Analysis of Black Hole Spacetimes in General Relativity

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Overview

(I) Introduction to Black Holes and the Stability Problem

(II) The Linear Scalar Wave Equation on Black Holes

(III) The Linear Stability of the Schwarzschild spacetime

(IV) The Teukolsky equation on slowly rotating Kerr

Parts III and IV are joint work with M. Dafermos and I. Rodnianski.

I: Introduction to Black Holes and the Stability Problem

General relativity as an evolution problem

In the minicourse of *J. Smulevici* we heard that general relativity is an evolutionary theory, i.e. it admits a Cauchy problem:

Given an initial data triple $\left(\Sigma^{(3)}, h_{\mu\nu}, K_{\mu\nu}\right)$ satisfying the *Einstein constraint equations* we can associate to it a 4-dimensional Lorentzian manifold $(\mathcal{M}, g)$ satisfying the vacuum Einstein equations

$$R_{\mu\nu}\left[g\right] = 0\,. \tag{1}$$

There is a notion of *maximum Cauchy development*.

General relativity is about studying the properties of this maximum development.

Special solutions and stability

It is clear that flat Minkowski space is a solution of the vacuum Einstein equations. The Cauchy problem gives us a framework to discuss the non-linear stability of this solution.

$\rightarrow$ *Stability of Minkowksi space*
(Christodoulou-Klainerman 1990, Lindblad-Rodnianski 2003)

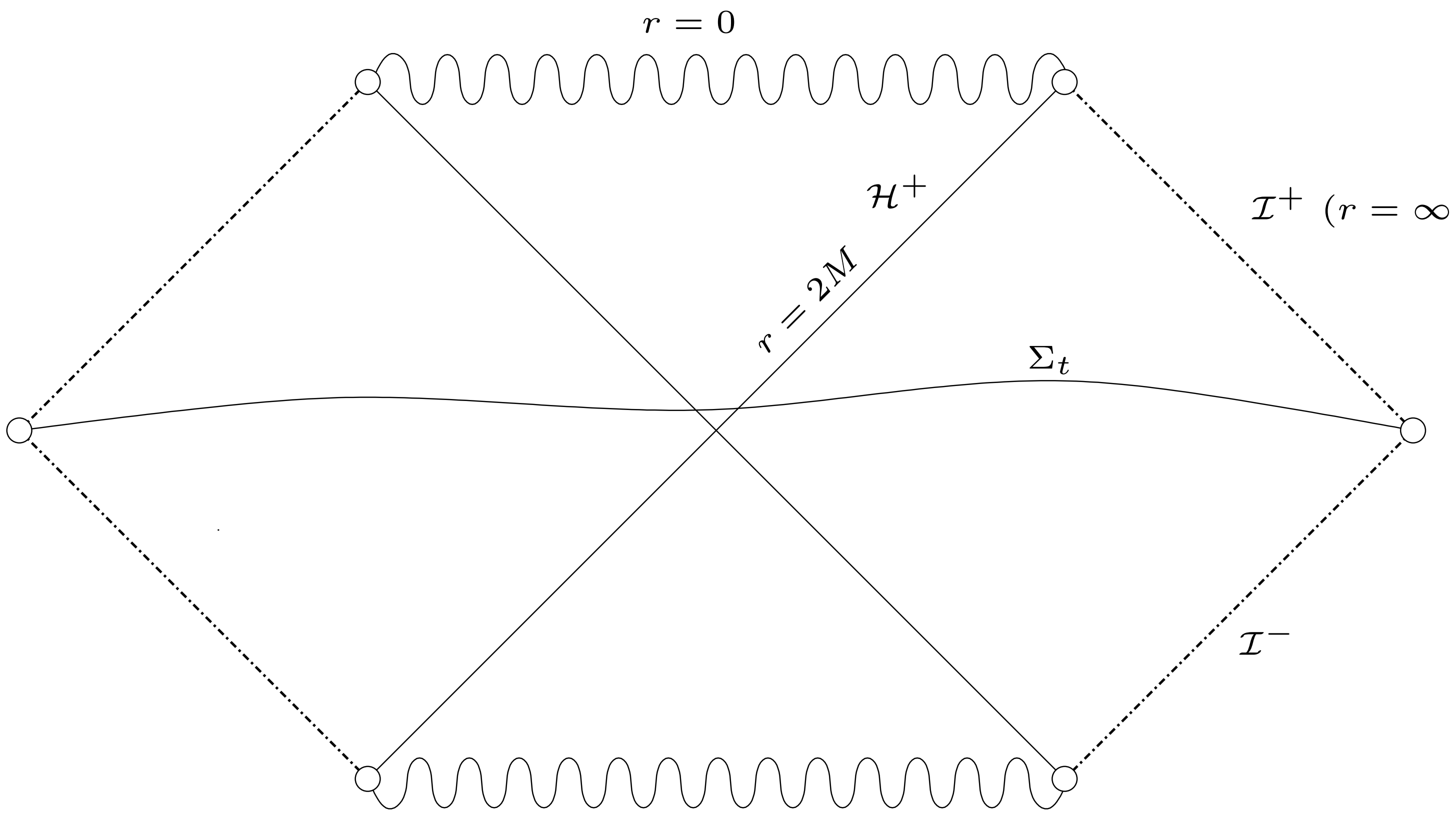
Black Holes

More complicated explicit solutions of $R_{\mu\nu}[g] = 0$ contain **black holes**. The most famous is the spherically symmetric Schwarzschild solution, written in local coordinates on next slide.

Any spherically symmetric vacuum spacetime is locally isometric to a member of the Schwarzschild family (Birkhoff's theorem)!

Schwarzschild spacetime (1916)

$$g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$



Geometry of the Schwarzschild family

- asymptotically flat
- static: Killing field T is timelike on the exterior; T becomes null on the event horizon $\mathcal{H}^+$
- red shift effect near the event horizon $\mathcal{H}^+$
- trapped null geodesics (photon sphere at $r = 3M$)

The Kerr family of black holes

The Schwarzschild family is contained in the two parameter family of *Kerr black hole solutions* written down explicitly in 1963.

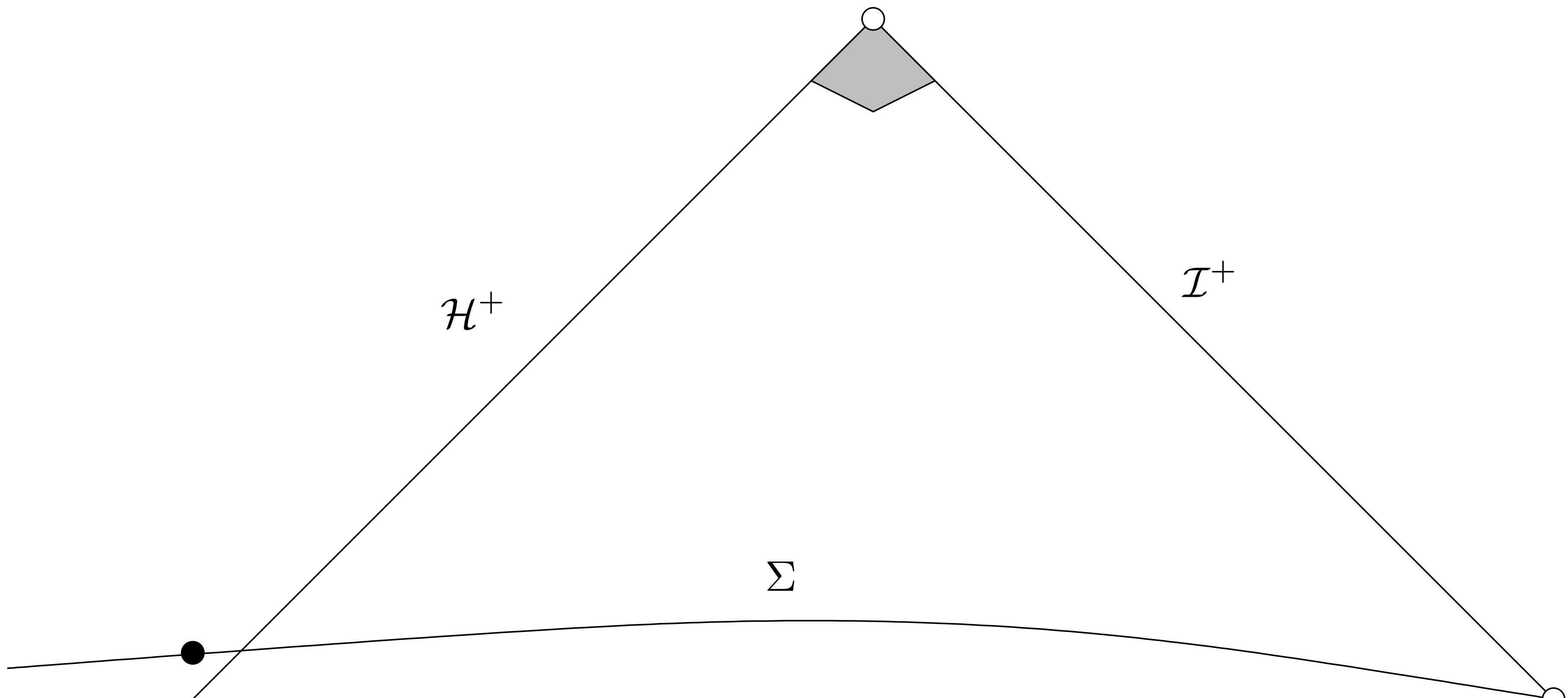
$$g_{M,a} = \ldots$$

These are stationary, axisymmetric and asymptotically flat (AF). Conjecture: Stationarity and AF make the family the *unique* solution of EVE. (Hawking, Carter-Robinson, Chrusciel, Alexakis et al.)

Understanding small perturbations of the Schwarzschild and the Kerr family is crucial to our understanding of general relativity.

This is the black hole stability problem.

Conjecture: The maximum development of initial data sufficiently close to sub-extremal Kerr initial data $(M_i, |a_i| < M_i)$ possesses a complete null infinity $\mathcal{I}^+$ such that the metric in $J^-(\mathcal{I}^+)$ asymptotically approaches another member of the Kerr family $(M_f, |a_f| < M_f)$.



The strategy to resolve the conjecture

1. Understand toy-problem $\Box_g \Psi = 0$ on black hole spacetimes.
   $\rightarrow$ interaction of dispersion with black hole geometry
2. Understand the linearisation of the Einstein equation (including difficult gauge issues).
3. Address non-linear difficulties.

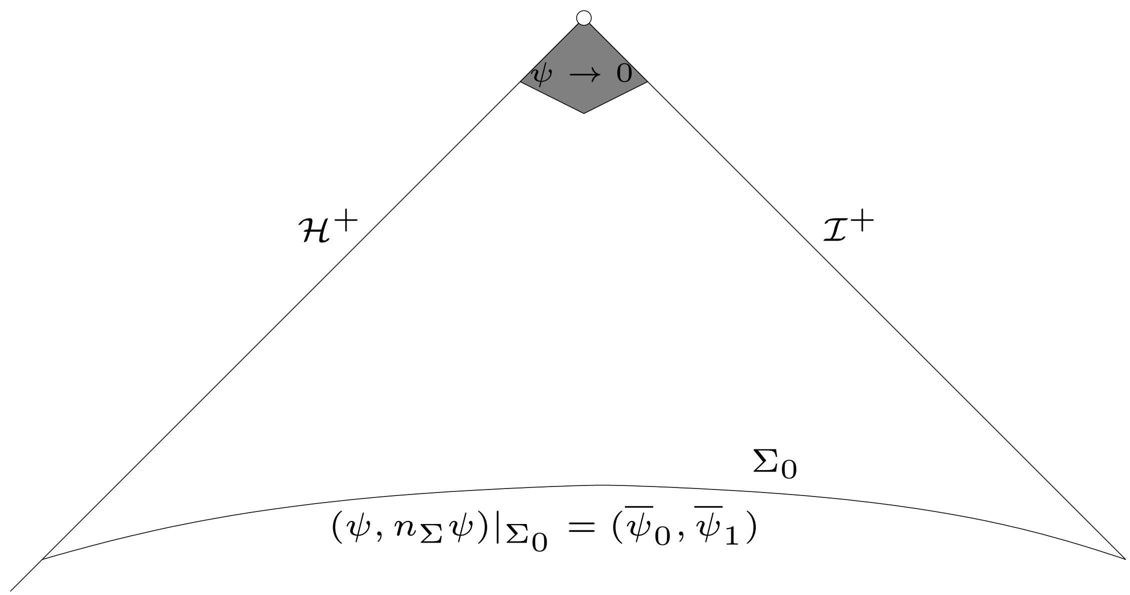
Part II of my talk: Step 1
Parts III and IV: Step 2.

Part II: The linear scalar problem

Theorem. *[Dafermos–Rodnianski–Shlapentokh-Rothman]* *Solutions of the linear wave equation*

$$\Box_{g_{M,a}}\psi = 0 \tag{2}$$

for $g_{M,a}$ a subextremal member of the Kerr-family decay inverse polynomially in time on the black hole exterior.



Historical remarks

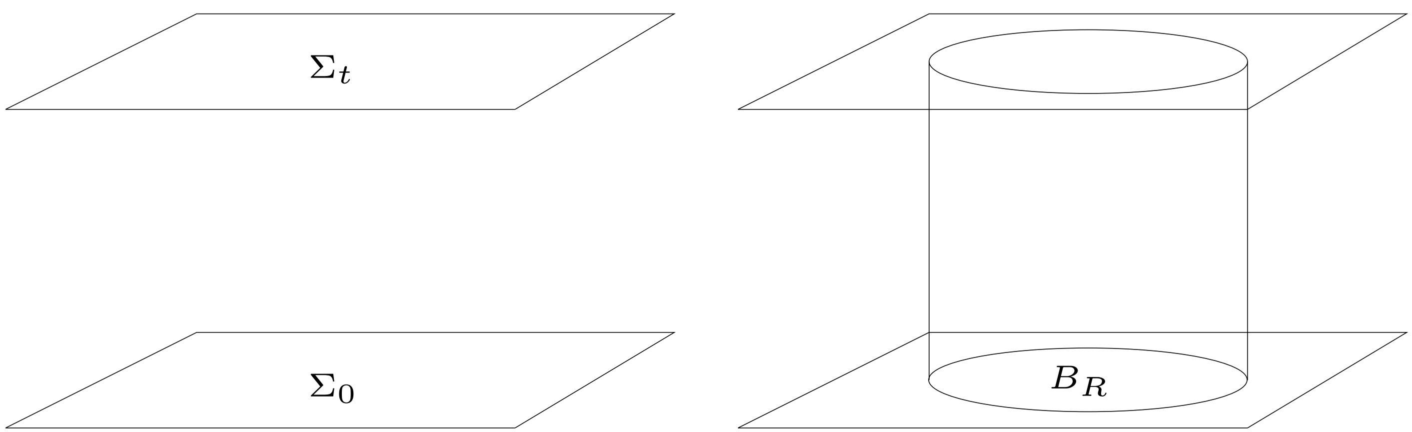
- Previous work (2005-2014): Dafermos–Rodnianski, Blue–Sterbenz, Tataru-Tohaneanu, Andersson–Blue; Kay–Wald (1987)
- Mode stability: Whiting, Shlapentokh-Rothman
- Theorem 1 fails in the extremal case $a = M$ (Aretakis)
- Maxwell's equations: ($a \ll M$ case) Andersson–Blue, Blue, Pasqualotto, Ma; Dirac equation Finster et al
- Generalizations to Kerr de Sitter (Bony-Haefner, Dafermos-Rodnianski, Dyatlov, Hintz-Vasy)
  and Kerr-anti de Sitter (Holzegel–Smulevici, Warnick, Gannot)

Ingredients from PDE theory for Step 1:

Recall Minkowski space: $\Box_\eta \psi = 0$. Two key estimates

$$\int_{\Sigma_t} (\partial_t \psi)^2 + |\nabla \psi|^2 = \int_{\Sigma_0} (\partial_t \psi)^2 + |\nabla \psi|^2 \quad \text{energy conservation}$$

$$\int_0^T dt \int_{\Sigma_t \cap \{r \le R\}} (\partial_t \psi)^2 + |\nabla \psi|^2 \le C_R \int_{\Sigma_0} (\partial_t \psi)^2 + |\nabla \psi|^2 \quad \text{ILED}$$



Already in the Schwarzschild case deriving analogues of these two estimates requires

- understanding of the *redshift* near $\mathcal{H}^+$ to prove boundedness
- understanding of *trapping* at the photon sphere to prove decay

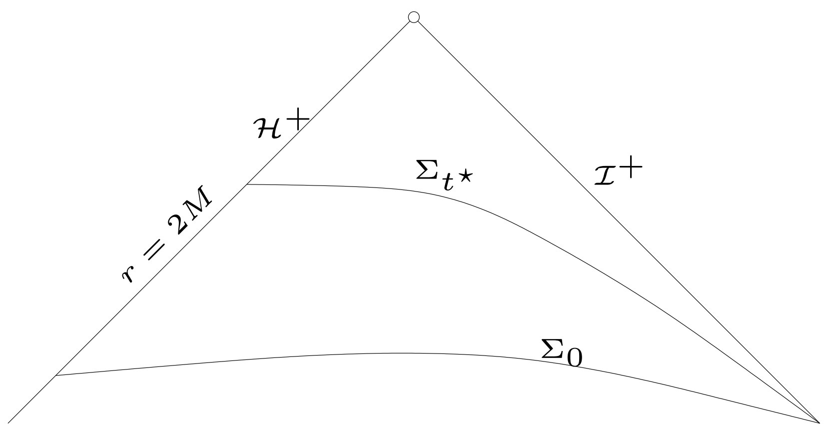
The Kerr case is much more complicated (some remarks later).

The redshift in Schwarzschild

The static Killing vector field $T = \partial_t$ gives rise to

$$\boxed{\int_{\Sigma_{t^\star}} |\partial\psi|^2 \left(1 - \frac{2M}{r}\right) \lesssim \int_{\Sigma_0} |\partial\psi|^2 \left(1 - \frac{2M}{r}\right)}$$

with $|\partial\psi|^2$ denoting (the sum of) all derivatives of ψ.



The redshift effect is about removing the degeneracy at $r = 2M$ to get a <u>non-degenerate</u> boundedness statement.

Trapping in Schwarzschild

- Null-geodesics can orbit in the timelike hypersurface $r = 3M$: This is the photon sphere.
- In the high frequency approximation, solutions to the wave equation travel along null-geodesics!

$\implies$ Non-degenerate decay estimates for Schwarzschild are necessarily associated with a loss of derivatives. (Sbierski 2013)

$$\boxed{\int_0^T dt^\star \int_{\Sigma_{t^\star}\cap\{r\leq R\}} |\partial\psi|^2 \leq C_R \int_{\Sigma_0} |\partial\psi|^2 + |\partial^2\psi|^2}$$

Remarks on the Kerr case

In the Kerr case one needs to understand the issue of *superradiance*.

In geometric language, $T = \partial_t$ is not timelike everywhere on the black hole exterior and **the first key estimate fails**!

One solution, developed by Dafermos–Rodnianski, is to frequency decompose the solutions exploiting a remarkable property of the Kerr geometry: "superradiant frequencies are not trapped".

Part III: Linear Stability of Schwarzschild

The linearization procedure

Consider a one-parameter family of Lorentzian metrics in double null-coordinates

$$\boldsymbol{g}\left(\epsilon\right) \doteq -4\boldsymbol{\Omega}^2\left(\epsilon\right)dudv + \boldsymbol{\not{g}}_{CD}\left(\epsilon\right)\left(d\theta^C - \boldsymbol{b}^C\left(\epsilon\right)dv\right)\left(d\theta^D - \boldsymbol{b}^D\left(\epsilon\right)dv\right)$$

with $\boldsymbol{g}\left(0\right)$ being the Schwarzschild metric of mass M.

- $\boldsymbol{\Omega}\left(\epsilon\right) = \Omega + \epsilon\overset{(1)}{\Omega} + \mathcal{O}\left(\epsilon^2\right)$, etc.
- express curvature components and connection-coefficients in associated null-frame
- write down Einstein equations to order ϵ

The result is a complicated system of equations of

- linearised null-curvature components (Newman-Penrose scalars) satisfying the Bianchi equations
- linearised connection coefficients satisfying transport equations

I won't show this system. Evolution is well-posed.

We would like to show boundedness and decay of *all* linearised quantities in terms of initial data of the linearised system.

The system of linearised Einstein equations: Key-observations

1. special solutions: pure gauge solutions
2. special solutions: linearised Kerr solutions
3. hierarchy of decoupled gauge invariant quantities

Gauge invariant quantities which decouple

It has long been known that the gauge invariant null-curvature components $\overset{\scriptscriptstyle(1)}{\alpha}$ and $\underline{\overset{\scriptscriptstyle(1)}{\alpha}}$ satisfy decoupled wave equations: The **Teukolsky** (or Bardeen-Press '73) **equation**:

$$\boxed{\Box_g \overset{\scriptscriptstyle(1)}{\alpha} + \left(1 - \frac{3M}{r}\right) \partial_t \overset{\scriptscriptstyle(1)}{\alpha} + V \overset{\scriptscriptstyle(1)}{\alpha} = 0.}$$

Only mode stability but not even uniform boundedness was known.

Hierarchy of gauge invariant quantities

There exists a second order differential operator which when applied to α or $\underline{\alpha}$ yields new quantities

$$P := \mathfrak{D}^2 \overset{\scriptscriptstyle(1)}{\alpha} \quad , \quad \underline{P} := \underline{\mathfrak{D}}^2 \overset{\scriptscriptstyle(1)}{\underline{\alpha}} \tag{3}$$

- the quantities P and $\underline{P}$ satisfy the Regge-Wheeler equation, which *does* admit both a good energy estimate and an ILED! **All the theory for $\Box_g \psi = 0$ applies.**

- the quantities P and $\underline{P}$ control α, $\underline{\alpha}$ respectively, in particular decay for P and $\underline{P}$ implies decay for α and $\underline{\alpha}$. **Simple transport equations.**

These transformations appear at the level of mode solutions in the work of Chandrasekhar.

Corollary. *(Dafermos–GH–Rodnianski 2016) Solutions to the Teukolsky equation decay inverse polynomially in time.*

Note this result holds independently of the whole system of gravitational perturbations.

Previous and related work:
Moncrief (1975), Martel-Poisson (2005), Sarbach–Tiglio (2001)
(metric perturbations; Regge-Wheeler and Zerilli)

From gauge invariant to all geometric quantities

- show $\boxed{\overset{(1)}{\alpha} = \overset{(1)}{\underline{\alpha}} = 0}$ globally $\Longrightarrow$ solution is pure gauge+Kerr
- need *quantitative* estimates of all geometric quantities
- This can be done. Remarkably, one can only show boundedness but not decay for some of the quantities. Why is that?

Solution decays to a pure gauge solution which is <u>dynamically determined</u> and can be <u>quantitatively estimated from data</u>.

Theorem (DHR 2016; Linear Stability of Schwarzschild)**.** *General solutions* $\mathscr{S}$ *of the system of gravitational perturbations on Schwarzschild arising from suitably normalised characteristic initial data*

- *remain uniformly bounded on the black hole exterior and in fact*
- *decay inverse polynomially to a linearised Kerr solution* $\mathscr{K}$ *after adding to* $\mathscr{S}$ *a dynamically determined pure gauge solution* $\mathscr{G}$ *which is itself uniformly bounded by initial data.*

Comments

1. Remarkably, a version of this result has recently obtained in harmonic gauge in the context of metric perturbations (Thomas Johnson)

2. **Non-linear applications** (work in progress with Martin Taylor (Imperial), Dafermos and Rodnianski): Prove non-linear stability of Schwarzschild for finite codimension manifold of initial data.

Part IV: The Teukolsky equation on Kerr spacetimes

The paper [DHR] provides a *complete* picture of linear stability of the Schwarzschild metric.

At the core of the analysis were the Teukolsky equations and the gauge invariant hierarchy.

The following two observations allow ourselves to immediately generalise the Teukolsky-part of the result of [DHR] to slowly roating Kerr spacetimes.

1. The extremal linearised curvature components still satisfy decoupled wave equations. (Teukolsky 1973)

2. The transformation theory can be straightforwardly generalised. Schematically

$$P = \mathfrak{D}\psi = \mathfrak{D}^2\alpha \quad , \quad \psi = \mathfrak{D}\alpha$$

now leads after a remarkable cancellation to

$$\Box^{RW}_{g_{M,a}} P = a\left(\partial_\phi \psi + \psi\right) + a\left(\alpha + \partial_\phi \alpha\right)$$

Of course for $a = 0$ there is complete decoupling!

As mentioned before, proving decay for the free wave equation on Kerr requires *frequency decomposition* and different multiplier estimates for different frequency ranges.

Compared with the Schwarzschild case, the estimates for the wave equation and the transport estimates are now *coupled.* The smallness factor of a allows one to immediately close the estimates. See independent recent work by Ma.

These observations lead directly to

Theorem. *[DHR 2017] Consider a Kerr spacetime* $(\mathcal{M}, g_{M,a})$ *with* $|a| \ll M$. *Solutions to the Teukolsky equation are uniformly bounded and in fact decay inverse polynomially on the black hole exterior.*