METRIC TENSORS ON THE SPACE OF INVERTIBLE QUANTUM STATES

SOME IDEAS BORN DURING AN UNCOUNTABLE NUMBER OF DISCUSSIONS WITH:

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THE MODEL QUANTUM SYSTEM

$$\mathcal{H}_n = \mathbb{C}^n$$

Is the Hilbert space of the system

$$\mathcal{B}(\mathcal{H}_n) = M_n(\mathbb{C})$$

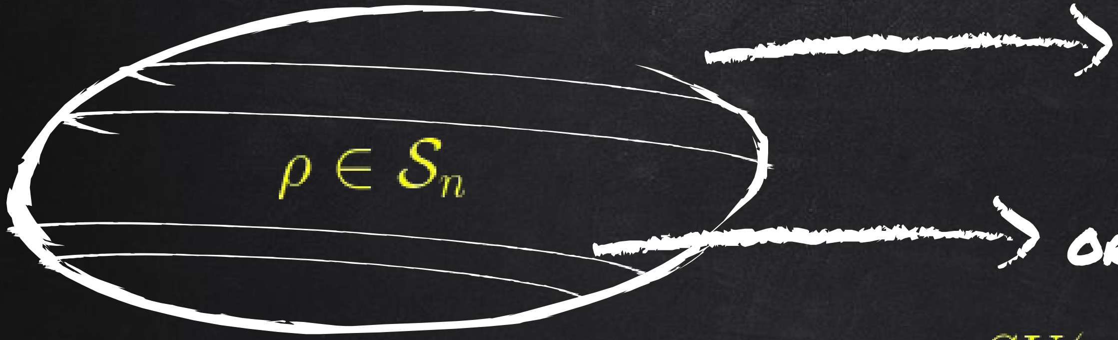
Is the C*-algebra of the system, observables are Hermitean matrices

$$\mathcal{S} = \{\rho \in \mathcal{M}_n(\mathbb{C}) \colon Tr(\rho) = 1\,, \;\rho \text{ is positive }\}$$

Is the space of states of the system

REMARK: every finite-dimensional quantum system (without superselection sectors) is isomorphic to this model quantum system once a basis in the Hilbert space of the system is selected. This means that the isomorphism is not canonical.

THE MODEL QUANTUM SYSTEM



$$\rho \in \mathcal{S}_n$$

INVERTIBLE QUANTUM STATES

ORBITS OF THE UNITARY ACTION:

$$SU(n) \times \mathcal{S}_n \ni (\mathbf{U}, \rho) \mapsto \mathbf{U}\,\rho\,\mathbf{U} \in \mathcal{S}_n$$

REMARK: It is a homogeneous space of the complex special linear group with respect to the following nonlinear action:

$$SL(n\,, \mathbb{C}) \times \mathcal{S}_n \ni (\mathbf{g}\,, \rho) \mapsto \frac{\mathbf{g}\,\rho\,\mathbf{g}^{\dagger}}{Tr(\mathbf{g}\,\rho\,\mathbf{g}^{\dagger})} \in \mathcal{S}_n$$

THE MODEL QUANTUM SYSTEM

Symmetries (if any) represent an useful tool when dealing with metric tensors. For instance, the Fubini-Study metric tensors on the space of pure quantum states is the real part of the Hemitean tensor:

$$h = \frac{\langle \mathrm{d}\psi | \mathrm{d}\psi \rangle}{\langle \psi | \psi \rangle} - \frac{\langle \mathrm{d}\psi | \psi \rangle \langle \psi | \mathrm{d}\psi \rangle}{\langle \psi | \psi \rangle^2}$$

REMARK: It is the unique metric tensors (up to a conformal factor) on the space of pure quantum states which is invariant w.r.t. the action of the unitary group.

THE MODEL QUANTUM SYSTEM

QUESTION: Is there a metric tensor on the space of invertible quantum states which is invariant w.r.t. the action of the unitary group? Is it unique?

ANSWER: There is an infinite number of metric tensors on the space of invertible quantum states that are invariant with respect to the unitary group.

REMARK: Actually, all these metric tensors satisfy the so-called monotonicity property (MP), a property which is more general than invariance w.r.t. the action of the unitary group.

THE MODEL QUANTUM SYSTEM

QUESTION: How can we obtain a metric tensor satisfying the monotonicity property?

In classical Information Geometry we take a divergence function S (often it is a relative entropy), derive it twice and then evaluate the result on the diagonal:

$$S(\mathbf{x}\,,\mathbf{y}) = \frac{1}{2}\,(x^j - y^j))^2 \Longrightarrow -\left(\frac{\partial^2 S}{\partial x^j \partial y^k}\right)_{x^j=y^j} = \delta_{jk} \Longrightarrow g := \delta_{jk}\,\mathrm{d}x^j \otimes \mathrm{d}y^k$$

We could take a quantum relative entropy and perform the same algorithm....

THE MODEL QUANTUM SYSTEM

Very often, the resulting quantum metrics satisfy the MP, however:

1) the algorithm is coordinate-based;
2) computational difficulties grow with the dimension of the Hilbert space;
3) we must prove that the resulting metric does actually satisfy the MP.

WHAT WE WILL SEE:

i) a coordinate-free algorithm of general validity which is well-suited for a generic n-level quantum system;
ii) when quantum divergence functions (quantum relative entropies) satisfy the data processing inequality (DPI), then the resulting quantum metrics satisfy the MP.

COVARIANT TENSORS FROM TWO-POINT FUNCTIONS

We start with a differential manifold M.

A vector field on M may be thought of as a derivation of the associative algebra of smooth functions, or as a section of the tangent bundle.
In a local chart $\{q^j\}$ on M a vector field X can be written as:

$$X = X^j(q)\,\frac{\partial}{\partial q^j}$$

REMARK: Vector fields are the coordinate-free version of the derivative operator and we will use them to give a coordinate-free algorithm to extract tensor fields from two-point functions.

COVARIANT TENSORS FROM TWO-POINT FUNCTIONS

Divergence functions are two-point functions, hence, the relevant manifold is M x M :

$$\begin{tikzcd}
\mathbb{R} & M \times M \arrow[l, "f_l = pr_l^* f"'] \arrow[r, "f_r = pr_r^* f"] \arrow[d, bend right, "pr_l"'] \arrow[d, bend left, "pr_r"] & \mathbb{R} \\
& M \arrow[d, "f"] & \\
& \mathbb{R} &
\end{tikzcd}$$

REMARK: The left functions f_l on M x M form a subalgebra of the algebra of smooth functions on M x M. The same is true for the right functions f_r.

COVARIANT TENSORS FROM TWO-POINT FUNCTIONS

From vector fields on M to vector fields on its double:

Left lift of a vector field on M

$$\mathfrak{X}(\mathcal{M}) \ni X \longrightarrow \mathbb{X}^l \in \mathfrak{X}(\mathcal{M} \times \mathcal{M}):$$

$$\mathbb{X}^l f_l = (X\, f)_l \;,\; \mathbb{X}^l f_r = 0$$

$$X = X^j(q)\frac{\partial}{\partial q^j} \longrightarrow \mathbb{X}^l = X^j(x)\frac{\partial}{\partial x^j}$$

Right lift of a vector field on M

$$\mathfrak{X}(\mathcal{M}) \ni X \longrightarrow \mathbb{X}^r \in \mathfrak{X}(\mathcal{M} \times \mathcal{M}):$$

$$\mathbb{X}^r f_r = (X\, f)_r \;,\; \mathbb{X}^r f_l = 0$$

$$X = X^j(q)\frac{\partial}{\partial q^j} \longrightarrow \mathbb{X}^r = X^j(y)\frac{\partial}{\partial y^j}$$

PROPOSITION: for every smooth function f, and for all vector fields X, Y on M, we have:

$$[\mathbb{X}_l \,, \mathbb{Y}_l] = ([X \,, Y])_l \;, \qquad [\mathbb{X}_r \,, \mathbb{Y}_r] = ([X \,, Y])_r \;, \qquad [\mathbb{X}_l \,, \mathbb{Y}_r] = 0 \,,$$

$$(f\,X)_l = f_l \mathbb{X}_l \,, \qquad (f\,X)_r = f_r \mathbb{X}_r \,, \qquad L_{\mathbb{X}_l} f_r = L_{\mathbb{X}_r} f_l = 0 \,.$$

COVARIANT TENSORS FROM TWO-POINT FUNCTIONS

QUESTION: How can we extract a covariant (0,2) tensor from a two-point function D?

To answer this question, let us consider the diagonal immersion of M into its double:

$$i_d \colon \mathcal{M} \to \mathcal{M} \times \mathcal{M} \,, \quad m \mapsto i_d(m) = (m\,, m)$$

Let X and Y be arbitrary vector fields on M, D a smooth function on its double, and define the following maps:

$$g_{ll}\,(X\,,Y) := i_d^*\left(L_{\mathbb{X}_l} L_{\mathbb{Y}_l} D\right)\,,\; g_{rr}\,(X\,,Y) := i_d^*\left(L_{\mathbb{X}_r} L_{\mathbb{Y}_r} D\right)\,,$$
$$g_{lr}\,(X\,,Y) := i_d^*\left(L_{\mathbb{X}_l} L_{\mathbb{Y}_r} D\right)\,,\; g_{rl}\,(X\,,Y) := i_d^*\left(L_{\mathbb{X}_r} L_{\mathbb{Y}_l} D\right)\,.$$

COVARIANT TENSORS FROM TWO-POINT FUNCTIONS

Without further assumptions on D, these maps do not define covariant (0,2) tensors.

PROPOSITION:

1) g_{lr} and g_{rl} are covariant (0,2) tensors, and $g_{lr}(X,Y)=g_{rl}(Y,X)$;

2) g_{ll} is a symmetric covariant (0,2) tensor if and only if :

$$i_d^*\left(L_{\mathbb{X}_l} D\right) = 0 \quad \forall X \in \mathfrak{X}(M)$$

3) g_{rr} is a symmetric covariant (0,2) tensor if and only if:

$$i_d^*\left(L_{\mathbb{X}_r} D\right) = 0 \quad \forall X \in \mathfrak{X}(M)$$

4) if the previous two conditions hold simultaneously for D, then:

$$g_{ll} = g_{rr} = -g_{lr} = -g_{rl}$$

COVARIANT TENSORS FROM TWO-POINT FUNCTIONS

DEFINITION: A smooth function D on the double of M such that:

$$i_d^*\left(L_{\mathbb{X}_l} D\right) = 0 \;\; \forall X \in \mathfrak{X}(M) \,, \qquad i_d^*\left(L_{\mathbb{X}_r} D\right) = 0 \;\; \forall X \in \mathfrak{X}(M)$$

is called a **potential function**, and we set $g = -g_{lr}$ for the symmetric covariant (0,2) tensor associated with D.

DEFINITION: A smooth function D on the double of M such that:

$$D(m_1\,, m_2) \geq 0\,, \qquad D(m_1\,, m_2) = 0 \iff m_1 = m_2$$

is called a **divergence function**.

REMARK: Every divergence function is a potential function, and the associated tensor g is positive-semidefinite. The converse is not true.

COVARIANT TENSORS FROM TWO-POINT FUNCTIONS

COORDINATES EXPRESSION: $\{q^j\}$, $\{x^j, y^j\}$ local charts on M and M x M:

$$g = \left(\frac{\partial^2 S}{\partial x^j \partial x^k}\right)\bigg|_d \mathrm{d}q^j \otimes_s \mathrm{d}q^k = \left(\frac{\partial^2 S}{\partial y^j \partial y^k}\right)\bigg|_d \mathrm{d}q^j \otimes_s \mathrm{d}q^k =$$

$$= - \left(\frac{\partial^2 S}{\partial x^j \partial y^k}\right)\bigg|_d \mathrm{d}q^j \otimes_s \mathrm{d}q^k$$

PROPOSITION: Let D be a **potential function** on M x M, then the tensor g is positive-semidefinite if and only if every point on the diagonal is a local minimum for D. In particular, g is a metric tensor if and only if every point of the diagonal is a nondegenerate local minimum for D.

COVARIANT TENSORS FROM TWO-POINT FUNCTIONS

QUESTION: What happens when we consider smooth maps between manifolds?

$$\phi\colon \mathcal{N} \to \mathcal{M}$$

$$\Phi\colon \mathcal{N} \times \mathcal{N} \to \mathcal{M} \times \mathcal{M}$$
$$(n_1\,, n_2) \mapsto \ \Phi(n_1\,, n_2) := (\phi(n_1)\,, \phi(n_2))$$

WE CAN PROVE THAT:

$$\begin{tikzcd}
\text{Symmetric covariant tensor on M} \arrow[r, "\text{Pullback by } \phi"] & \text{Symmetric covariant tensor on N} \\
\text{Potential function on MxM} \arrow[u, "\text{extract}"'] \arrow[r, "\text{Pullback by } \Phi"] & \text{Potential function on NxN} \arrow[u, "\text{extract}"']
\end{tikzcd}$$

IS COMMUTATIVE!

METRICS ON THE SPACE OF INVERTIBLE QUANTUM STATES

Back to the quantum model, we consider the family $\{\mathcal{S}_n\}_{n\in\mathbb{N}_2}$ of manifolds of invertible quantum states, where $\mathbb{N}_2$ is the set of natural numbers without 0 and 1.

QUANTUM STOCHASTIC MAPS:

$$\phi \colon M_n(\mathbb{C}) \to M_m(\mathbb{C})$$

is a Quantum Stochastic map if it is a linear completely-positive trace-preserving map (CPTP map) such that:

$$\phi(\mathcal{S}_n) \subset \mathcal{S}_m$$

REMARK: Quantum Stochastic maps are the quantum analogue of the Markov maps in classical theory of probability. These maps are related with coarse-graining.

METRICS ON THE SPACE OF INVERTIBLE QUANTUM STATES

MONOTONICITY PROPERTY OF METRIC TENSORS

$\{g_n\}_{n\in\mathbb{N}_2}$ is a family of metric tensors on $\{\mathcal{S}_n\}_{n\in\mathbb{N}_2}$

A family of quantum metric tensors satisfies the monotonicity property (MP) if:

$$g_n(X\,,X)\geq\phi^* g_m\,(X\,,X)$$

for every n,m, for every vector field X, and for every quantum stochastic map:

$$\phi\colon\, M_n(\mathbb{C})\to M_m(\mathbb{C})$$

REMARK: The MP entails the fact that distances between quantum states do not increase under coarse-graining.

METRICS ON THE SPACE OF INVERTIBLE QUANTUM STATES

DATA PROCESSING INEQUALITY FOR DIVERGENCE FUNCTIONS

$\{D_n\}_{n\in\mathbb{N}_2}$ is a family of divergence functions on $\{\mathcal{S}_n \times \mathcal{S}_n\}_{n\in\mathbb{N}_2}$

A family of divergence functions satisfies the data processing inequality (DPI) if:

$$D_n(\rho\,,\widetilde{\rho}) \geq \Phi^* D_m(\rho\,,\widetilde{\rho}) \equiv D_m(\phi(\rho)\,,\phi(\widetilde{\rho}))$$

for every n,m, and for every quantum stochastic map:

$$\phi\colon M_n(\mathbb{C}) \to M_m(\mathbb{C})$$

REMARK: From an information-theoretic perspective, the DPI states that information can not increase after coarse-graining.

METRICS ON THE SPACE OF INVERTIBLE QUANTUM STATES

PROPOSITION: Given a family of quantum divergence functions satisfying the DPI, the family of quantum metric tensors that can be extracted from it satisfies the MP.

PROOF:

$$D_n(\rho\,,\widetilde{\rho}) \geq D_m(\phi(\rho)\,,\phi(\widetilde{\rho})) \geq 0 \Longrightarrow D^{\phi}_{nm}(\rho\,,\widetilde{\rho}) := D_n(\rho\,,\widetilde{\rho}) - D_m(\phi(\rho)\,,\phi(\widetilde{\rho})) \geq 0$$

$D^{\phi}_{nm}(\rho\,,\widetilde{\rho})$ is a non-negative potential function vanishing on the diagonal.

g^{ϕ}_{nm} is a positive semidefinite covariant (0,2) tensor.

$$g^{\phi}_{nm}(X,Y) = -i_d^*\left(L_{\mathbb{X}_l}\,L_{\mathbb{Y}_r}(D_n - \Phi^* D_m)\right) = g_n(X,Y) - \phi^* g_m(X,Y) \geq 0$$

$$g_n(X,Y) \geq \phi^* g_m(X,Y)$$

Q.E.D.

METRICS ON THE SPACE OF INVERTIBLE QUANTUM STATES

We studied the family of q-z-Rényi relative entropies (Audenaert, Datta):

$$D_n^{q,z}(\rho\,,\widetilde{\rho}) = \frac{1}{q(1-q)}\left(1 - Tr\left[\left(\rho^{\frac{q}{z}}\,\widetilde{\rho}^{\frac{1-q}{z}}\right)^z\right]\right), \quad q \in \mathbb{R}\,, z \in \mathbb{R}_0^+$$

This family of quantum divergence functions satisfies the DPI when z ≥ 0 and 0 ≤ q ≤1

REMARK: some interesting limiting cases are:

When q=z=1 we recover Von Neumann's relative entropy.
When z=1 we recover the q-Rényi relative entropies.
When z=q we recover the q-quantum Rényi divergence.
When z=1 and q=1/2 we recover the Wigner-Yanase-Dyson skew information.

METRICS ON THE SPACE OF INVERTIBLE QUANTUM STATES

To perform coordinate-free computations we decided to work in the space:

$$M_n = SU(n) \times \Delta_n$$

Special unitary group

$$M_n(\mathbb{C}) \ni \mathbf{U} : \mathbf{U}\mathbf{U}^\dagger = \mathbb{I}, \quad |det(\mathbf{U})| = 1$$

Open interior of the n-dimensional simplex

$$\mathbb{R}^n \ni \vec{p} : \sum_{j=1}^{n} p^j = 1 , \quad p^j > 0$$

SURJECTIVE SUBMERSION:

$$\pi_n : \mathcal{M}_n \ni (\mathbf{U} , \vec{p}) \mapsto \rho = \pi_n(\mathbf{U} , \vec{p}) = \mathbf{U}\,\rho_0\,\mathbf{U}^\dagger \in \mathcal{S}_n, \text{with } diag(\vec{p}) = \rho_0$$

The kernel of the differential at each point is given by the Hermitean matrices commuting with ρ_0

Unfolding of the manifold of invertible quantum states by means of the spectral decomposition

METRICS ON THE SPACE OF INVERTIBLE QUANTUM STATES

The family of q-z-Rényi relative entropies gives rise to a family of potential functions on $\{M_n\}$ by means of the pullback through π_n:

$$\mathbb{D}_n^{q,z}(\mathbf{U},\rho_0;\mathbf{V},\widetilde{\rho}_0) = \frac{1}{q(1-q)}\left(1 - Tr\left[\left((\mathbf{U}\,\rho_0\,\mathbf{U}^\dagger)^{\frac{q}{z}}(\mathbf{V}\,\widetilde{\rho}_0\,\mathbf{V}^\dagger)^{\frac{1-q}{z}}\right)^z\right]\right)$$

REMARK: Being parallelizable, M_n has a basis of globally defined vector fields and differential one-forms, and this global differential calculus allows us to perform calculations in every dimension without the need to introduce coordinates:

$$\mathrm{d}\rho = \mathbf{U}\,\mathrm{d}\rho_0\,\mathbf{U}^\dagger + \mathbf{U}\,\left[\mathbf{U}^\dagger\mathrm{d}\mathbf{U}\,,\rho_0\right]\,\mathbf{U}^\dagger = \mathbf{U}\,\mathbf{e}_j\,\mathbf{U}^\dagger\,\mathrm{d}p^j + \mathbf{U}\,\left[\tau_k\,,\rho_0\right]\,\mathbf{U}^\dagger\,\theta^k$$

$\mathrm{d}\rho_0 = \mathbf{e}_j\mathrm{d}p^j$ → One-form on Δ_n

Diagonal matrix with 1 in the j-th place

$\mathbf{U}^\dagger\mathrm{d}\mathbf{U} = \tau_k\theta^j$ → Left-invariant one-form on SU(n)

Left Maurer-Cartan form

Orthonormal basis in the Lie algebra of SU(n)

METRICS ON THE SPACE OF INVERTIBLE QUANTUM STATES

$$g_n^{q,z} = \sum_{\alpha=1}^{n} p_\alpha \mathrm{d}\ln p_\alpha \otimes \mathrm{d}\ln p_\alpha + \frac{z}{q(1-q)} \sum_{j,k=1}^{n^2-1}{}' \mathcal{C}_{jk}\, \theta^j \otimes \theta^k$$

Where ∑' denotes the summation over all indexes except those pertaining to the Cartan subalgebra of the Lie algebra of SU(n), and:

$$\mathcal{C}_{jk} = \sum_{\alpha,\beta=1}^{n}{}' \mathcal{E}_{\alpha\beta}\, \Re\left[M_j^{\alpha\beta} M_k^{\beta\alpha}\right] \qquad \mathcal{E}_{\alpha\beta} := \frac{(p_\alpha - p_\beta)(p_\alpha^{\frac{q}{z}} - p_\beta^{\frac{q}{z}})(p_\alpha^{\frac{1-q}{z}} - p_\beta^{\frac{1-q}{z}})}{(p_\alpha^{\frac{1}{z}} - p_\beta^{\frac{1}{z}})}$$

$$\tau_k = \sum_{\alpha,\beta=1}^{n} M_k^{\alpha\beta} \mathbf{e}_{\alpha\beta} \longrightarrow$$

with τ_k a basis of the Lie algebra of SU(n), and $e_{\alpha\beta}$ the matrix with 1 in the (α,β) place, and 0 elsewhere

METRICS ON THE SPACE OF INVERTIBLE QUANTUM STATES

After a patient calculation, we obtain the symmetric covariant (0,2) tensor:

$$g_n^{q,z} = \sum_{\alpha=1}^{n} p_\alpha \mathrm{d}\ln p_\alpha \otimes \mathrm{d}\ln p_\alpha + \frac{z}{q(1-q)} \sum_{j,k=1}^{n^2-1}{}' \mathcal{C}_{jk}\,\theta^j \otimes \theta^k$$

Fisher–Rao metric: "Classical-like" contribution depending only on the eigenvalues of the quantum states

Purely quantum contribution depending on eigenvalues and phases of the quantum states, it is tangent to the orbit of the action of the unitary group

Final remarks

- We introduced a coordinate-free algorithm to extract symmetric covariant (0,2) tensors from two-point functions. It can be used to extract symmetric covariant (0,3) tensors (skewness tensors). How far can we go?

- In this formalism, it naturally follows that a family of quantum divergences satisfying the **DPI** gives rise to a family of metric tensors satisfying the **MP**.

- We computed the family of metric tensors associated with the family of q-z-Rényi relative entropies for a quantum system with arbitrary finite dimension. What about the family of skewness tensors?

- Is it possible to look at this family of quantum relative entropies as the Hamilton principal function of some Lagrangian function on the space of invertible quantum states? This would be a sort of dynamical characterization of quantum relative entropies.

THANK YOU FOR YOUR ATTENTION

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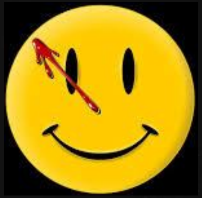

Extra graphics