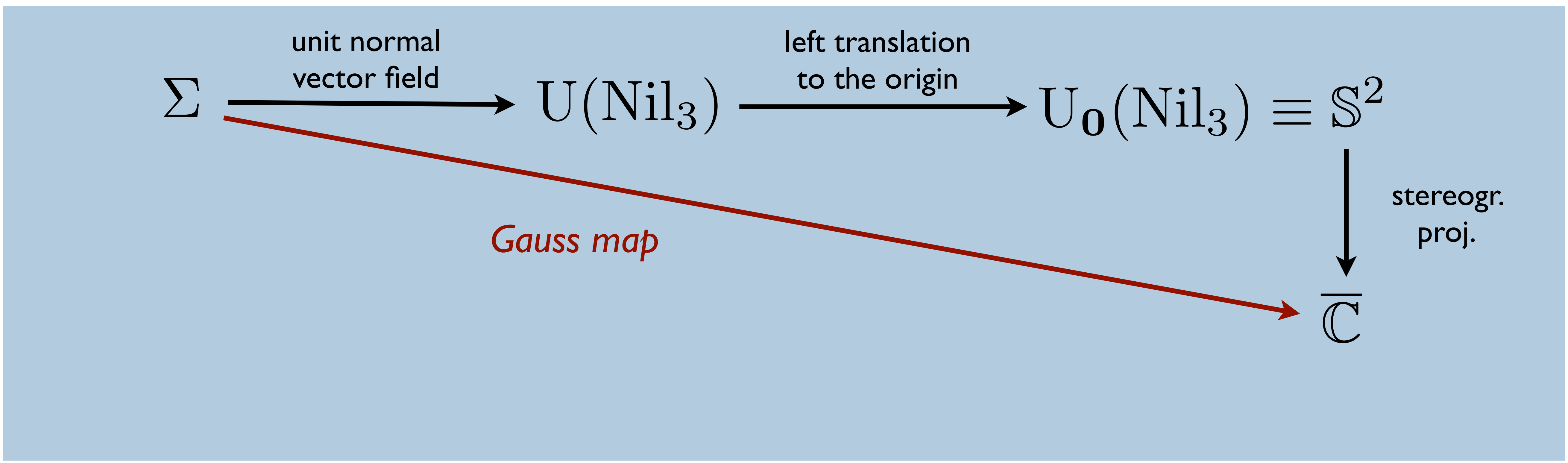

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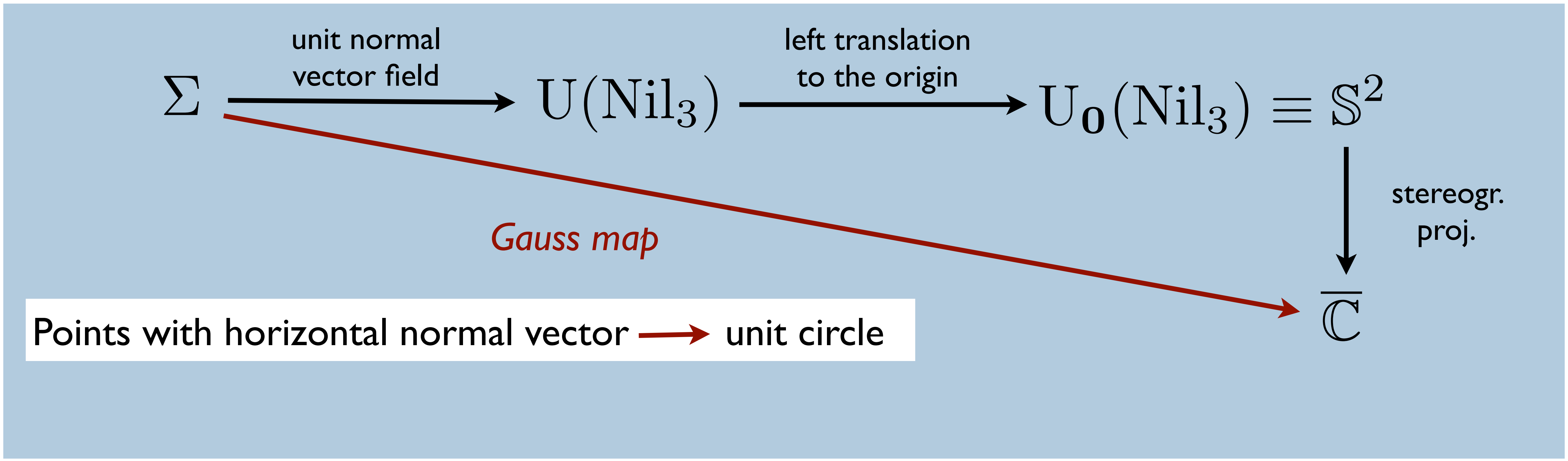

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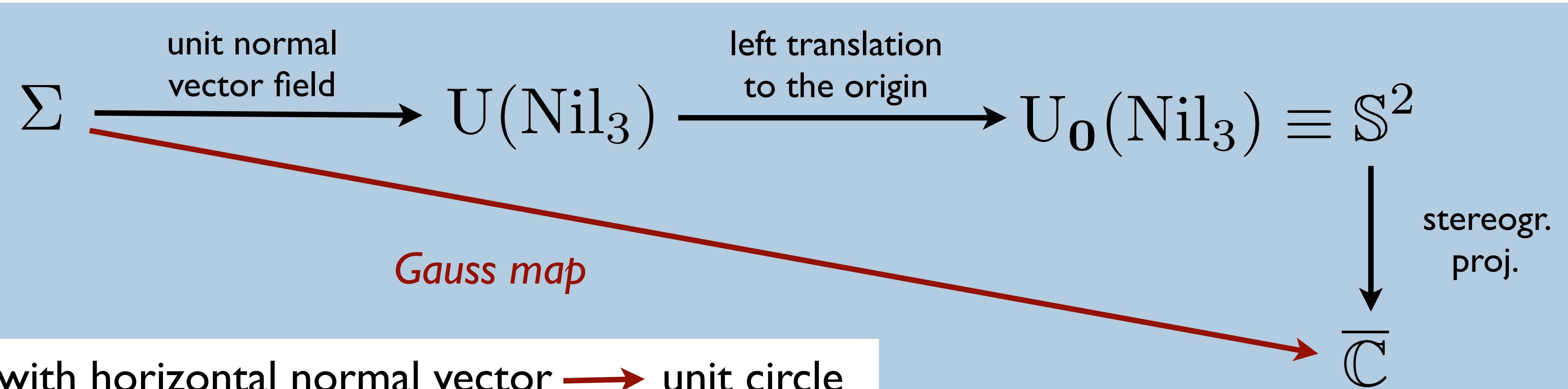

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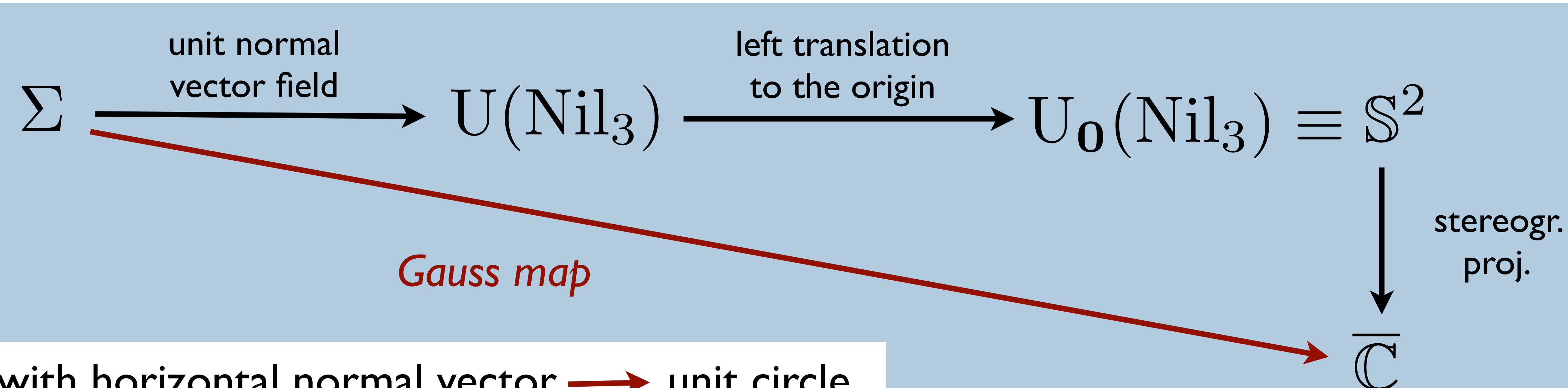

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Daniel (2011) : G is harmonic into $\mathbb{H}^2$ for CRITICAL CMC local graphs in Nil3

What is left...

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t = 0	$\mathbb{H}^2 \times \mathbb{R}$	~~$\mathbb{R}^3$~~	$\mathbb{S}^2 \times \mathbb{R}$
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UNIFIED DEFINITION in all the spaces ?

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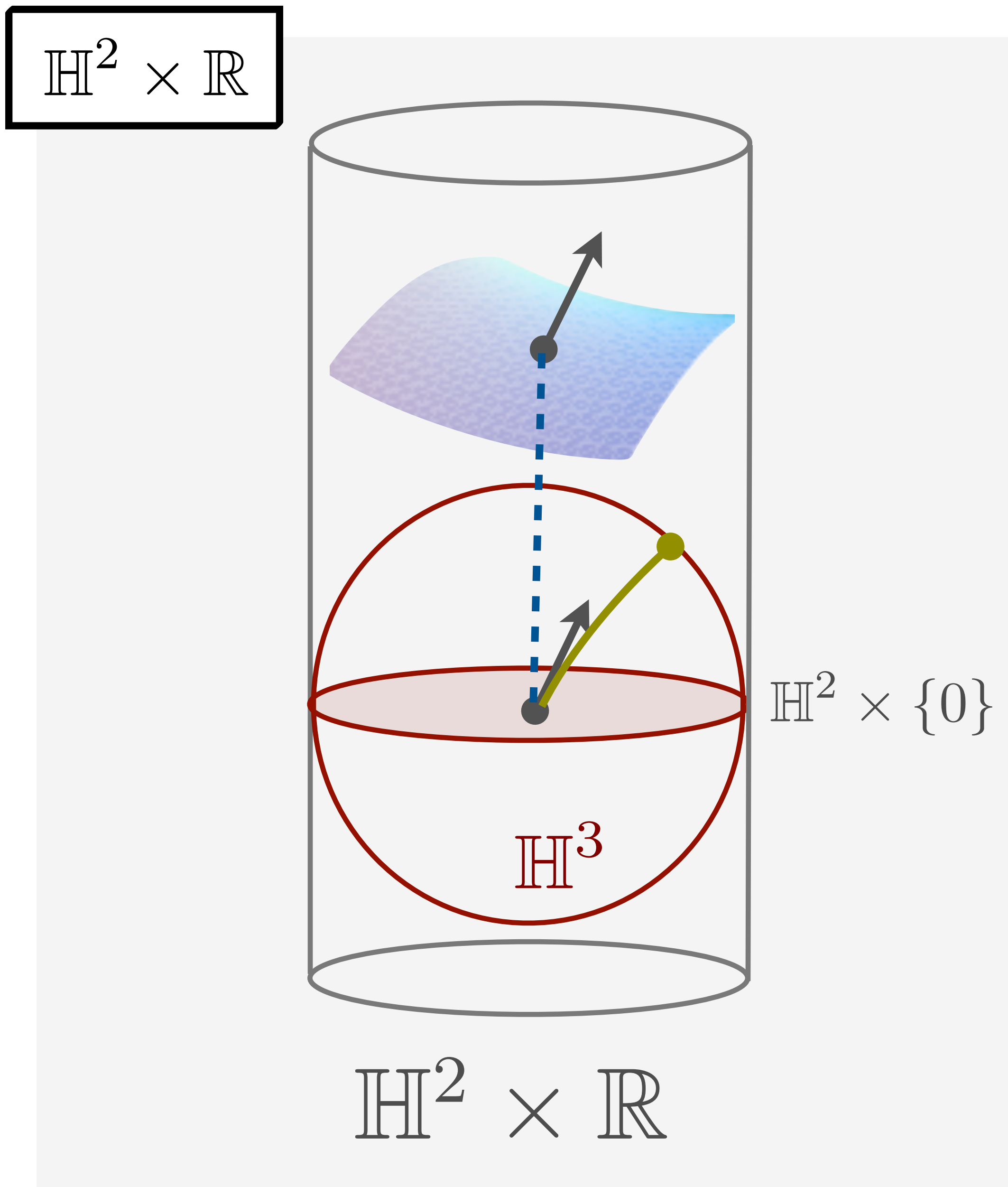

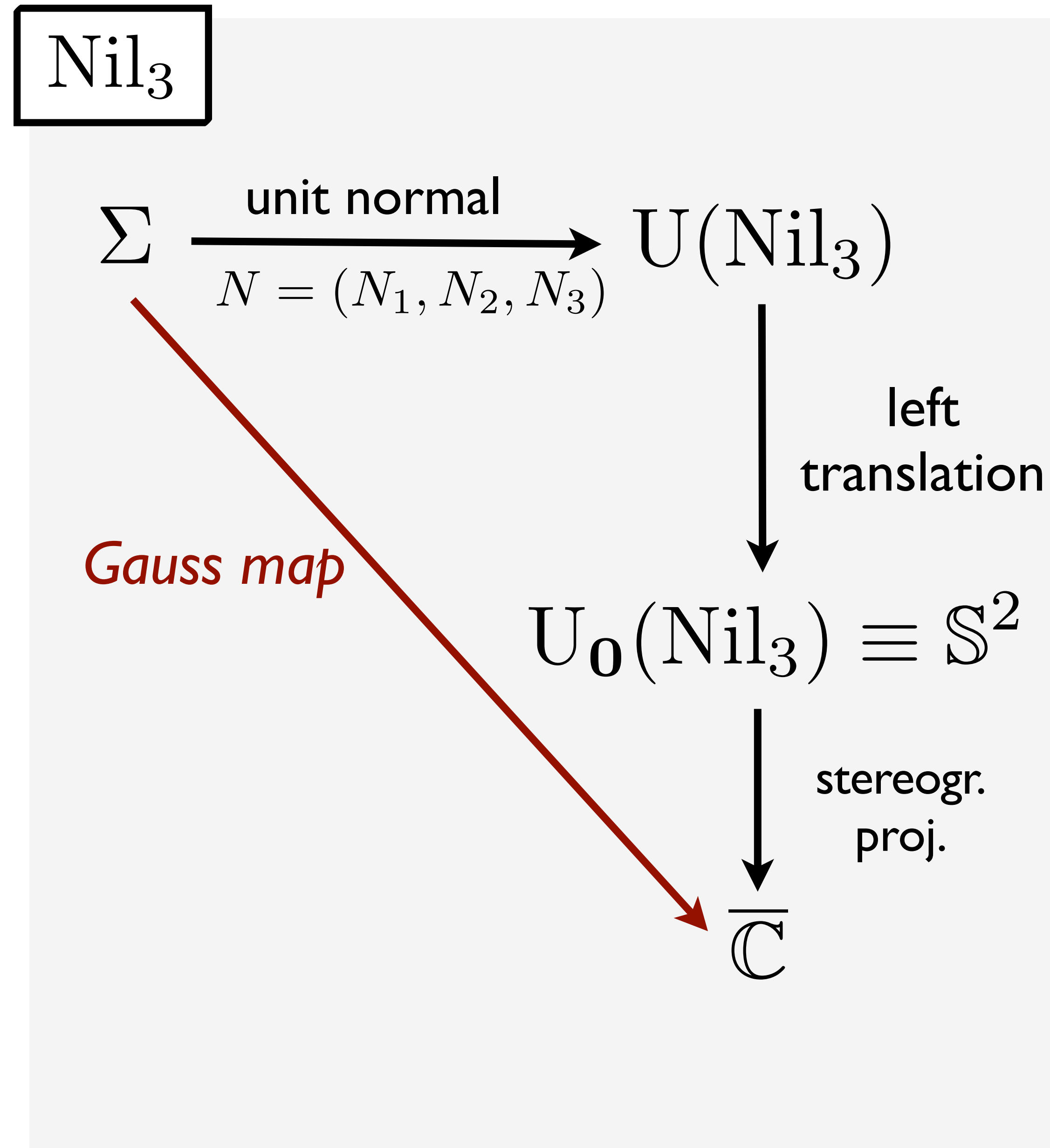

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  - Different behavior when prescribing the Gauss map:
    - $\ast$ In Nil_3 there is only one surface for each Gauss map
    - $\ast$ In $\mathbb{H}^2 \times \mathbb{R}$ there is a 2-parametric family of surfaces with the same Gauss map

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  - They relate well with the ambient isometries.

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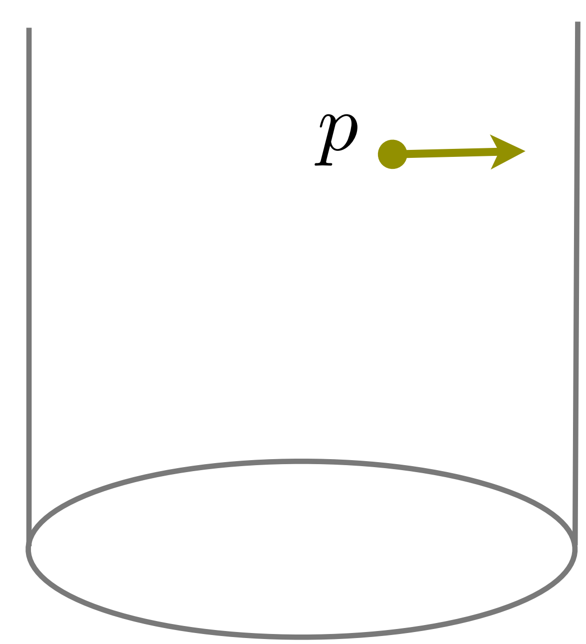

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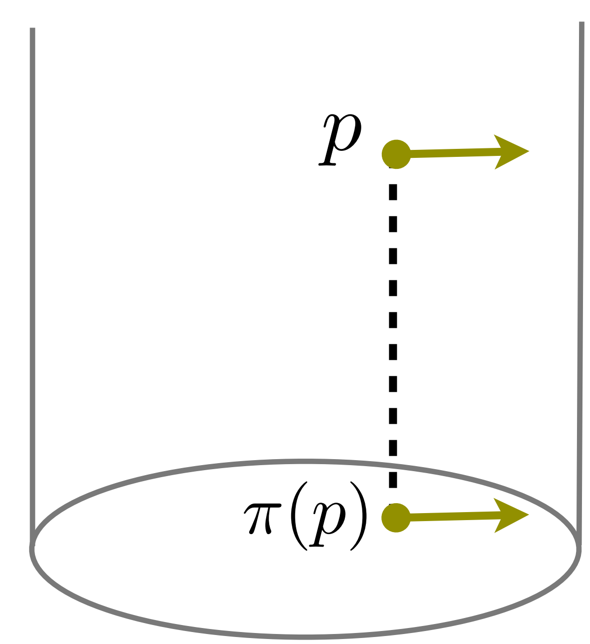

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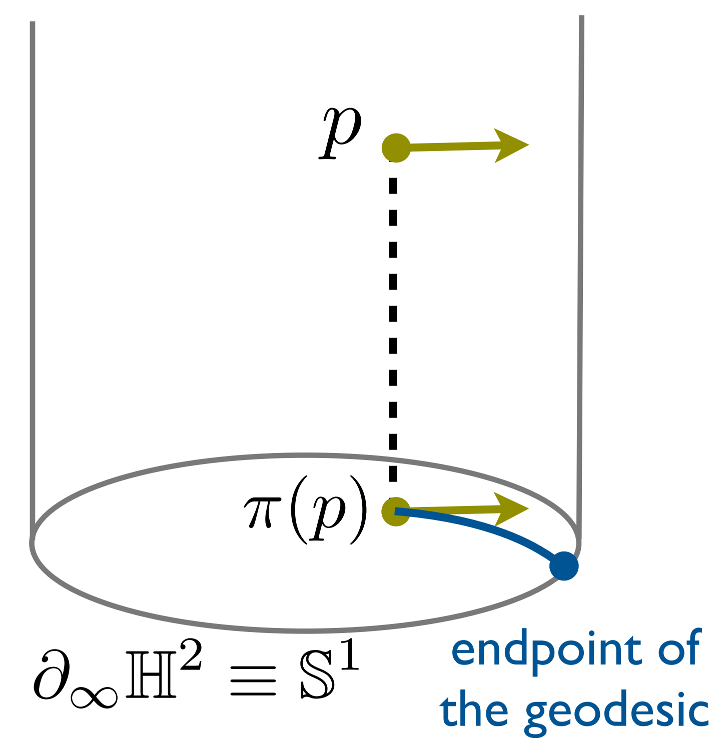

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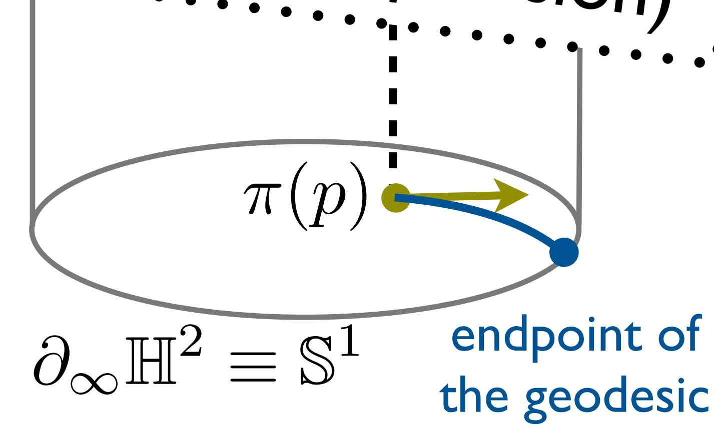
PROPOSITION

There is only one way of doing this (and it has a nice explicit expression)



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In particular, G is harmonic into $\mathbb{D} \equiv \mathbb{H}^2$ for CRITICAL CMC local graphs.

Representation formula

$X : \Sigma \to \mathbb{E}^3(\kappa, \tau)$ CRITICAL CMC inmersion (doesn't have to be a local graph)

$G : \Sigma \longrightarrow \overline{\mathbb{C}}$ its Gauss map

Daniel, --, Mira : The surface can be recovered from the Gauss map by means of

$$\begin{cases} \zeta_z = \dfrac{2}{c+i\tau} \dfrac{(1-c\,\zeta\,\bar{G})^2}{(1-|G|^2)^2} G_z \\ \zeta_{\bar{z}} = \dfrac{-2}{c-i\tau} \dfrac{(G-c\,\zeta)^2}{(1-|G|^2)^2} \bar{G}_{\bar{z}} \\ (x_3)_z = \dfrac{-2}{c+i\tau} \dfrac{(\bar{G}-c\bar{\zeta})(1-c\,\zeta\,\bar{G})}{(1-c^2|\zeta|^2)(1-|G|^2)^2} G_z + \dfrac{i\tau}{2} \dfrac{\zeta\,\bar{\zeta}_z - \bar{\zeta}\,\zeta_z}{1-c^2|\zeta|^2} \end{cases}$$

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How many *different* immersions X are there ?

- For **k<0** there is a 2-parametric family of surfaces with the same Gauss map
- For **k=0** there is only one surface for each Gauss map

Summarizing...

There exist a (unified)
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(*harmonic* for *critical* CMC surfaces)

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MAIN THEOREM (--, Daniel, Mira):

There exists a unified definition for the Gauss map of a surface in $\mathbb{E}^3(\kappa, \tau)$ s.t.:

1) Two surfaces are tangent at one point iff their Gauss maps agree at this point.

2) If the surface is a local graph, then G lies in the unit disc.

3) If in addition the surface has critical CMC, then G is harmonic into $\mathbb{H}^2$ and *nowhere antiholomorphic.*

Using coordinates w.r.t the *canonical frame:*

$$G = \frac{N_1 + iN_2 + c\zeta(1 + N_3)}{c\bar{\zeta}(N_1 + iN_2) + 1 + N_3}$$

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CONVERSELY, any *nowhere antiholom.* harmonic map from a simply connected surface into $\mathbb{H}^2$ is the Gauss map of a critical CMC local graph in $\mathbb{E}^3(\kappa, \tau)$

the GAUSS MAP for CMC SURFACES in HOMOGENEOUS SPACES

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Obrigada !

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