Around the dynamics of Painlevé's equations

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Painlevé (classical) equations

(P_I) $y'' = 6y^2 + x$

(P_{II}) $y'' = 2y^3 + xy + \alpha$

(P_{III}) $y'' = \frac{1}{y}(y')^2 - \frac{1}{x}y' + \frac{1}{x}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$

(P_{IV}) $y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4xy^2 + 2(x^3 - \alpha)y + \frac{\beta}{y}$

(P_V) $y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{1}{x}y' + \frac{(y-1)^2}{x^2}\left(\alpha y + \frac{\beta}{y}\right)\frac{\gamma y}{x} + \frac{\delta y(y+1)}{y-1}$

$$(P_{VI})\quad y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)(y')^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y' + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha + \frac{\beta x}{y^2} + \frac{\gamma(x-1)}{(y-1)^2} + \frac{\delta x(x-1)}{(y-x)^2}\right)$$

$\alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{C}$

Painlevé (classical) equations

Purpose: To work out the dynamics associated with these equations.

Applications:

(1) to determine when the Galois-Malgrange pseudogroup is maximal

(2) to obtain accurate asymptotic estimates for solutions

Galois-Malgrange pseudogroup

Galois-Malgrange pseudogroup: *generalization of the differential Galois group associated with linear equations (algebraic definition whose dynamical meaning is not obvious)*

- *(Virtually) non-solvabe differential Galois group* $\Rightarrow$ *the solutions cannot be "integrated by quadratures"*
- *Painlevé equations are not linear: problem of irreducibility of Painlevé "transcendents"*

Irreducibility of Painlevé transcendents

- Umemura notion of irreducibility $\Rightarrow$ Umemura results + works by the Japanese school

- Galois-Malgrange pseudogroup

  (1) general definition implying all other forms of irreducibility
  (2) more directly related with dynamics in that it is defined as a "closure" for holonomy groups in the spirit of dynamical interpretations of Galois theory
  (3) coincides with the Galois group if the equation is linear (Malgrange theorem, not trivial)
  (4) it can also be viewed as a "measure" - albeit a coarse one - of the dynamical complexity of the equations

Galois-Malgrange pseudogroup

How to compute Galois-Malgrange pseudogroup?

For order 2 equations, we have

Theorem (Casale)

The Galois-Malgrange pseudogroup is maximal if and only if

(1) *there is no algebraic codimension 1 (multi)foliation invariant by the equation and*

(2) *there is no algebraic transverse (multi)affine structure invariant by the equation*

Galois-Malgrange pseudogroup

Consider the Painlevé equation P_I

$$y'' = 6y^2 + x\,.$$

This equation is equivalent to the vector field

$$X_I = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + (6y^2 + x)\frac{\partial}{\partial z}$$

in $\mathbb{C}^3$.

The Galois-Malgrange pseudogroup is maximal if and only if

(1) *there is no algebraic codimension* 1 *(multi)foliation whose leaves contain the orbits of* X_I

(2) *there is no algebraic transverse (multi)affine structure for* $\mathcal{F}_I$*, the foliation associated to* P_I

Galois-Malgrange pseudogroup

Toy-model case: **Airy equation**

$$y'' = xy$$

whose associated vector field is

$$X_A = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z}$$

Galois-Malgrange pseudogroup

Some known results:

- **Airy equation:** the Galois-Malgrange pseudogroup is maximal ("classical" result with algebraic computations)
- P_I **equation:** the Galois-Malgrange pseudogroup is maximal (Casale)
- P_{II}-P_{VI} **equation:** the Galois-Malgrange pseudogroup is maximal for "generic values" of the parameters (Casale, Roques, etc)
- P_{VI} **equation:** complete characterization by Cantat-Loray

Problems about "asymptotic estimates"

P_I, P_{II}, P_{IV} and modified P_{III}, P_V equations

- *they have meromorphic solutions defined on $\mathbb{C}$ (i.e. the solutions are holomorphic functions $\varphi : \mathbb{C} \to \mathbb{CP}(1)$)*
- *they have an essential singularity at "$T = \infty$"*

$$\downarrow$$

* *study of their asymptotic behavior (Nevalinna theory)*
* *purely dynamical problem: ergodic properties*

Remark

The above mentioned questions will not be posed for P_{VI} - solutions do not have a maximal domain of definition on $\mathbb{C}$.

Airy Equation

Vector field in correspondence to the Airy equation

$$X_A = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z}$$

Alternative (more dynamical-geometric proof) proof of the maximality of the Galois-Malgrange pseudogroup (coinciding with the differential Galois group).

Airy Equation

Theorem

Consider the (meromorphic) extension of X_A to $\mathbb{CP}(3) = \mathbb{C}^3 \cup \Delta_\infty$. There are two points $p, q \in \Delta_\infty$ such that the following holds: given two neighborhoods V_p, V_q of p, q on $\mathbb{CP}(3)$, respectively, and an integral curve ϕ for X_A, we have

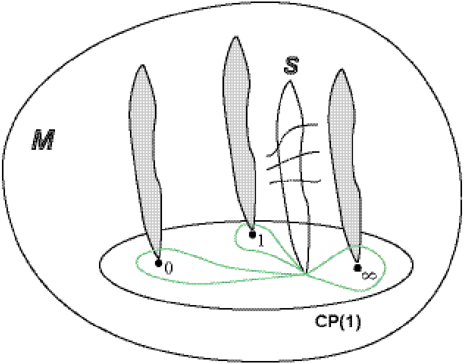
$$\lim_{r\to\infty} \frac{Area(\{T \in B(r) : \phi(T) \in V_p \cup V_q\})}{Area(B(r))} = 1$$

Remark

X_A is "very non-ergodic" - its solutions are "confined" in a probabilistic sense.

Global properties of Painlevé equations

Riemann-Hilbert picture (Okamoto compactification) for P_{VI}



Global properties of Painlevé equations

Holonomy group: *generated by two independent holonomy (algebraic) maps* $f = h_{\gamma_0}$ *and* $g = h_{\gamma_\infty}$

$$f : S \to S$$
$$g : S \to S$$

Let $\bar{S}$ *stands for the compactification of* S. *Then* $f, g : \bar{S} \to \bar{S}$ *are* **birational maps**.

Cantat-Loray viewpoint

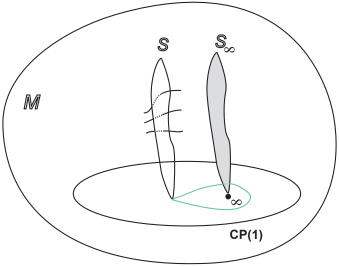
Dynamical study if the action of $\Gamma =< f, g >$ **on** $\bar{S}$

- **If** there exists codimension 1 foliation invariant by $\mathcal{F}_{P_{VI}}$, **then** there exists a 1-dimensional foliation on $\bar{S}$ invariant by Γ.
- **If** there exists a transverse affine structure for $X_{P_{VI}}$, **then** there exists a (singular) affine structure on $\bar{S}$ invariant by Γ

By studying the dynamics of Γ *Cantat and Loray rule out the existence of these objects for all parameters but the case corresponding to Picard solutions.*

There is no codimension 1 *foliation invariant by* $X_{P_{VI}}$*, not even among foliations with entire coefficients.*

Riemann-Hilbert picture (Okamoto compactification) for P_I/P_{II}



base simply connected ⇒ *the holonomy maps are trivial*

⇒ *plenty of invariant* **holomorphic** *foliations /*

transverse affine structures

Our approach

Problem: How to detect if, say an invariant codimension 1 foliation, is algebraic?

- Notion of dynamics at infinity
- incidentally this "dynamics" at infinity is exactly the object that controls the asymptotic behaviour of solutions
- add a fiber over the "missing points" (if they were missing) and study the foliation in the neighbourhood of this fiber

The case of Airy equation

The Airy equation defined a 1-dimensional foliation $\mathcal{F}$ on $\mathbb{C}^3$, induced by

$$X_A = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z}$$

In standard coordinates for $\mathbb{CP}(3)$ around Δ_∞

$$x_1 = \frac{1}{x_1}, \quad y_1 = \frac{y}{x}, \quad z_1 = \frac{z}{x}$$

where $\Delta_\infty \subseteq \{x_1 = 0\}$, we have

$$X_{A,1} = \frac{1}{x_1}\left[-x_1^3\frac{\partial}{\partial x_1} + (x_1 z_1 + y_1 x_1^2)\frac{\partial}{\partial y_1} + (y_1 - x_1^2 z_1)\frac{\partial}{\partial z_1}\right].$$

The case of Airy equation

$\mathcal{F}$ extends to a (singular) foliation on $\mathbb{C}^3 \cup \Delta_\infty \simeq \mathbb{CP}(3)$

$$X_{A,1}|_{\Delta_\infty} = y_1 \frac{\partial}{\partial z}$$

Some issues on $\mathcal{F}$:

- *global dynamics on Δ_∞ is "simple"*
- *$\mathcal{F}$ has singular points that may "conceal" non-trivial dynamics*
- *$\mathcal{F}$ possesses compact leaves on Δ_∞ which may carry holonomy*
- *there is a foliated affine structure on the leaves of $\mathcal{F}$*

Singularities:

In coordinates (x_1, y_1, z_1) *for* $\mathbb{CP}(3)$, *where* $\{x_1 = 0\} \subseteq \Delta_\infty$

$$\text{Foliation:} \quad X_{A,1} = -x_1^3 \frac{\partial}{\partial x_1} + (x_1 z_1 + y_1 x_1^2) \frac{\partial}{\partial y_1} + (y_1 - x_1^2 z_1) \frac{\partial}{\partial z_1} \,.$$

1. $\{x_1 = 0, y_1 = 0\} = C_1$ $\rightarrow$ *curve of singularities*
2. *nilpotent singularities*

In coordinates (x_2, y_2, z_2) *for* $\mathbb{CP}(3)$, *where* $\{x_2 = 0\} \subseteq \Delta_\infty$

$$\text{Foliation:} \quad X_{A,2} = (y_2^2 - x_2 y_2 z_2) \frac{\partial}{\partial x_2} - y_2^2 z_2 \frac{\partial}{\partial y_2} + (x_2 - y_2 z_2^2) \frac{\partial}{\partial z_2} \,.$$

1. $\{x_2 = 0, y_2 = 0\} = C_2$ $\rightarrow$ *curve of singularities*
2. *plenty of degenerate singularities*
3. *reduction of singularities*

Resolution of singularities:

1. 2 ramified blow-ups along C_1 and C_2 - $\Delta_\infty \cup \Delta_1 \cup \Delta_2$ total divisor
2. the resulting foliation possesses only 3 singular points: q, p_1, p_2
3. $q \in \Delta_\infty \cap \Delta_1 \cap \Delta_2$ - simple singular point
4. $p_1 \in \Delta_1 \cap \Delta_2$; $p_2 \in \Delta_2 \setminus (\Delta_\infty \cup \Delta_1)$ - saddle-node singularities
5. $L_1 = \Delta_1 \cap \Delta_2$ - rational invariant curve
6. there exists $L_2 \subseteq \Delta_2$ containing p_1 and p_2 - invariant rational curve

The invariant rational curves L_1 and L_2 form a sort of "attractor" of the dynamics (every leaf intersects a neighbourhood of this curve)

Fundamental dynamical issue:
To understand the dynamics of the foliation on a neighborhood of $L_1 \cup L_2$.

- *one non-trivial holonomy map* $h : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$
- *highly resonant saddle-nodes (eigenvalues:* $0, 1, -1$*)*