Variational Integrators for Euler-Poincaré equations

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Motivation

[Geometric mechanics]: Symmetry-preserving **discretizations of variational principles lead to Numerical algorithms** that approximate trajectories of the corresponding dynamical system with good long-term energy conservation properties.

Objective: Create an algorithm that generates discrete fields approximating solutions of Euler-Poincaré field equations:

$$\left(\frac{\partial \ell}{\partial q}(q,\chi)\right)\circ \pi_q - \mathrm{div}_\chi \left(\frac{\partial \ell}{\partial \chi}(q,\chi)\right) = 0 \qquad\qquad (\ell \in \mathcal{C}^\infty(H\mathrm{Str}\times_X \mathrm{CP}))$$

Unknown flat principal connection χ, and χ-parallel H-structure q on principal G-bundle P

Mechanism: Solve discrete Euler-Poincaré field equations arising in H-reduced discrete variational principles (solutions have conservation of discrete Noether currents).

Tasks: – Create discrete Euler-Poincaré equations by means of a covariant discretization of Euler-Poincaré variational principle.

– Devise algorithm to generate solutions of discrete Euler-Poincaré equations.

Idea: Generalize to field theories the arguments used for reduced discrete mechanics on Lie groups (Kobalirov,Marsden).

Contents

Reduced discrete mechanics on Lie group G

Determination of **unknown trajectory** $g(t)\colon \mathbb{R}_t \to G$

Critical for action functional: (Variational principle)

$$\mathbb{L}_K(g(t)) = \int_{t=t_{ini}}^{t=t_{end}} \mathcal{L}(t, g(t), \dot{g}(t))dt, \qquad K = [t_{ini}, t_{end}] \subset \mathbb{R}_t$$

Dynamics is encoded by a **fixed Lagrangian function** $\mathcal{L}(t, g, \dot{g})\colon \mathbb{R} \times TG \to \mathbb{R}$.

Particular situation:

Left-action morphisms $l_h\colon g \in G \mapsto hg \in G$ are symmetries of the Lagrangian

$$\mathcal{L}(t, A_g) = \mathcal{L}(t, (d_g l_h)A_g), \qquad \forall A_g \in T_gG,\ h \in G$$

Left trivialisation $A_g \in TG \to (d_e l_g)^{-1} A_g \in T_eG = \mathrm{Lie}\, G \Rightarrow$ identification $TG/G \simeq \mathrm{Lie}\, G$

Reduced Lagrangian $\ell(t, \xi)\colon \mathbb{R} \times \mathrm{Lie}\, G \to \mathbb{R}$

$$\mathcal{L}(t, A_g) = \ell(t, (d_e l_g)^{-1} A_g) \qquad \mathcal{L}(t, g, \dot{g}) = \ell(t, g^{-1}\dot{g})$$

Unknown reduced trajectory $\xi(t) = (d_e l_{g(t)})^{-1}((d/dt)g(t)) = (g(t))^{-1}\dot{g}(t)$ on $\mathrm{Lie}\, G$

Reduced discrete mechanics on Lie group G

Reduced variational principles in mechanics: [Crouch, Lewis, Munthe-Kaas, Owren, Kobilarov]

Determination of unknown reduced trajectories $\xi(t)\colon \mathbb{R}_t \to \operatorname{Lie} G$

Critical for action functional:

$$\mathbb{L}_K(\xi(t)) = \int_{t=t_{ini}}^{t=t_{end}} \ell(t,\xi(t))dt, \qquad K = [t_{ini}, t_{end}] \subset \mathbb{R}_t$$

with respect to **particular variations** $\operatorname{Var}_\xi = \{\delta_a \xi = \dot{a} + [a,\xi]\}$ parameterized by particular curves $a(t)$ on $\operatorname{Lie} G$ with compact support.

Euler-Poincaré $\dfrac{d}{dt}\left(\dfrac{\partial \ell}{\partial \xi}(t,\xi(t))\right) = \left(\dfrac{\partial \ell}{\partial \xi}(t,\xi(t))\right) \circ \operatorname{Ad}_{\xi(t)} \in (\operatorname{Lie} G)^*$ (1st order)

$\star$ Discrete analogue?

Reduced discrete mechanics on Lie group G

Discretisation of timeline manifold: Fix increasing sequence of temporal events $(t_k)_{k\in\mathbb{Z}}$ with time-steps $h_k = t_{k+1} - t_k > 0$

Retraction mapping: Fix $\tau\colon \operatorname{Lie} G \to G$ to linearize elements close to $e \in G$

Approximate solution $g(t)$ by discrete sequence $(g_k)_{k\in\mathbb{Z}}$ on G, where g_k is considered approximation to $g(t_k) \Rightarrow$ Approximate $\xi(t_k)$ and the action functional by:

$$\xi_k = \frac{\tau^{-1}(g_k^{-1} g_{k+1})}{h_k}, \qquad \mathbb{L}_K(\xi(t)) \simeq \mathbb{L}^d_K(g_k)_{k\in\mathbb{Z}} = \sum_{[t_k, t_{k+1}]\subset K} \ell(t_k, \xi_k) h_k$$

Criticality: Discrete Euler-Poincaré equations (sequence in $(\operatorname{Lie} G)^*$)

$$0 = \left(\frac{\partial \ell}{\partial \xi}(t_{k-1}, \xi_{k-1})\right) \circ (d\tau)^{-1}_{h_{k-1}\xi_{k-1}} \circ \operatorname{Ad}_{\tau(h_{k-1}\xi_{k-1})} - \left(\frac{\partial \ell}{\partial \xi}(t_k, \xi_k)\right) \circ (d\tau)^{-1}_{h_k \xi_k} \quad \forall k \in \mathbb{Z}$$

Can be expressed as:

$$0 = \operatorname{Ad}^*_{\tau(h_{k-1}\xi_{k-1})} \mu_{k-1} - \mu_k$$

$$\mu_k = \left(\frac{\partial \ell}{\partial \xi}(t_k, \xi_k)\right) \circ (d\tau)^{-1}_{h_k \xi_k}$$

Reduced discrete mechanics on Lie group G

Solving Discrete Euler-Poincaré equations

$$0 = \mathrm{Ad}^*_{\tau(h_{k-1}\xi_{k-1})}\,\mu_{k-1} - \mu_k$$

Discrete Legendre mapping $Leg_{\tau,\ell}\colon (k,\xi_k) \in \mathbb{Z} \times \mathrm{Lie}\,G \mapsto (k,\mu_k) \in \mathbb{Z} \times (\mathrm{Lie}\,G)^*$

$$\mu_k = \left(\frac{\partial \ell}{\partial \xi}(t_k,\xi_k)\right) \circ (d\tau)^{-1}_{h_k\xi_k}$$

Integrating discrete Euler-Poincaré equations relies in:

- From μ_{k-1} and using $Leg^{-1}_{\tau,\ell}$ one may obtain ξ_{k-1}.
- From ξ_{k-1}, μ_{k-1} and using discrete E.P. one may obtain μ_k.

Iterative application of the first two steps allows to recover $(\mu_k,\xi_k)_{k\in\mathbb{N}}$ from μ_0 or ξ_0.

- From (g_k,ξ_k), and using definition $\xi_k = \frac{\tau^{-1}(g_k^{-1}g_{k+1})}{h_k}$ one may obtain g_{k+1}.

Iteration allows to recover $(g_k,\xi_k,\mu_k)_{k\in\mathbb{N}}$ from

either (g_0,g_1), or (g_0,ξ_0) or (g_0,μ_0) (Initial data)

Euler-Poincaré in field equations

Fields: Sections $y(x) \in \Gamma(Y)$ of bundle $\pi\colon Y \to X$ (x^ν, y^i) fibered local coordinates

X oriented by volume element $\mathrm{vol}_X = dx^1 \wedge \ldots \wedge dx^n \in \Omega^n(X)$

$j\pi\colon JY \to X$ associated **jet bundle** (x^ν, y^i, y^i_ν) induced local coordinates

Lagrangian function $\mathcal{L}(x^\nu, y^i, y^i_\nu)\colon JY \to \mathbb{R} \Rightarrow$ Lagrangian density $\mathcal{L} \cdot \mathrm{vol}_X \Rightarrow$

Action functional $\mathbb{L}_K(y) = \int_K \mathcal{L} \circ jy \cdot \mathrm{vol}_X$ on compact domains $K \subset X$

Necessary condition for $y \in \Gamma(Y)$ to minimize $\mathbb{L}_K$ (with respect to variations of y vanishing at ∂K) is $0 = \mathcal{EL}(y) \in \Gamma(y^* V^* Y)$ (**Euler-Lagrange equations**)

$$\mathcal{EL}_x(y) = \left[\frac{\partial \mathcal{L}}{\partial y^i}(j_x y) - \sum_\nu \left(\frac{d}{dx^\nu}\right)_x \frac{\partial \mathcal{L}}{\partial y^i_\nu}(jy)\right] dy^i \in V^*_{y(x)}Y$$

Infinitesimal symmetries have corresponding conserved currents (Noether)

Euler-Poincaré in field equations

Particular case: Smooth proper free action $\lambda\colon G\times P\to P$ of Lie group G on manifold P.

$\pi^G\colon p\in P\mapsto Gp\in P/G=X$ **principal G-bundle**

$\pi_{\mathrm{Gau}}\colon\ \mathrm{Gau}\,P\to X$ bundle of G-covariant automorphisms of P

$$\phi_x\colon P_x\leftrightarrow P_x,\quad \phi_x(gp_x)=g\phi_x(p_x),\quad \forall g\in G,\,\phi_x\in(\mathrm{Gau}\,P)_x$$

$\pi_{\mathrm{Ad}}\colon\ \mathrm{Ad}\,P=VP/G\to X$ bundle of π^G-vertical G-invariant vector fields on P

$$a_x\in\mathfrak{X}(P_x),\,\lambda_g a_x=a_x\,\forall g\in G,\,\forall a_x\in(\mathrm{Ad}\,P)_x$$

$\exp\colon(\epsilon,A_x)\in\mathbb{R}\times\mathrm{Ad}\,P\to\exp\epsilon A_x\in\mathrm{Gau}\,P$ ($\mathrm{Id}^*\,V\,\mathrm{Gau}\,P\simeq\mathrm{Ad}\,P$)

$$0\to P\underset{X}{\times}\mathrm{Ad}\,P\to VP\to 0$$

$$(p_x,A_x)\mapsto(d/d\epsilon)_{\epsilon=0}(\exp\epsilon A_x)(p_x)$$

Euler-Lagrange equations for $\mathcal{L}\colon JP\to\mathbb{R}$:

$VP\simeq P\times_X\mathrm{Ad}\,P\Rightarrow p^*V^*P\simeq\mathrm{Ad}^*\,P\Rightarrow\mathcal{EL}(p)\in\Gamma(\mathrm{Ad}^*\,P)$

Euler-Poincaré in field equations

Existence of symmetries for a Lagrangian $\mathcal{L}\colon JP \to \mathbb{R}$? Two particular cases:

- Subgroup of the infinite-dimensional gauge group $\Gamma(\operatorname{Gau} P)$, given by Gauge transformations $\phi\colon P \to P$ such that $\mathcal{L} \circ j\phi = \mathcal{L}$.
- Subgroup H of the Lie group G given by group elements $h \in G$ such that $\mathcal{L} \circ j\lambda_h = \mathcal{L}$.

Reduction by closed subgroup $H \subseteq G$ $\qquad JP/H \simeq (P/H) \times_{P/G} (JP/G)$

• $\pi_{H\mathrm{Str}}\colon H\mathrm{Str} = P/H \to X$ bundle of H-structures

$\pi^H\colon P \to P/H$ principal H-bundle $P^{H\mathrm{Str}}$ with $H\mathrm{Str}$ as base manifold

$\pi^H_{\mathrm{Ad}}\colon \operatorname{Ad} P^{H\mathrm{Str}} = VP/H \to H\mathrm{Str}$ bundle of π^H-vertical H-invariant vector fields on P

$$0 \to \operatorname{Ad} P^{H\mathrm{Str}} \to H\mathrm{Str} \underset{X}{\times} \operatorname{Ad} P \to VH\mathrm{Str} \to 0$$

• $\pi_{\mathrm{CP}}\colon JP/G = \mathrm{CP} \to P/G = X$. Its sections $\chi \in \Gamma(JP/G)$ are in one-to-one correspondence with principal connections on P. (CP bundle of principal connections)

CP is an affine bundle modelled on $T^*X \otimes \operatorname{Ad} P$, therefore $V\mathrm{CP} \simeq \mathrm{CP} \times_X (T^*X \otimes \operatorname{Ad} P)$

$$q^* VH\mathrm{Str} \simeq \operatorname{Ad} P / q^* \operatorname{Ad} P^{H\mathrm{Str}} \qquad \chi^* V\mathrm{CP} \simeq T^*X \otimes \operatorname{Ad} P$$

Euler-Poincaré in field equations

Particular case: Smooth proper free action of Lie group G on manifold P.

$\pi \colon P \to P/G = X$ principal G-bundle;

Closed subgroup $H \subset G$ acts as symmetries of Lagrangian $\mathcal{L} \colon JP \to \mathbb{R}$.

Reduction of configuration bundle:

$\pi_{\mathrm{CP}} \colon JP/G = \mathrm{CP} \to P/G = X$ bundle of principal connections

$\pi_{H\mathrm{Str}} \colon H\mathrm{Str} = P/H \to X$ bundle of H-structures

$JP/H \simeq H\mathrm{Str} \times_X \mathrm{CP} \to X$ bundle of H-reduced fields.

Reduction of a field:

$p \in \Gamma(P) \Rightarrow \chi = \pi^G \circ jp \in \Gamma(JP/G) = \Gamma(\mathrm{CP})$ induced principal connection

$p \in \Gamma(P) \Rightarrow q = \pi^H \circ p \in \Gamma(P/H) = \Gamma(H\mathrm{Str})$ induced H-structure

(q, χ) generated from $p \in \Gamma(P) \Rightarrow$ Flat connection, Parallel H-structure:

$$\mathrm{Curv}\chi = 0, \qquad d^{\chi} q = 0$$

Local reconstruction of p is possible from $(q, \chi) \in \Gamma(H\mathrm{Str} \times_X \mathrm{CP})$ if and only if the connection is flat and the H-structure is parallel.

Reduced variational principle (Recall $JP/H = H\mathrm{Str}\times_X \mathrm{CP}$)

Choice of reduced Lagrangian $\ell\colon H\mathrm{Str}\times_X \mathrm{CP}\to\mathbb{R}$ (or H-invariant $\mathcal{L}\colon JP\to\mathbb{R}$)

Choice of admissible H-reduced fields $(q,\chi)\in\Gamma(H\mathrm{Str}\times_X \mathrm{CP})$ of the particular form

Flatness: $\mathrm{Curv}\chi = 0,$ Parallelism: $d^\chi q = 0$

Choice of admissible infinitesimal variations $(\delta_a q,\delta_a\chi)\in\Gamma(q^*VH\mathrm{Str})\oplus\Gamma(\chi^*V\mathrm{CP})\simeq$ $\Gamma(\mathrm{Ad}\,P/q^*\,\mathrm{Ad}\,P^{H\mathrm{Str}})\oplus\Gamma(T^*X\otimes\mathrm{Ad}\,P)$ of the particular form

$\delta_a q = \pi_q(a), \quad \delta_a\chi = d^\chi a,$ compactly supported $a\in\Gamma(\mathrm{Ad}\,P)$

Relevant difference:

Substitution of arbitrary fields $p(x)\in\Gamma(P)$ and variations with compact support $\delta p\in\Gamma^c(p^*VP)$ by H-reduced field $(q(x),\chi(x))\in\Gamma(H\mathrm{Str}\times_X \mathrm{CP})$ with a flatness+parallelism constraint, and a reduced family of infinitesimal variations.

The new formulation admits new global fields, not represented by global “potentials” $p\in\Gamma(P)$

Euler-Poincaré in field equations

Theorem [Castrillon,Ratiu 2003]

Fix $\ell\colon H\mathrm{Str}\times_X \mathrm{CP}\to\mathbb{R}$ (H-reduced lagrangian), $\mathcal{L}=\ell\circ\pi^H\colon JP\to\mathbb{R}$

The following are **equivalent for** $p\in\Gamma(P)$ **and induced** $(q,\chi)\in\Gamma(H\mathrm{Str}\times_X \mathrm{CP})$

1. $\delta\int\mathcal{L}\circ jp\cdot\mathrm{vol}_X=0$ holds for variations δp with compact support.
2. $p\in\Gamma(P)$ satisfies **Euler-Lagrange equations** $0=\mathcal{EL}(p)\in\Gamma(p^*VP)$ assoc.to $\mathcal{L}$.
3. $\delta\int\ell\circ(q,\chi)\cdot\mathrm{vol}_X=0$ holds for the subset $\mathrm{Var}_{q,\chi}$ of variations with the form

   $\delta_a q=\pi_q(a)\in\Gamma(\mathrm{Ad}\,P/q^*\,\mathrm{Ad}\,P^{H\mathrm{Str}})=\Gamma(q^*VH\mathrm{Str})$,

   $\delta_a\chi=d^\chi a\in\Gamma(T^*X\otimes\mathrm{Ad}\,P)=\Gamma(\chi^*\mathrm{CP})$

   with compactly supported $a\in\Gamma(\mathrm{Ad}\,P)$
4. (q,χ) satisfy **Euler-Poincaré equations** $0=\mathcal{EP}(q,\chi)$, where

$$\mathcal{EP}(q,\chi)=\left(\frac{\partial\ell}{\partial q}(q,\chi)\right)\circ\pi_q-\mathrm{div}_\chi\left(\frac{\partial\ell}{\partial\chi}(q,\chi)\right)\in\Gamma(\mathrm{Ad}^*\,P)$$

Here $\pi_q\colon\ \mathrm{Ad}\,P\to\mathrm{Ad}\,P/q^*\,\mathrm{Ad}\,P^{H\mathrm{Str}}\simeq q^*VH\mathrm{Str}$, $d^\chi\colon\ J\,\mathrm{Ad}\,P\to T^*X\otimes\mathrm{Ad}\,P\simeq\chi^*V\mathrm{CP}$

Discrete model of space: CFK simplicial partition of space.

Discrete model of timeline for mechanics: Totally ordered countable set V. (Abstract but indexed by points $x = k \in \mathbb{Z}$ of the real line $\mathbb{R}$). Specific events given as monotone sequence $(t_k)_{k\in\mathbb{Z}}$, generating a partition of smooth timeline into temporal intervals.

Discrete model of base manifold for field theories? Specific nodes as $(x_v)_{v\in\mathbb{Z}^n} \subset X$

Simplicial complex structure on $\mathbb{Z}^n$: Removing from $\mathbb{R}^n$ hyperplanes

$$x^i = c \in \mathbb{Z}, \qquad x^{i_1} - x^{i_2} = c \in \mathbb{Z}$$

Taking the closure of its connected components: Partition into affine simplices

$$\bar{K}_{v,\sigma} = \{v + \epsilon_1 e_{\sigma(1)} + \ldots + \epsilon_n e_{\sigma(n)},\, 1 \geq \epsilon_1 \geq \epsilon_2 \geq \ldots \geq \epsilon_n \geq 0\}$$

$$\mathrm{Ext}(\bar{K}_{v,\sigma}) = \{v_0, v_1, \ldots, v_n\}, \qquad v_0 = v,\, v_k = v_{k-1} + e_{\sigma(k)}$$

$(v \in \mathbb{Z}^n$, $\sigma \in \mathrm{Sym}_n)$

Abstract Coxeter-Freudenthal-Kuhn (CFK) facet: Sequence of $n+1$ points $(v_0, \ldots, v_n) \subset \mathbb{Z}^n$ with $v_k = v_{k-1} + e_{\sigma(k)}$ for some permutation $\sigma \in \mathrm{Sym}_n$.

Discrete model of timeline for mechanics: Totally ordered countable set V. (Abstract but indexed by points $x = k \in \mathbb{Z}$ of the real line $\mathbb{R}$). Specific events given as monotone sequence $(t_k)_{k\in\mathbb{Z}}$, generating a partition of smooth timeline into temporal intervals.

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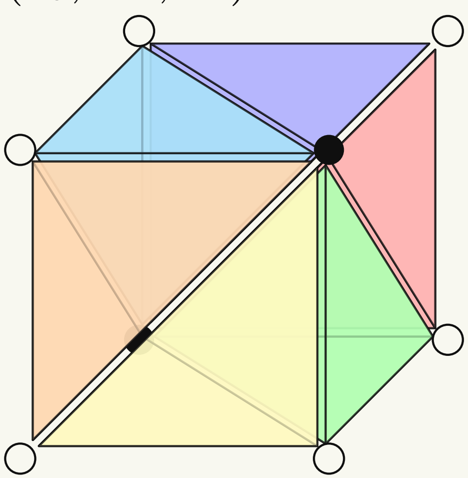
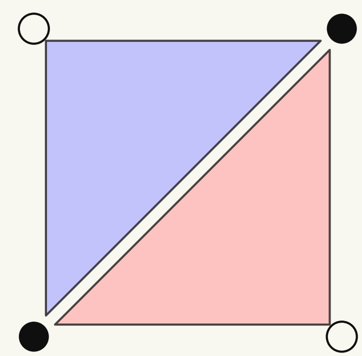




3D and 2D CFK simplicial partitions on a single cube/square, respectively

Abstract Coxeter-Freudenthal-Kuhn (CFK) facet: Sequence of $n+1$ points $(v_0, \ldots, v_n) \subset \mathbb{Z}^n$ with $v_k = v_{k-1} + e_{\sigma(k)}$ for some permutation $\sigma \in \mathrm{Sym}_n$.

Abstract CFK k-dimensional simplex: Subsequence of $k+1$ points $(v_{i_0}, \ldots, v_{i_k}) \subset \mathbb{Z}^n$ ($0 \leq i_0 < \ldots < i_k \leq n$) of some abstract CFK facet $(v_0, \ldots, v_n)$

CFK Simplicial Complex: Family $\mathcal{V}$ of all abstract CFK simplices.

Set $\mathcal{V}$ of all abstract CFK simplices.

$V^k = \{\alpha \in \mathcal{V} : \ \dim \alpha = k\} \subset \mathcal{V}$ ($V = V^0 = \mathbb{Z}^n$)

$\beta = (v_0, \ldots, v_n) \in V^n \subset \mathcal{V}$ CFK **facet**.

$\alpha = (v_0, v_1) \in V^1 \subset \mathcal{V}$ CFK **edge**.

Topology arises from a natural adherence notion: $\alpha \prec \beta$ (being a subsequence)

DEFINE: $V^{\times k} = V \times \ldots \times V$ (k+1) copies OBSERVE: $V^k \subset V^{\times k}$

Natural projectors $\pi_{i_0 \ldots i_k} : V^j \to V^k$ ($0 \leq i_0 < \ldots < i_k \leq j$)

$\{\pi_i(\beta)\}_{0 \leq i \leq k}$ adherent vertices $\{\pi_{i_0 i_1}(\beta)\}_{0 \leq i_0 < i_1 \leq k}$ adherent edges of $\beta \in V^k$

Discrete variational principles on simplicial complexes

Discrete bundle on V: Projection $Y_d \to V$ whose fibers Y_v are smooth manifolds.

Vertical bundle $VY_d \to Y_d$, with fiber $(VY_d)_{y_v} = T_{y_v} Y_v$

Discrete field on Y_d: Section $y_d \colon v \in V \mapsto y_v \in Y_d$

Infinitesimal variation of $y_d \in \Gamma(Y_d)$: Section of discrete bundle $y_d^* VY_d \to V$

Any discrete bundle $Y_d \to V$ induces an

Extended bundle $Y_d^k \to V^k$ (Restriction of $(Y_d)^{\times k} \to V^{\times k}$ to $V^k \subset V^{\times k}$)

Discrete jet bundle: $Y_d^n \to V^n$.

Jet extension of discrete fields: $y_d \in \Gamma(Y_d) \Rightarrow y_d^n \in \Gamma(Y_d^n)$

$\beta = (v_0, \ldots, v_n) \in V^n \subset V^{\times n} \Rightarrow y_\beta^n = (y_{v_0}, \ldots, y_{v_n}) \in Y_d^n \subset (Y_d)^{\times n}$

Discrete Lagrangian $\mathcal{L}_d \colon Y_d^n \to \mathbb{R}^n$. Family $(\mathcal{L}_\beta)_{\beta \in V^n}$ of smooth functions

$$\mathcal{L}_\beta \colon Y_\beta^n = Y_{v_0} \times Y_{v_1} \times \ldots \times Y_{v_n} \to \mathbb{R}, \qquad \beta = (v_0, \ldots, v_n) \in V^n \subset V^{\times n}$$

Differential at $y_\beta^n = (y_0, \ldots, y_n) \in Y_d^n$ of a Discrete Lagrangian $\mathcal{L}_d \colon Y_d^n \to \mathbb{R}$:

$$d_{y_\beta^n} \mathcal{L}_\beta = (d^0_{y_\beta^n} \mathcal{L}_\beta, \ldots, d^n_{y_\beta^n} \mathcal{L}_\beta) \in T^*_{y_0} Y_{\pi_0(\beta)} \oplus \ldots \oplus T^*_{y_n} Y_{\pi_n(\beta)}$$

Action functional associated to $\mathcal{L}_d$ and finite domain $K \subset V^n$:

$$\mathbb{L}_K \colon y_d \in \Gamma(Y_d) \mapsto \sum_{\beta \in K} (\mathcal{L}_d \circ y_d^n)(\beta)$$

$$d_{y_d}\mathbb{L}_K \colon \delta y_d \in \Gamma(y_d^* V Y_d) \mapsto \sum_{\beta \in K} (d\mathcal{L}_d \circ \delta y_d^n)(\beta)$$

Criticality: For any given discrete bundle $Y_d \to V$ on the n-dimensional CFK simplicial complex, and any given discrete Lagrangian $\mathcal{L}_d \colon Y_d^n \to \mathbb{R}$ we say $y_d \in \Gamma(Y_d)$ is critical for the variational principle associated to $\mathcal{L}_d$, with fixed boundary variations, if $\langle d_{y_d}\mathbb{L}_K, \delta y_d \rangle$ vanishes for infinitesimal variations $\delta y_d \in \bigoplus_{v \in \mathrm{int}\, K} T_{y_v} Y_v \subset \Gamma(y_d^* V Y_d)$ with support interior to K, for each finite domain $K \subset V^n$.

THEOREM: A section $y_d \in \Gamma(Y_d)$ is **critical** for the variational principle associated to $\mathcal{L}_d$, with fixed boundary variations, if and only if the **discrete Euler-Lagrange tensor** $\mathcal{E}\mathcal{L}_d(y_d) \in \Gamma(y_d^* V^* Y_d)$ vanishes, where:

$$\mathcal{E}\mathcal{L}_v(y_d) = \sum_{\beta \in \mathrm{Star}_v^n} d_{y_\beta^n}^{I(v,\beta)} \mathcal{L}_\beta \in T_{y_v}^* Y_v$$

H-reduction of discrete jet bundle

Particular case: Discrete principal G-bundle $P_d \to V$.

Difference of two elements p_0, p_1 on P_d can be computed as:

- Group element $g \in G$ such that $gp_0 = p_1$ (**group difference** $g = p_1 p_0^{-1}$).

  Existence only if $\pi(p_0) = \pi(p_1)$. Uniqueness.
- G-covariant morphism $\psi \colon P_d \to P_d$ such that $\psi(p_0) = p_1$ (**gauge difference**).

  Existence; Not unique but on the G-orbit of p_0 determined by $\psi(\bar{p}_0) = (\bar{p}_0 p_0^{-1})p_1$.

For $p_0 \in P_v$ and $p_1 \in P_d$ denote $p_0^{-1}p_1$ the uniquely defined G-covariant morphism defined on the single G-orbit P_v such that $\psi(p_0) = p_1$.

Use reversed notation $\bar{p}\psi$ instead of $\psi(\bar{p})$, $\psi \circ \bar{\psi} = \bar{\psi}\psi$

Ehresmann's bundle $\operatorname{End} P_d$ of fiber-to-fiber endomorphisms associated to $P_d \to V$:

$\operatorname{End} P_d = \{\psi \colon P_v \to P_d$ domain a single fiber, $\psi \circ \lambda_g = \lambda_g \circ \psi\}$ groupoid

Source+Target: $(s,t) \colon \operatorname{End} P_d \to V \times V$ $\operatorname{Dom}(\psi) = P_{s(\psi)}, \quad \operatorname{Img}(\psi) = P_{t(\psi)}$

Restrict $\operatorname{End} P_d \to V \times V$ to diagonal $\Rightarrow \operatorname{Gau} P_d \to V$ gauge bundle.

H-reduction of discrete jet bundle

$\operatorname{End} P_d = \{\psi\colon P_v \to P_d$ domain a single fiber, $\psi \circ \lambda_g = \lambda_g \circ \psi\}$

Source+Target: $(s,t)\colon \operatorname{End} P_d \to V \times V$

Fibered product $(\operatorname{End} P_d)^{\times_s k-1}$ of k fiber-to-fiber endomorphisms with common source.

$$(s, t_1, \ldots, t_k)\colon (\operatorname{End} P_d)^{\times_s k-1} \to V^{\times k}$$

Extension to simplices: Restrict to $V^k \subseteq V^{\times k} \Rightarrow (s, t_1, \ldots, t_k)\colon \operatorname{End}^k P_d \to V^k$, .

$\operatorname{End}^n P_d \to V^n$ **extended Ehresmann bundle on facets** of the CFK simplicial complex

$$(\psi_i\colon P_{v_0} \to P_{v_i})_{i=1\ldots n},\ (v_0, v_1, \ldots, v_n) \in V^n \subset V \times \ldots \times V$$

$\operatorname{End}^1 P_d \to V^1$ **Ehresmann bundle on edges** of the CFK simplicial complex

$$\psi\colon P_{v_0} \to P_{v_1}, \quad (v_0, v_1) \in V^1 \subset V \times V$$

PROPOSITION: The Gauge difference mapping $(p_0, p_1) \in P_d^1 \to p_0^{-1} p_1 \in \operatorname{End}^1 P_d$ has as fibers the orbits of G acting diagonally $(p_0, p_1) \mapsto (g p_0, g p_1)$ on P_d^1.

$\pi^G\colon P_d^1 \to P_d^1/G \simeq \operatorname{End}^1 P_d$

H-reduction of discrete jet bundle

PROPOSITION: The projectors $\pi_0 \colon P_d^n \to P_d$, and $\pi^G \circ \pi_{0i} \colon P_d^n \to \operatorname{End}^1 P_d$ determine a natural identification $P_d^n \simeq P_d \times_{(\pi,s)} \operatorname{End}^n P_d$. Under this identification the diagonal action $\lambda_g^{\times n}$ on P_d^n is identified by the action $\lambda_g \times \mathrm{Id}_{\operatorname{End}^n P_d}$.

COROLLARY: For any closed subgroup $H \subseteq G$

$$\boxed{P_d^n/H \simeq H\mathrm{Str}_d \underset{(\pi_{H\mathrm{Str}},s)}{\times} \operatorname{End}^n P_d}$$

Where $\pi_{H\mathrm{Str}} \colon H\mathrm{Str}_d = P_d/H \to V$ is the discrete bundle of H-structures, $\operatorname{End}^n P_d \to V^n$ is the extended Ehresmann bundle on facets, and we consider a fibered product over the source mapping $s \colon \operatorname{End}^n P_d \to V$, leading to a bundle on the set of facets $V \times_s V^n = V^n$.

Remark: $s \colon \operatorname{End}^n P_d \to V$ is the analogue for discrete field theories of the bundle of principal connections $\mathrm{CP} \to X$ that exists in the smooth field theories on principal G-bundles.

H-reduction of discrete jet bundle

Calculus of variations: Introduce the discrete Lagrangian function $\mathcal{L}_d \colon P_d^n \to \mathbb{R}$

We call trivialised form associated to $\mathcal{L}_d(p_0, p_1, \dots, p_n)$, the function $\mathcal{L}_d(p_0, \psi_1, \dots, \psi_n)$ determined by $\mathcal{L}_d$ using the natural identification $P_d^n \simeq P_d \times_{(\pi,s)} \operatorname{End}^n P_d$

Discrete Lagrangian functions that are **invariant for H acting diagonally** on P_d^n are in one-to-one correspondence with smooth functions $\ell_d(q_0, \psi_1, \dots, \psi_n)$ defined on the bundle of H-reduced discrete jets

$$RJP_d = H\mathrm{Str}_d \underset{(\pi_{H\mathrm{Str}}, s)}{\times} \operatorname{End}^n P_d \to V^n$$

These $\ell_d \colon RJP_d \to \mathbb{R}$ are called **H-reduced discrete Lagrangian functions**

RECALL $VP_d/G = \operatorname{Ad} P_d \to P_d/G = V$ discrete adjoint bundle associated to P_d

PROPOSITION: The differential of the Gauge difference mapping $\pi^G \colon P_d^1 \to \operatorname{End}^1 P_d$ determines a natural identification (source trivialisation):

$$s^* \operatorname{Ad} P_d \simeq V \operatorname{End}^1 P_d$$

H-reduction of discrete jet bundle

Calculus of variations: Introduce the discrete Lagrangian function $\mathcal{L}_d \colon P_d^n \to \mathbb{R}$

We call trivialised form associated to $\mathcal{L}_d(p_0, p_1, \ldots, p_n)$, the function $\mathcal{L}_d(p_0, \psi_1, \ldots, \psi_n)$ determined by $\mathcal{L}_d$ using the natural identification $P_d^n \simeq P_d \times_{(\pi,s)} \operatorname{End}^n P_d$

Discrete Lagrangian functions that are **invariant for H acting diagonally** on P_d^n are in one-to-one correspondence with smooth functions $\ell_d(q_0, \psi_1, \ldots, \psi_n)$ defined on the bundle of H-reduced discrete jets

$$RJP_d = H\mathrm{Str}_d \underset{(\pi_{H\mathrm{Str}}, s)}{\times} \operatorname{End}^n P_d \to V^n$$

These $\ell_d \colon RJP_d \to \mathbb{R}$ are called **H-reduced discrete Lagrangian functions**

RECALL $VP_d/G = \operatorname{Ad} P_d \to P_d/G = V$ discrete adjoint bundle associated to P_d

RECALL $P_d \to P_d/H = H\mathrm{Str}_d$ principal H-bundle $P^{H\mathrm{Str}}$

PROPOSITION: The differential of the quotient mapping $\pi^H \colon P_d \to H\mathrm{Str}_d$ determines an exact sequence

$$0 \to \operatorname{Ad} P^{H\mathrm{Str}} \to \pi_{H\mathrm{Str}}^* \operatorname{Ad} P_d \to VH\mathrm{Str}_d \to 0$$

H-reduction of discrete jet bundle

Calculus of variations: Introduce the discrete Lagrangian function $\mathcal{L}_d \colon P_d^n \to \mathbb{R}$

We call trivialised form associated to $\mathcal{L}_d(p_0, p_1, \ldots, p_n)$, the function $\mathcal{L}_d(p_0, \psi_1, \ldots, \psi_n)$ determined by $\mathcal{L}_d$ using the natural identification $P_d^n \simeq P_d \times_{(\pi,s)} \operatorname{End}^n P_d$

Discrete Lagrangian functions that are **invariant for H acting diagonally** on P_d^n are in one-to-one correspondence with smooth functions $\ell_d(q_0, \psi_1, \ldots, \psi_n)$ defined on the bundle of H-reduced discrete jets

$$RJP_d = H\mathrm{Str}_d \underset{(\pi_{H\mathrm{Str}}, s)}{\times} \operatorname{End}^n P_d \to V^n$$

These $\ell_d \colon RJP_d \to \mathbb{R}$ are called **H-reduced discrete Lagrangian functions**

PROPOSITION: Source trivialisation $V \operatorname{End}^1 P_d \simeq s^* \operatorname{Ad} P_d$ determines

$V_{(q_0, \psi_1, \ldots, \psi_n)} RJP_d = \left(\operatorname{Ad} P_{v_0} / \operatorname{Ad} P_{q_0}^{H\mathrm{Str}}\right) \oplus \bigoplus_{i=1}^n \operatorname{Ad} P_{v_0}$

and a natural immersion

$V^*_{(q_0, \psi_1, \ldots, \psi_n)} RJP_d \subseteq \operatorname{Ad}^* P_{v_0} \oplus \bigoplus_{i=1}^n \operatorname{Ad}^* P_{v_0} = \bigoplus_{i=0}^n \operatorname{Ad}^* P_{v_0}$

(first component vanishes on $\operatorname{Ad} P_{q_0}^{H\mathrm{Str}} \subset \operatorname{Ad} P_{v_0}$).

H-reduction of discrete jet bundle

Calculus of variations: Introduce the discrete Lagrangian function $\mathcal{L}_d\colon P_d^n \to \mathbb{R}$

We call trivialised form associated to $\mathcal{L}_d(p_0, p_1, \ldots, p_n)$, the function $\mathcal{L}_d(p_0, \psi_1, \ldots, \psi_n)$ determined by $\mathcal{L}_d$ using the natural identification $P_d^n \simeq P_d \times_{(\pi,s)} \operatorname{End}^n P_d$

Discrete Lagrangian functions that are **invariant for H acting diagonally** on P_d^n are in one-to-one correspondence with smooth functions $\ell_d(q_0, \psi_1, \ldots, \psi_n)$ defined on the bundle of H-reduced discrete jets

$$RJP_d = H\mathrm{Str}_d \underset{(\pi_{H\mathrm{Str}}, s)}{\times} \operatorname{End}^n P_d \to V^n$$

These $\ell_d\colon RJP_d \to \mathbb{R}$ are called H**-reduced discrete Lagrangian functions**

$V^*_{(q_0,\psi_1,\ldots,\psi_n)} RJP_d \subseteq \operatorname{Ad}^* P_{v_0} \oplus \bigoplus_{i=1}^n \operatorname{Ad}^* P_{v_0} = \bigoplus_{i=0}^n \operatorname{Ad}^* P_{v_0}$

DEFINE: For any H-reduced discrete lagrangian function $\ell_d\colon RJP_d \to \mathbb{R}$, its differential $d_{rj}\ell_d$ at any H-reduced discrete jet $rj = (q_0, \psi_1, \ldots, \psi_n) \in RJP_d$ determines $n+1$ linear forms on $\operatorname{Ad} P_{s(rj)}$ that we denote:

$$\partial^0_{rj}\ell_d,\ \partial^{01}_{rj}\ell_d,\ \ldots,\ \partial^{0n}_{rj}\ell_d \in \operatorname{Ad}^* P_{s(rj)}$$

Discrete Euler-Poincaré equations in H-reduced coordinates

Theorem: If $\mathcal{L}_d\colon P_d^n \to \mathbb{R}$ is the discrete Lagrangian function $\mathcal{L}_d = \ell_d \circ \pi^H$ determined using the natural projector $P_d^n \to P_d^n/H = RJP_d$ and any H-reduced discrete Lagrangian function $\ell_d\colon RJP_d \to \mathbb{R}$, then the discrete Euler-Lagrange tensor $\mathcal{EL}_d(p) \in \Gamma(p^* V^* P_d)$ associated to $\mathcal{L}_d$ and a section $p \in \Gamma(P_d)$, using the identification $VP_d \simeq P_d \times_V \operatorname{Ad} P_d$, takes the specific form:

$$\mathcal{EL}_v(p) = \sum_{\pi_0(\beta)=v} \partial^0_{rj_\beta} \ell_\beta + \sum_{i=1}^n \sum_{\pi_0(\beta)=v} \partial^{0i}_{rj_\beta} \ell_\beta - \sum_{i=1}^n \sum_{\pi_i(\beta)=v} \operatorname{Ad}^*_{\psi_{\pi_{0i}(\beta)}} \partial^{0i}_{rj_\beta} \ell_\beta \in \Gamma(\operatorname{Ad}^* P_d)$$

where $rj_\beta \in RJP_d$ is the π^H-projection of $p^n_\beta \in P^n_d$ and $\operatorname{Ad}^*_\psi\colon \operatorname{Ad}^* P_{s(\psi)} \to \operatorname{Ad}^* P_{t(\psi)}$ is transpose to $\operatorname{Ad}^{-1}_\psi\colon \operatorname{Ad} P_{t(\psi)} \to \operatorname{Ad} P_{s(\psi)}$, induced by $\psi^{-1}\colon P_{t(\psi)} \to P_{s(\psi)}$

DEFINE: For any $q \in \Gamma(H\mathrm{Str}_d)$ and $\psi \in \Gamma(End^1 P_d)$ call **Discrete Euler-Poincaré tensor** $\mathcal{EP}_d(q,\psi) \in \Gamma(\operatorname{Ad}^* P_d)$ associated to (q,ψ):

$$\mathcal{EP}_v(q,\psi) = \sum_{\pi_0(\beta)=v} \partial^0_{rj_\beta} \ell_\beta + \sum_{i=1}^n \sum_{\pi_0(\beta)=v} \partial^{0i}_{rj_\beta} \ell_\beta - \sum_{i=1}^n \sum_{\pi_i(\beta)=v} \operatorname{Ad}^*_{\psi_{\pi_{0i}(\beta)}} \partial^{0i}_{rj_\beta} \ell_\beta$$

$\beta = (v_0, v_1, \ldots, v_n) \Rightarrow rj_\beta = (q_{v_0}, \psi_{v_0 v_1}, \ldots, \psi_{v_0 v_n})$

Discrete Euler-Poincaré equations in H-reduced coordinates

Any discrete field on a discrete principal G-bundle $p \in \Gamma(P_d)$ determines a discrete principal connection $\psi \in \Gamma(\mathrm{End}^1 P_d)$ and a discrete H-structure $q \in \Gamma(H\mathrm{Str}_d)$ by:

$$\psi_\alpha = \pi^G(p_{s(\alpha)}, p_{t(\alpha)}) = p_{s(\alpha)}^{-1} p_{t(\alpha)} \qquad q_v = \pi^H(p_v) = Hp_v$$

DEFINE: We call H**-reduced discrete field** any pair $(q, \psi) \in \Gamma(H\mathrm{Str}_d) \times \Gamma(\mathrm{End}^1 P_d)$.

The H-reduced field (q, ψ) associated to $p \in \Gamma(P_d)$ is called the **projected field**.

REMARK: For projected fields there holds (compatibility conditions)

- **Parallelism**: $q_{v_0}\psi_{v_0 v_1} = q_{v_1}$ for any edge $(v_0, v_1) \in V^1$
- **Flatness**: $\psi_{v_0 v_1}\psi_{v_1 v_2} = \psi_{v_0 v_2}$ for any 2-simplex $(v_0, v_1, v_2) \in V^2 \subset V^{\times 2}$

Parallelism+Flatness conditions $\Rightarrow (q, \psi)$ is called **admissible**

Discrete Euler-Poincaré equations in H-reduced coordinates

Variational principle:

Choice of discrete H-reduced Lagrangian $\ell_d \colon RJP_d \to \mathbb{R}$

Choice of admissible discrete H-reduced fields $(q, \psi) \in \big(\Gamma(H\mathrm{Str}_d) \times \Gamma(\mathrm{End}^1 P_d)\big)_{Adm}$

Flatness + Parallelism

Choice of subset of admissible variations $\delta_a rj = (\delta_a q, \delta_a \psi) \in \mathrm{Var}_{q,\chi}$ with the form

$\delta_a \psi_\alpha = a_{s(\psi_\alpha)} - \mathrm{Ad}_{\psi_\alpha^{-1}} a_{t(\psi_\alpha)} \in \Gamma(\psi^* V \, \mathrm{End}^1 P_d) \simeq \Gamma(s^* \, \mathrm{Ad}\, P_d)$,

$\delta_a q_v = \pi_q(a) \in \Gamma(q^* V \, H\mathrm{Str}_d) \simeq \Gamma(\mathrm{Ad}\, P_d / q^* \, \mathrm{Ad}\, P^{H\mathrm{Str}})$

with compactly supported $a \in \Gamma(\mathrm{Ad}\, P_d)$

Seek admissible discrete H-reduced fields (q, ψ) such that its H-reduced discrete jet extension $rj \in \Gamma(RJP_d)$ is a critical point of the discrete action $\sum_K \ell_d(rj)$, for admissible discrete variations $\delta_a rj$ with $a \in \bigoplus \mathrm{Ad}\, P_v \subset \Gamma(\mathrm{Ad}\, P_d)$, vanishing at the boundary of K.

Discrete Euler-Poincaré equations in H-reduced coordinates

Theorem [CasimRodr]

Fix $\ell_d \colon H\mathrm{Str}_d \times_s \mathrm{End}^n P_d \to \mathbb{R}$ (discrete H-reduced lagrangian), $\mathcal{L}_d = \ell_d \circ \pi^H \colon JP_d \to \mathbb{R}$

The following are equivalent for $p_d \in \Gamma(P_d)$ and associated $(q, \psi) \in \Gamma(H\mathrm{Str}_d \times_s \mathrm{End}^n P_d)$

1. $\delta \int \mathcal{L}_d \circ p_d^n = 0$ holds for variations δp_d with compact support.
2. $p_d \in \Gamma(P_d)$ satisfies **discrete Euler-Lagrange equations** $0 = \mathcal{E}\mathcal{L}_d(p_d) \in \Gamma(p_d^* V P_d)$ associated to $\mathcal{L}_d$.
3. $\delta \int \ell_d \circ rj(q, \psi) \cdot \mathrm{vol}_X = 0$ holds for the subset $\mathrm{Var}_{q,\chi}$ of variations with the form
   $\delta_a \psi_\alpha = a_{s(\psi_\alpha)} - \mathrm{Ad}_{\psi_\alpha^{-1}} a_{t(\psi_\alpha)} \in \Gamma(\psi^* \mathrm{End}^1 P_d) \simeq \Gamma(s^* \mathrm{Ad}\, P_d)$,
   $\delta_a q_v = \pi_q(a) \in \Gamma(q^* V H\mathrm{Str}_d) \simeq \Gamma(\mathrm{Ad}\, P_d / q^* \mathrm{Ad}\, P^{H\mathrm{Str}})$
   with compactly supported $a \in \Gamma(\mathrm{Ad}\, P_d)$
4. (q, ψ) satisfy **discrete Euler-Poincaré equations** $0 = \mathcal{E}\mathcal{P}_d(q, \psi)$ associated to ℓ_d

Covariant discretization for reduced field theories

We have results concerning a smooth variational field theory and the associated variational reduced field theory.

We have results concerning a discrete variational field theory and the associated discrete variational reduced field theory.

Can we give methods to generate discrete formalism from smooth ones? In such a way that symmetries are preserved?

Yes, we can [Casimiro, Rodrigo 2017] but...

Due to time constraints: Not to be treated here.

$\star$ From a single H-covariant Lagrangian $\mathcal{L}$ for field theories on a principal G-bundle $P \to X$, possibility to generate 4 related variational principles: Classified into smooth/discrete and unreduced/H-reduced, preserving gauge symmetries.

Integration of discrete EP equations

Consider the H-reduced, discrete case on a discrete principal G-bundle $\pi\colon P_d \to X$

Discrete H-reduced fields: Pair of sections $(q, \psi) \in \Gamma(H\mathrm{Str}_d) \times \Gamma(\mathrm{End}^1 P_d)$

Admissible discrete H-reduced fields: Flatness and parallelism

- Parallelism: $q_{v_0}\psi_{v_0 v_1} = q_{v_1}$ for any edge $(v_0, v_1) \in V^1$
- Flatness: $\psi_{v_0 v_1}\psi_{v_1 v_2} = \psi_{v_0 v_2}$ for any 2-simplex $(v_0, v_1, v_2) \in V^2 \subset V^{\times 2}$

Discrete H-reduced Lagrangian $\ell_d\colon H\mathrm{Str}_d \times_s \mathrm{End}^n P_d \to \mathbb{R}$ generates a variational principle for admissible discrete H-reduced fields.

Critical discrete H-reduced fields characterized by $0 = \mathcal{EP}_d(q, \psi)$

- Disc.E.-P.Tensor $\mathcal{EP}_d(q, \psi) \in \Gamma(\mathrm{Ad}^* P_d)$ associated to (q, ψ):

$$\mathcal{EP}_v(q,\psi) = \sum_{\pi_0(\beta)=v} \partial^0_{rj_\beta}\ell_\beta + \sum_{i=1}^{n}\sum_{\pi_0(\beta)=v} \partial^{0i}_{rj_\beta}\ell_\beta - \sum_{i=1}^{n}\sum_{\pi_i(\beta)=v} \mathrm{Ad}^*_{\psi_{\pi_{0i}(\beta)}} \partial^{0i}_{rj_\beta}\ell_\beta$$

$$\beta = (v_0, v_1, \ldots, v_n) \in V^n \Rightarrow rj_\beta = (q_{v_0}, \psi_{v_0 v_1}, \ldots, \psi_{v_0 v_n}) \in RJP_\beta$$

Integration of discrete EP equations

Integration Problem: Recover unknown $(q, \psi) \in \Gamma(H\mathrm{Str}_d) \times \Gamma(\mathrm{End}^1 P_d)$ from:

- Parallelism: $q_{v_0}\psi_{v_0 v_1} = q_{v_1}$ for any edge $(v_0, v_1) \in V^1$
- Flatness: $\psi_{v_0 v_1}\psi_{v_1 v_2} = \psi_{v_0 v_2}$ for any 2-simplex $(v_0, v_1, v_2) \in V^2 \subset V^{\times 2}$
- Criticality

  $0 = \sum_{\pi_0(\beta)=v} \partial^0_{rj_\beta}\ell_\beta + \sum_{i=1}^n \sum_{\pi_0(\beta)=v} \partial^{0i}_{rj_\beta}\ell_\beta - \sum_{i=1}^n \sum_{\pi_i(\beta)=v} \mathrm{Ad}^*_{\psi_{\pi_{0i}(\beta)}} \partial^{0i}_{rj_\beta}\ell_\beta$

  for any vertex $v \in V$

Propagate field values from an initial band: Decompose $V = \mathbb{Z}^n$ into slices

$$S_c = \{(k_1, \ldots, k_n) \in \mathbb{Z}^n \colon k_1 + \ldots + k_n = c\} \qquad (c \in \mathbb{Z})$$

Consider a vertex $u \in \mathbb{Z}^n$ in a given slice S_{c+n}. Assume that all values $q_w, \psi_{v_0 w}$ are known for vertices w in the region $k_1 + \ldots + k_n < c + n$.

What can be said about the values q_u and $\psi_{v_0 u}$?

Integration of discrete EP equations

Slices $S_c \subset \mathbb{Z}^n$ defined by $k_1 + \ldots + k_n = c$

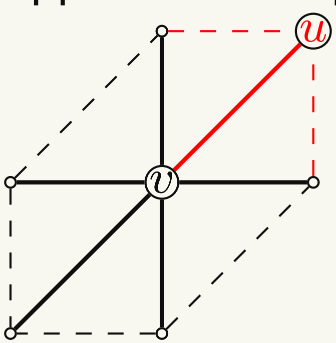
Assume $q_w, \psi_{v_0 w}$ **known for** $w \in V = \mathbb{Z}^n$ **in the region** $k_1 + \ldots + k_n < c + n$

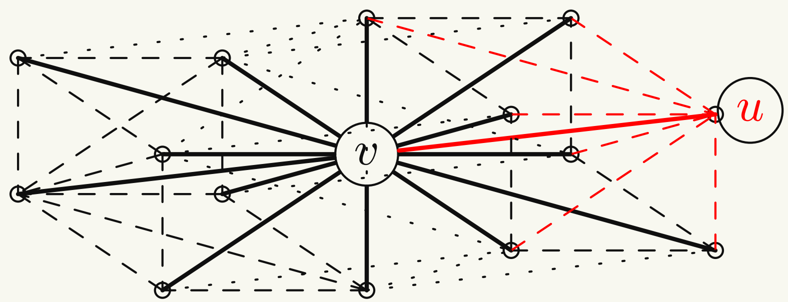
Can we **generate** $q_u, \psi_{v_0 u}$ for u in the **region** $k_1 + \ldots + k_n = c + n$**?**

Take Euler-Poincaré equations at $v = u - (1, \ldots, 1)$ (hence $v \in S_c$)

$$0 = \sum_{\pi_0(\beta)=v} \partial^0_{rj_\beta} \ell_\beta + \sum_{i=1}^{n} \sum_{\pi_0(\beta)=v} \partial^{0i}_{rj_\beta} \ell_\beta - \sum_{i=1}^{n} \sum_{\pi_i(\beta)=v} \mathrm{Ad}^*_{\psi_{\pi_{0i}(\beta)}} \partial^{0i}_{rj_\beta} \ell_\beta$$

Expression $0 = \mathcal{EP}_v(q, \psi)$ only depends on rj_β when $\pi_i(\beta) = v$ for some i. This implies $\pi_0(\beta) \in S_{c-i}$, and rj_β only depends in determined configurations q_w, $\psi_{v_0 w}$ with w in the region $k_1 + \ldots + k_n < c + n$, **plus the particular undetermined configuration** ψ_{vu} (that appears in components rj_β when $\pi_0(\beta) = v$).





Integration of discrete EP equations

Decompose Euler-Poincaré equations at v into component that depends on rj_β for $\pi_0(\beta) = v$ and another one that depends on rj_β for $\pi_i(\beta) = v$, $i = 1 \ldots n$:

$$\overbrace{\sum_{\pi_0(\beta)=v} \partial^0_{rj_\beta} \ell_\beta + \sum_{i=1}^{n} \sum_{\pi_0(\beta)=v} \partial^{0i}_{rj_\beta} \ell_\beta}^{\mathrm{Leg}_v} = \overbrace{\sum_{i=1}^{n} \sum_{\pi_i(\beta)=v} \mathrm{Ad}^*_{\psi_{\pi_{0i}(\beta)}} \partial^{0i}_{rj_\beta} \ell_\beta}^{\mathrm{Mom}_v}$$

$q_w, \psi_{v_0 w}$ known in the region $k_1 + \ldots + k_n < c + n \Rightarrow$, Right hand side Mom_v is known. Left hand side Leg_v depends on q_v and on ψ_α for edges $\alpha \in V^1$ with source $s(\alpha) = v$. All these components are also known, except for the particular component ψ_{vu} with $u = v + (1, \ldots, 1)$.

If $\dim G = m$ (and consequently $\dim \mathrm{Ad}^* P_v = m$, $\dim \mathrm{End}^1 P_{uv} = m$) we have a **system of m equations with ψ_{vu} as m-dimensional unknown** that taking into account the dimensions, in some regular cases, will determine a unique solution.

Integration of discrete EP equations

DEFINE: We call space of **discrete H-reduced forward configurations** at $v \in V$ the manifold $\mathrm{Forw}_v^H = H\mathrm{Str}_v \times \prod_{s(\alpha)=v} \mathrm{End}^1 P_\alpha$.

DEFINE: We call **discrete Legendre mapping** associated to a discrete H-reduced Lagrangian ℓ_d, the mapping $\mathrm{Leg}\colon \mathrm{Forw}_d^H \to \mathrm{Ad}^* P_d$ defined on each fiber by:

$$\mathrm{Leg}_v\colon (q_v, (\psi_\alpha)_{s(\alpha)=v}) \in \mathrm{Forw}_v^H \mapsto \sum_{\pi_0(\beta)=v} \partial^0_{rj_\beta} \ell_\beta + \sum_{i=1}^{n} \sum_{\pi_0(\beta)=v} \partial^{0i}_{rj_\beta} \ell_\beta \in \mathrm{Ad}^* P_v$$

where $rj_\beta = (q_v, (\psi_\alpha)_{\alpha \prec \beta})$.

The central step in solving discrete Euler-Poincaré equations lies in the determination of a single component $\psi_{\Delta v}$, for some edge $\Delta v = (v, v + (1, \ldots, 1)) \in V^1$, using the remaining available configurations (using an underdetermined forward configuration)

DEFINE: We call space of **underdetermined discrete H-reduced forward configurations** at $v \in V$ the manifold $\mathrm{UForw}_v^H = H\mathrm{Str}_v \times \prod_{\substack{s(\alpha)=v \\ \alpha \neq \Delta v}} \mathrm{End}^1 P_\alpha$.

$$\boxed{\mathrm{Forw}_d^H = \mathrm{UForw}_d^H \times_V \Delta^* \mathrm{End}^1 P_d} \qquad (\Delta\colon V \to V^1)$$

Integration of discrete EP equations

Legendre mapping is a morphism of discrete bundles

$$\mathrm{Leg}\colon \mathrm{UForw}_d^H \times_V \Delta^* \operatorname{End}^1 P_d \to \operatorname{Ad}^* P_d$$

DEFINE: We call **integrator for the Legendre mapping**

$\mathrm{Leg}\colon \mathrm{UForw}_d^H \times \Delta^* \operatorname{End}^1 P_d \to \operatorname{Ad}^* P_d$, any mapping

$$\Phi\colon \mathrm{UForw}_d^H \times \operatorname{Ad}^* P_d \to \Delta^* \operatorname{End}^1 P_d$$

such that, for any $uf_v \in \mathrm{UForw}_d^H$, $\theta_v \in \operatorname{Ad}^* P_v$ there holds,

$$\mathrm{Leg}(uf_v, \Phi(uf_v, \theta_v)) = \theta_v$$

If, moreover, $\mathrm{Leg}(uf_v, \cdot)\colon \Delta^* \operatorname{End}^1 P_d \to \operatorname{Ad}^* P_d$ is injective , we say the integrator is a strong integrator (in this case the integrator is unique).

Local existence of integrator iff the following linear morphism is non-degenerate

$$\frac{\partial \mathrm{Leg}_v}{\partial \psi_{\Delta v}}\colon\ \operatorname{Ad} P_v \to \operatorname{Ad}^* P_v$$

THEOREM

Let Φ be an integrator for the Legendre mapping. Consider a locally defined admissible H-reduced field $(q,\psi)_c \in \Gamma(B_c, H\mathrm{Str}_d) \times \Gamma(B_c^1, \mathrm{End}^1 P_d)$, **defined on vertices and edges included in the initial condition band** $B_c = S_{c-n} \cup S_{c-n+1} \cup \ldots \cup S_{c+n-1}$.

Consider at each $v \in S_c$ the momentum and underdetermined H-reduced forward configuration $\mu_v \in \mathrm{Ad}^* P_v$, $uf_v \in \mathrm{UForw}_v^H$ determined from $(q,\psi)_c$

For $u \in S_{c+n}$ and $v = u - (1,\ldots,1)$, **the values**

$$\psi_{vu} = \Phi_v(uf_v, \mu_v), \quad q_u = q_v \psi_{vu}, \quad \psi_{v_0 u} = \psi_{vv_0}^{-1} \psi_{vu} \qquad (1)$$

extend $(q,\psi)_c$ **to** $(q,\psi)_{c+1} \in \Gamma(B_{c+1}, H\mathrm{Str}_d) \times \Gamma(B_{c+1}^1, \mathrm{End}^1 P_d)$, an admissible H-reduced field on the following band $B_{c+1} = S_{c+n} \sqcup B_c \setminus S_{c-n}$.

The discrete field so defined in $B_c \cup C_{c+1}$ **satisfies Euler-Poincaré equations** $0 = \mathcal{EP}_v(q,\psi)$ **are satisfied at each vertex** $v \in S_c$.

(Uniqueness of solution if Φ is a strong integrator)

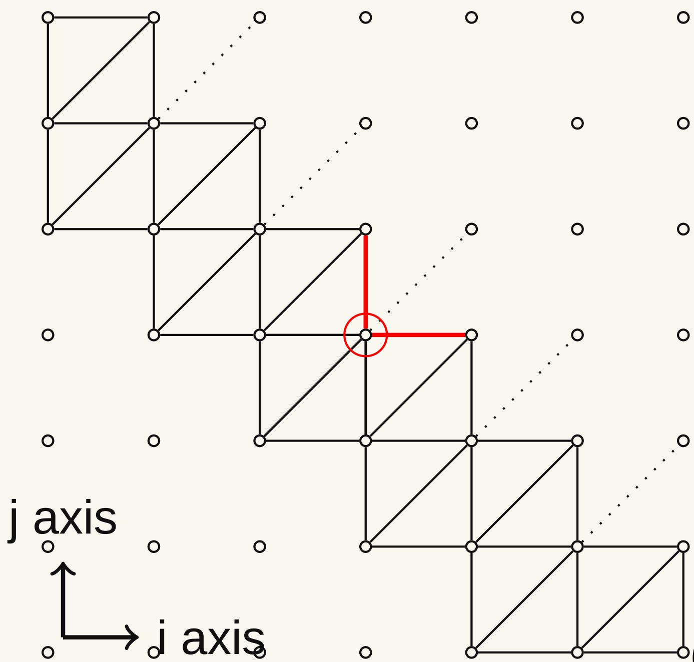
Integration algorithm

Diagonal slice $S_c \subset V = \mathbb{Z}^n$ given by $k_1 + \ldots + k_n = c$ ($c \in \mathbb{Z}$)

Initial data at B_c: Admissible discrete H-reduced field $(q, \psi)_c$ defined on the set $B_c = \cup_{k=-n}^{n-1} S_{c+k}$ (initial band)

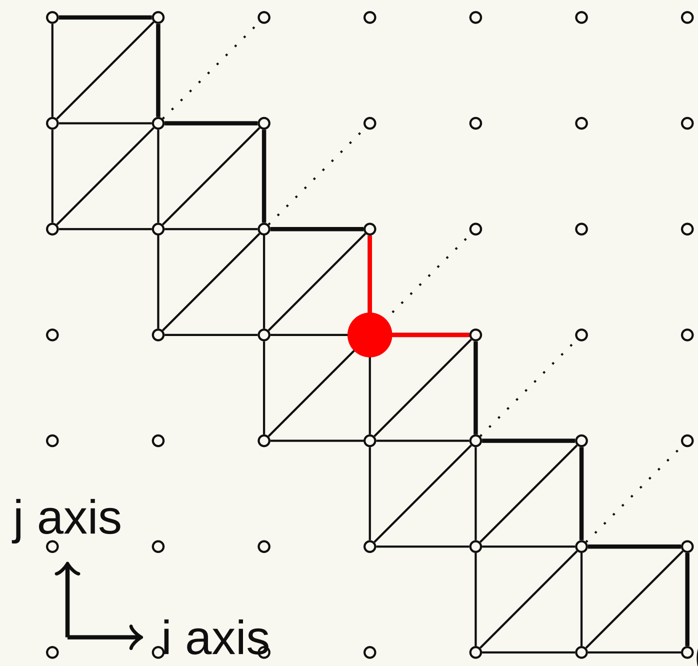
THEOREM: If the Legendre mapping is regular (existence of integrator Φ) then for any initial data at B_c there exists compatible initial data at B_{c+1} determining an admissible discrete field on $\cup_{k=-n}^{k=n} S_{c+k}$, for which Euler-Poincaré equations hold at any vertex $v \in S_c$ in its central slice.

Integration algorithm



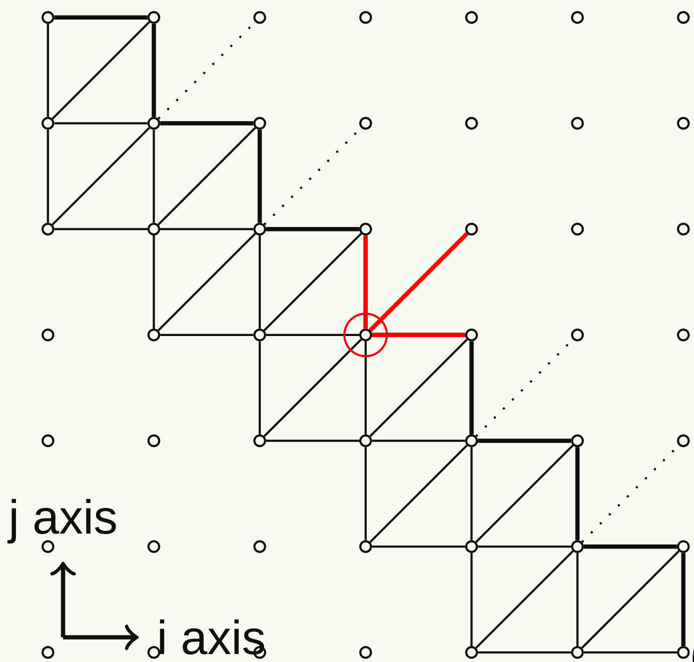
(For each $v \in S_c$)

1.- Extract discrete H-reduced underdetermined forward configuration $uf_v \in \mathrm{UForw}_v^H$



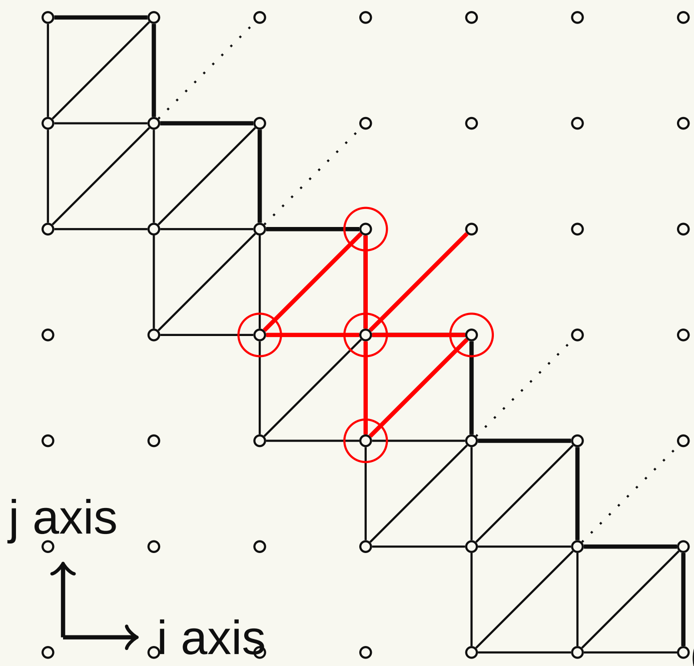
(For each $v \in S_c$)

2.- Compute $\mu_v \in \mathrm{Ad}^* P_v$ using the momentum mapping. Obtention of element $(uf_v, \mu_v) \in \mathrm{UForw}_v^H \times \mathrm{Ad}^* P_v$.



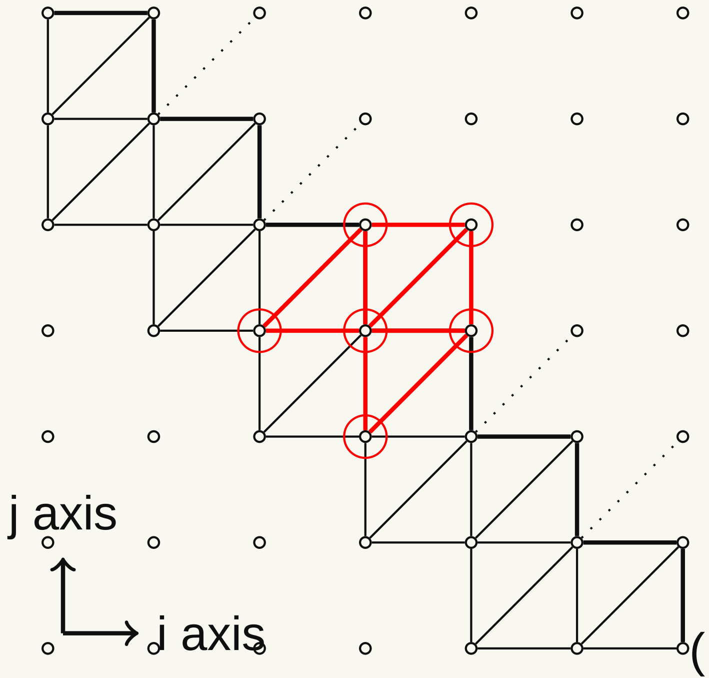
(For each $v \in S_c$)

3.- Compute $\psi_{vu} = \Phi_v(uf_v, \mu_v) \in \mathrm{End}^1 P_{\Delta v}$ for $u = v + (1, \ldots, 1)$ using the integrator



(For each $v \in S_c$)

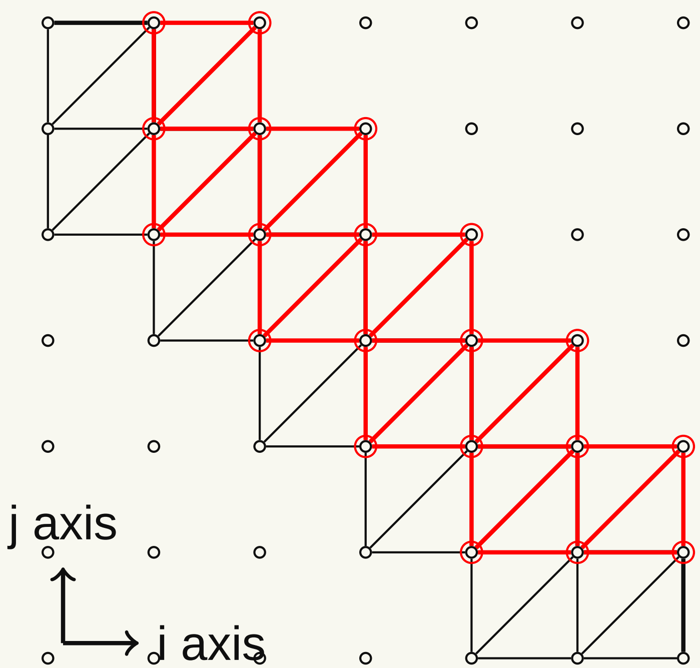
4.- Values q_u and ψ_{wu} for $w, u \in \cup_{k=-n}^{n-2} S_{c+1+k}$ are determined by initial data.



(For each $v \in S_c$)

5.- Values q_u and ψ_{wu} for $u = v + (1, \ldots, 1)$ and $w \in \cup_{k=-n}^{n-2} S_{c+1+k}$ are now determined by parallelism and trivial holonomy, using ψ_{vu}.

Integration algorithm



(For each $v \in S_c$)

6.- Collect all H-structures and fiber-to-fiber endomorphisms into new admissible discrete field $(q, \psi)_{c+1}$ defined on $B_{c+1} = \cup_{k=-n}^{n-1} S_{(c+1)+k}$ (new initial band). **Initial data at B_{c+1}**

Integration algorithm

Initial Data: Known $(q,\psi)_c$ defined on $B_c = \cup_{k=-n}^{n-1} S_{c+k}$ (initial condition band)

For each $v \in S_c$

1. Extract undetermined flat configuration $uf_v \in \mathrm{UFlat}_v^H$
2. Compute $\mu_v \in \mathrm{Ad}^* P_v$ using the momentum mapping. Obtention of element in $\mathrm{UForw}_v^H \times \mathrm{Ad}^* P_v$.
3. Compute $\psi_{\Delta v} = \Phi(uf_v, \mu_v) \in \mathrm{End}^1 P_{\Delta v}$ using the integrator
4. Values q_u and ψ_{wu} for $w, u \in \cup_{k=-n}^{n-2} S_{c+1+k}$ are already determined by initial data.
5. Values q_u and ψ_{wu} for $u = v + (1, \ldots, 1)$, $w \in \cup_{k=-n}^{n-2} S_{c+1+k}$ are now determined by parallelism and trivial holonomy.
6. Collect all H-structures and fiber-to-fiber endomorphisms into new admissible discrete field $(q,\psi)_{c+1}$ defined on $B_{c+1} = \cup_{k=-n}^{n-1} S_{(c+1)+k}$ (new initial band).
   New initial data at B_{c+1}

Iterate.

THANK YOU!

Some selected relevant bibliography:

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- *Euler-Poincaré reduction for discrete field theories.* J. Vankerschaver, 2007
- *Discrete geometric optimal control on Lie groups* M. B. Kobilarov, J. E. Marsden 2011
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- *Variational Integrators for reduced field equations* A.C. Casimiro, C. Rodrigo 2017 (submitted).

Variational Integrators for Euler-Poincaré equations

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Academia Militar

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