Is the world more classical or more quantum?

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1. Introduction

2. Historical background

3. Geometry of the $\mathcal{D}_{2n,\mathbb{K}}$ state space

4. The 4×4 case revisited

Quantum mechanical formalism

An adequate model of the *quantum event algebra* is the projection lattice $\mathcal{P}(\mathcal{H})$ of a separable Hilbert space $\mathcal{H}$.

Gleason's theorem

If $\dim(\mathcal{H}) > 2$, then each state of $\mathcal{P}(\mathcal{H})$ is of the form

$$\mathbb{P}_\rho(Q) = \mathrm{tr}(\rho Q) \qquad \forall Q \in \mathcal{P}(\mathcal{H}),$$

where $\rho \in C_1(\mathcal{H}) \cap \mathcal{B}(\mathcal{H})^+$ and $\mathrm{tr}(\rho) = 1$.

The state space of an n-level non-relativistic quantum system is

$$\overline{\mathcal{D}_{n,\mathbb{K}}} = \{\rho \in \mathbb{K}^{n\times n} \mid \rho = \rho^*,\ \rho > 0,\ \mathrm{tr}(\rho) = 1\} \quad \mathbb{K} = \mathbb{R}, \mathbb{C}.$$

The interior $\mathcal{D}_{n,\mathbb{K}}$ is an $n + \binom{n}{2} d - 1$ dimensional smooth manifold.

The space of qubits and rebits $(n = 2)$

Bloch parametrization

The affine space of 2×2 self-adjoint trace one matrices can be parametrized as

$$\rho(x) = \frac{1}{2}\left(I + \sum_{j=1}^{3} x_j \sigma_j\right),$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices.

One can check easily that a trace one matrix $\rho(x)$ is positive semi definite if and only if $||x|| \leq 1$. We have

$$\rho(x) \in \mathcal{D}_{2,\mathbb{C}} \Leftrightarrow ||x|| \leq 1$$

and thus the space of qubits $(\mathcal{D}_{2,\mathbb{C}})$ /rebits $(\mathcal{D}_{2,\mathbb{R}})$/ can be identified with the unit ball /disk/.

The state space of retrits $(n = 3)$

Retrits are elements of $\mathcal{D}_{3,\mathbb{R}}$ that is a 5-dimensional manifold.

Let $0 < a, b, c < 1$ be fixed such that $a + b + c = 1$ and introduce the parametrization

$$\rho(a, b, c, g, h, f) = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$$

$$F = \frac{f}{\sqrt{bc}} \quad G = \frac{g}{\sqrt{ca}} \quad H = \frac{h}{\sqrt{ab}}.$$

The matrix $\rho(a, b, c, g, h, f)$ is positive definite if and only if

$$-1 < F, G, H < 1 \quad F^2 + G^2 + H^2 - 2FGH < 1$$

holds.

The state space of retrits $(n = 3)$

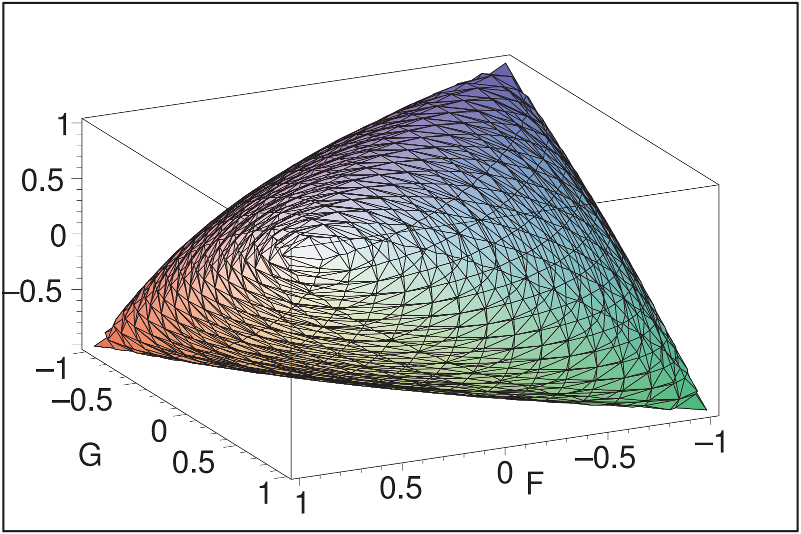

Figure: Region of the parameter space that represents density matrices.

Metrics on the state space

Flat metric

The Hilbert–Schimdth metric is defined as

$$g_{HS}(\rho)(A,B) = \mathrm{tr}(AB).$$

Clear geometric interpretation, but no obvious statistical meaning.

Monotone metrics

Due to Petz's theorem, they are of the form

$$(g_f)_n(\rho)(A,B) = \mathrm{tr}\left(A\left(\mathrm{R}_\rho^{\frac{1}{2}} f(\mathrm{L}_\rho \mathrm{R}_\rho^{-1})\mathrm{R}_\rho^{\frac{1}{2}}\right)^{-1}(B)\right)$$

where L_ρ and R_ρ are the left and right multiplication operators and $f : [0,\infty) \to \mathbb{R}^+$ is a symmetric and normalized operator monotone function.

An important example: $g_{f_{GM}}(\rho)(A,B) = \mathrm{tr}(A\rho^{-1/2}B\rho^{-1/2}) \quad \rho \in \mathcal{D}_{n,\mathbb{K}}$, where $f_{GM}(x) = \sqrt{x}$.

Some comments on monotone metrics

1. Monotone metrics are non-commutative generalizations of the Fisher information matrix known from classical information geometry.
2. Quantum mechanical state spaces endowed with monotone metrics are called quantum statistical manifolds.
3. There is no natural/canonical monotone metric which would be preferred due to its physical importance.
4. Not easy to calculate with monotone metrics. For example, we cannot compute...
   - the volume of $(\mathcal{D}_{n,\mathbb{K}}, g_f)$ manifolds for $n > 2$.
   - the geodesic lines of $(\mathcal{D}_{n,\mathbb{K}}, g_f)$ for $n > 2$.

   We don't understand the behavior of the scalar curvature (Petz's conjecture).

ENTANGLED AND SEPARABLE QUANTUM STATES

POSTULATE

If a composite system consists of the subsystems described by the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, then the composite system is described by $\mathcal{H}_1 \otimes \mathcal{H}_2$.

SEPARABLE STATES

A state ρ of a composite system $\mathcal{D}_{n\times m,\mathbb{K}}$ is called *separable (or classically correlated)* if it can be written as a convex combination of product states i.e.

$$\rho = \sum_{k=1}^{r} p_k \rho_k^1 \otimes \rho_k^2,$$

where $\rho_k^1 \in \mathcal{D}_{n,\mathbb{K}}$, $\rho_k^2 \in \mathcal{D}_{m,\mathbb{K}}$ and $(p_k)_{1\leq k\leq r}$ is a probability distribution.

Physical meaning and importance of quantum entanglement

Some comments

- Non-separable states are called *entangled.*
- In the definition of separability, the product $\rho_k^1 \otimes \rho_k^2$ means that the first subsystem is in ρ_k^1 and the second is in ρ_k^2.
- Every classical state (multivariate probability distribution with finite support) is separable thus the existence of entangled states is characteristic for composite quantum systems.

Applications of entanglement

- Quantum algorithms like superdense coding and quantum teleportation.
- Entanglement is used in some protocols of quantum cryptography.
- Most researchers believe that entanglement is necessary to realize quantum computing.

THE SEPARABILITY PROBABILITY

The problem of deciding whether a state is separable or not is called *the separability problem*.

$$\mathcal{D}_{n\times m,\mathbb{K}} = \mathcal{D}^{\mathrm{sep}}_{n\times m,\mathbb{K}} \,\dot{\cup}\, \mathcal{D}^{\mathrm{ent}}_{n\times m,\mathbb{K}}$$

Ioannou has showed that separability testing is NP-hard in general [2].

What proportion of quantum states are separable?

Let μ be a Borel measure on $\mathcal{D}_{n\times m,\mathbb{K}}$ and define

$$\mathcal{P}_{\mathrm{sep},\mu}(\mathcal{D}_{n\times m,\mathbb{K}}) = \frac{\mu(\mathcal{D}^{\mathrm{sep}}_{n\times m,\mathbb{K}})}{\mu(\mathcal{D}_{n\times m,\mathbb{K}})}.$$

Three main reasons of importance – *philosophical, practical and physical* – for examining such questions (Życzkowski, Horodecki, Lewstein and Sanpera, 1998, [1]).

The separability probability

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The 4×4 case

Since 1998, the minimal non-trivial case ($m = n = 2$) with the flat metric has been studied by several authors including Życzkowski, Slater, Dunkl, Fei, Joynt, Khvelidze, Rogojin, Milz, Strunz, etc.

Using moment reconstruction and the Zeilberger's algorithm, Slater obtained

$$\begin{aligned} P(\alpha) &= \sum_{i=0}^{\infty} f(\alpha+i) \\ f(\alpha) &= \frac{q(\alpha)2^{-4\alpha-6}\Gamma(3\alpha+\frac{5}{2})\Gamma(5\alpha+2)}{3\Gamma(\alpha+1)\Gamma(2\alpha+3)\Gamma(5\alpha+\frac{13}{2})} \\ q(\alpha) &= 185000\alpha^5 + 779750\alpha^4 + 1289125\alpha^3 + 1042015\alpha^2 \\ &\quad + 410694\alpha + 63000 \end{aligned}$$

which gives back $\mathcal{P}_{sep}(\mathbb{R}) = \mathbf{\frac{29}{64}}$, $\mathcal{P}_{sep}(\mathbb{C}) = \frac{8}{33}$ for $\alpha = \frac{1}{2}$ and 1, respectively [5]. Fei validated these results by Monte–Carlo simulations.

We should emphasize here that what Slater and the others did is not an exact mathematical proof. It is just a simulation result.

Milz and Strunz's conjecture (2015)

Block matrix form of a general 4×4 quantum state

$$\rho = \begin{pmatrix} D_1 & C \\ C^* & D_2 \end{pmatrix} \quad C \in \mathbb{K}^{2\times 2},\ D_1, D_2 \in \mathcal{M}^{\mathrm{sa}}_{2,\mathbb{K}} \quad \mathrm{tr}(D_1 + D_2) = 1$$

Peres–Horodecki criterion (PPT)

$$\rho \in \mathcal{D}^{\mathrm{sep}}_{4,\mathbb{K}} \Leftrightarrow \begin{pmatrix} D_1^T & C^T \\ (C^*)^T & D_2^T \end{pmatrix} > 0$$

Let $D \in \mathcal{D}_{2,\mathbb{K}}$ and

$$\mathcal{D}_{4,\mathbb{K}}(D) = \{\rho \in \mathcal{D}_{4,\mathbb{K}} | D_1 + D_2 = D\}$$

Conjecture (Milz and Strunz)

The volume of $\mathcal{D}_{4,\mathbb{K}}(D)$ equipped with the Hilbert–Schmidt measure is a simple polynomial of the radius of D in the Bloch-ball. Furthermore, the probability to find a state with a positive partial transpose in $\mathcal{D}_{4,\mathbb{K}}(D)$ is independent from D.

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Analytical proof for X-states and numerical verification in general settings

X-states are states of the form

$$\rho_X = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{23} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}.$$

S. Milz, W. T. Strunz [4]

"While the eigenvalues of X-states can be easily expressed analytically, the eigenvalues of a general two-qubit state could, in principle, be calculated. So far, however, a direct derivation of the volume $V_{\mathrm{HS}}^{(2\times 2)}(r)$ from these expressions is beyond reach."

The manifold $\mathcal{D}_{n,\mathbb{K}} \times \mathcal{E}_{n,\mathbb{K}} \times \mathrm{B}_1\left(\mathbb{K}^{n\times n}\right)$

Semi-symmetric product of positive matrices

$$A \oslash B = (A^{1/2}BA^{1/2})^{1/2} \quad A, B > 0$$

For $D \in \mathcal{D}_{n,\mathbb{K}}$, $Z \in \mathcal{E}_{n,\mathbb{K}} =]-I, I[$ and $X \in \mathrm{B}_1\left(\mathbb{K}^{n\times n}\right)$, we define

$$\phi_n\left(D, Z, X\right) = S_n(D, Z)\begin{bmatrix} I & X \\ X^* & I \end{bmatrix} S_n(D, Z),$$

where

$$S_n(D, Z) = \begin{bmatrix} D \oslash \frac{I+Z}{2} & 0 \\ 0 & D \oslash \frac{I-Z}{2} \end{bmatrix}.$$

Theorem

The map $\phi : \Pi_{n,\mathbb{K}} \to \mathcal{D}_{2n,\mathbb{K}}\ (D, Z, X) \mapsto \phi_n(D, Z, X)$ establishes a diffeomorphism and $\mathrm{tr}_2 \circ \phi = \mathrm{pr}_1$ holds, where

$$\Pi_{n,\mathbb{K}} = \mathcal{D}_{n,\mathbb{K}} \times \mathcal{E}_{n,\mathbb{K}} \times \mathrm{B}_1\left(\mathbb{K}^{n\times n}\right).$$

Characterization of PPT states

Let us denote the states with positive partial transpose by $\mathcal{D}_{2n,\mathbb{K}}^{\mathrm{PPT}}$.

$$\begin{pmatrix} D_1 & C \\ C^* & D_2 \end{pmatrix} \in \mathcal{D}_{2n,\mathbb{K}}^{\mathrm{PPT}} \Leftrightarrow \begin{pmatrix} D_1 & C^* \\ C & D_2 \end{pmatrix} > 0$$

Note that $\mathcal{D}_{2n,\mathbb{K}}^{\mathrm{sep}} \subseteq \mathcal{D}_{2n,\mathbb{K}}^{\mathrm{PPT}}$ and $\mathcal{D}_{2n,\mathbb{K}}^{\mathrm{sep}} = \mathcal{D}_{2n,\mathbb{K}}^{\mathrm{PPT}} \Leftrightarrow n = 2, 3$.

Theorem

The manifold $\mathcal{D}_{2n,\mathbb{K}}^{\mathrm{PPT}}$ is diffeomorphic to

$$\Pi_{n,\mathbb{K}}^{\mathrm{PPT}} = \mathcal{D}_{n,\mathbb{K}} \times \left\{ (Z, X) \in \mathcal{E}_{n,\mathbb{K}} \times \mathrm{B}_1\left(\mathbb{K}^{n \times n}\right) \,\middle|\, 1 > \left\| \Sigma\left(\sqrt{\tfrac{I-Z}{I+Z}}\right)^{-1} X \Sigma\left(\sqrt{\tfrac{I-Z}{I+Z}}\right) \right\| \right\},$$

where $\Sigma\left(\sqrt{\frac{I-Z}{I+Z}}\right)$ is a diagonal matrix that contains the eigenvalues of $\sqrt{\frac{I-Z}{I+Z}} > 0$ in its diagonal in a decreasing order.

Heuristic explanation of the result

For a 4×4 quantum state, the previous result has the following implications:

Roughly speaking a 4×4 quantum state of the form

$$\begin{bmatrix} D \oslash \frac{I+Z}{2} & 0 \\ 0 & D \oslash \frac{I-Z}{2} \end{bmatrix} \begin{bmatrix} I & X \\ X^* & I \end{bmatrix} \begin{bmatrix} D \oslash \frac{I+Z}{2} & 0 \\ 0 & D \oslash \frac{I-Z}{2} \end{bmatrix}$$

is separable if and only if X is "small enough".

- On the singular value ratio of $\sqrt{\frac{I-Z}{I+Z}}$ depends, how small X should be.
- The reduced state D does not influence the separability. $\Rightarrow$ Milz's conjecture

Pullback of the volume forms

We consider the map $\phi : \Pi_{n,\mathbb{K}} \to \mathcal{D}_{2n,\mathbb{K}}$ and the corresponding pullback metrics.

Hilbert–Schmidt metric

$$\sqrt{\det(\phi^* G_{HS}(D,Z,X))} = \frac{\sqrt{n}}{2^{n-1}} \det(D)^{(3n-1)\frac{d}{2}+1} \det(I - Z^2)^{\frac{nd}{2}}$$

Monotone metric associated to the geometric mean

$$\sqrt{\det(\phi^* G_{f_{GM}}(D,Z,X))} = \frac{\det(I-Z^2)^{(3n-1)\frac{d}{2}-\frac{1}{2}}}{2^{\binom{2n}{2}d-\frac{1}{2}} \det(D)^{(n-1)\frac{d}{2}} \det(I-XX^*)^{(n-1)\frac{d}{4}+\frac{1}{2}}}$$

These volume forms arise as products where each factor depends only on one component which means integrals involving them can be also factorized.

Rebit-rebit and qubit-qubit systems

Combining together the previously mentioned ideas, we get the volume of the separable 4×4 states w.r.t. the Hilbert–Schmidt metric.

$$\mathrm{Vol}\left(\mathcal{D}^{\mathrm{sep}}_{4,\mathbb{K}}(D)\right) = \frac{\det(D)^{4d-\frac{d^2}{2}}}{2^{6d}} \times \int\limits_{]-I,I[} \det(I-Y^2)^d \times \chi_{\mathbf{d}} \circ \sigma\left(\sqrt{\frac{I-Y}{I+Y}}\right) \mathrm{d}\lambda_{d+2}(Y)$$

$$\mathrm{Vol}\left(\mathcal{D}^{\mathrm{sep}}_{4,\mathbb{K}}\right) = \int\limits_{\mathcal{D}_{2,\mathbb{K}}} \mathrm{Vol}\left(\mathcal{D}^{\mathrm{sep}}_{4,\mathbb{K}}(D)\right) \mathrm{d}\lambda_{d+1}(D),$$

The main point here is the function $\chi_{\mathbf{d}}$.

From this, we can conclude that the conjecture of Milz and Strunz holds true.

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Separability functions I.

The function $\chi_d : (0, \infty) \to (0, \infty)$ measures the intersection of the unit ball of $\mathbb{K}^{2\times 2}$ w.r.t. the standard induced matrix norm and its image under some special similarity transform.

More precisely

$$\chi_d(\varepsilon) = \int\limits_{\mathrm{B}_1(\mathbb{K}^{2\times 2})} \mathbf{1}_{\|V_\varepsilon^{-1} X V_\varepsilon\| < 1} \,\mathrm{d}\lambda_{4d}(X)$$

$$V_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}, \ \varepsilon \in (0, \infty) \text{ and } d = \dim_{\mathbb{R}}(\mathbb{K}).$$

Until now, we could compute χ_d analytically only for $d = 1$.

$$\tilde{\chi}_1(\varepsilon) = \chi_1(\varepsilon)/\chi_1(1) = \frac{4}{\pi^2} \int\limits_0^\varepsilon \left(s + \frac{1}{s} - \frac{1}{2}\left(s - \frac{1}{s}\right)^2 \log\left(\frac{1+s}{1-s}\right)\right) \frac{1}{s} \,\mathrm{d}s$$

SEPARABILITY FUNCTIONS II.

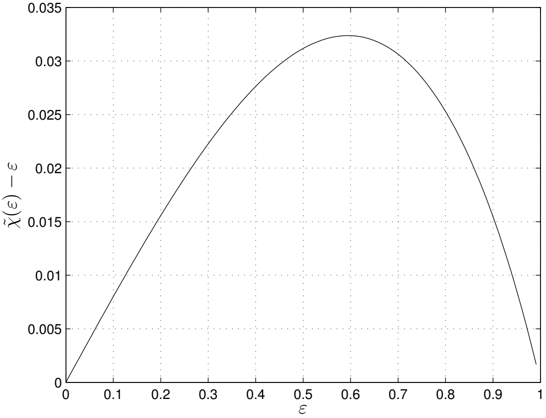

FIGURE: The graph of $\varepsilon \mapsto \tilde{\chi}_1(\varepsilon) - \varepsilon$.

Separability probabilities

Using some unitary invariance argument, the original 9-dimensional integral can be simplified to a double integral.

In the Hilbert–Schmidt case, we have got

$$\mathcal{P}_{\mathrm{sep}}(\mathbb{R}) = \frac{\int\limits_{-1}^{1}\int\limits_{-1}^{x} \tilde{\chi}_1\left(\sqrt{\frac{1-x}{1+x}}\Big/\sqrt{\frac{1-y}{1+y}}\right)(1-x^2)(1-y^2)(x-y)\,\mathrm{d}y\,\mathrm{d}x}{\int\limits_{-1}^{1}\int\limits_{-1}^{x}(1-x^2)(1-y^2)(x-y)\,\mathrm{d}y\,\mathrm{d}x} = \mathbf{\frac{29}{64}}$$

which is the first analytical result in this direction and actually it proves the conjecture of Slater, Dunkl and Fei.

Generalization to quantum statistical manifolds

Thanks to the factorization of the volume form, we could compute the separability probability not only for $(\mathcal{D}_{4,\mathbb{K}}, g_{HS})$, but also for the quantum statistical manifold $(\mathcal{D}_{4,\mathbb{K}}, g_{f_{GM}})$.

We have got

$$\mathcal{P}_{\mathrm{sep},g_{f_{GM}}}(\mathbb{R}) = \frac{\int\limits_{-1}^{1}\int\limits_{-1}^{x} \tilde{\chi}_1\left(\sqrt{\frac{1-x}{1+x}} \Big/ \sqrt{\frac{1-y}{1+y}}\right)(1-x^2)^{-\frac{1}{4}}(1-y^2)^{-\frac{1}{4}}(x-y)\,\mathrm{d}y\,\mathrm{d}x}{\int\limits_{-1}^{1}\int\limits_{-1}^{x}(1-x^2)^{-\frac{1}{4}}(1-y^2)^{-\frac{1}{4}}(x-y)\,\mathrm{d}y\,\mathrm{d}x}$$

$$= \int_0^1 \frac{8\left(8\left(t^4+t^2\right)E\left(1-\frac{1}{t^2}\right)-\left(t^2+3\right)\left(3t^2+1\right)K\left(1-\frac{1}{t^2}\right)\right)}{\pi\sqrt{t}\left(t^2-1\right)^3}\tilde{\chi}_1(t)\,\mathrm{d}t$$

$$\approx 0.26223.$$

For more details see [3].

Thank you for your attention!

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