A stellar model with diffusion in general relativity

J. Geom. Phys. 120 (2017) 62–72

Artur Alho
(Joint work with Simone Calogero)

Center for Mathematical Analysis, Geometry and Dynamical Systems, Technical University of Lisbon

XXVI International Fall Workshop on Geometry and Physics, Braga - September 5th, 2017

Introduction

Structure of singularities formed in the gravitational collapse of bounded matter distributions

- Are such singularities naked, i.e. visible to far-away observers?
- Are they safely hidden inside a black-hole?

Collapse of a homogenous dust ball: Oppenheimer-Snyder (1937)

- Consists of a collapsing Friedmann-Lemaître-Robertson-Walker interior matched at a comoving boundary with a Schwarzschild exterior .

Collapse of an inhomogeneous dust ball: Christodoulou (1984)

- Consists of a Lemaître-Tolman-Bondi interior matched at a comving boundary with a Schwarzschild exterior.
- In contrat with Oppenheimer-Snyder, inhomogeneous dust collapse leads to the formation of naked singularities.

Diffusion in General Relativity

- Gravitational collapse of matter subject to diffusion?
- Mathematically the inclusion of diffusion terms introduce a regularizing effect in the equations, which might prevent the formation of naked singularities
- Diffusion is the cause for several physical processes
  - Heat conduction
  - Brownian motion
- At the microscopic level diffusion is due to random collisions between the particles of the system with those of the background substance
  - Stochastic differential equations
- At the macroscopic scale, random effects are averaged, and diffusion is described by an effective and deterministic theory
  - Relativistic kinetic Fokker-Planck equation for distribution function f.
- There are two theories:
  - Kinetic theory based on a Fokker-Planck equation for the particle density in the phase-space:
    S. Calogero, JCAP 11/2011, 016 (2011)
  - Fluid theory which is the formal macroscopic limit of the kinetic theory:
    S. Calogero J. Geom. Phys. **62**, 2208–2213 (2012)

Fluid theory

- The energy-momentum tensor and energy current for a perfect fluid

$$T^{\mu\nu} = \rho u^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu)$$
$$J^\mu = n u^\mu$$

- For a perfect-fluid undergoing velocity diffusion

$$\nabla_\mu T^{\mu\nu} = \sigma J^\nu$$
$$\nabla_\mu J^\mu = 0$$

- σ is the diffusion constant and measures the average energy transferred per unit time from the background substance to a fluid particle.

- Projecting parallel and orthogonal to u^μ

$$\nabla_\mu(\rho u^\mu) + p\nabla_\mu u^\mu = \sigma n,$$
$$(\rho + p)u^\mu \nabla_\mu u^\nu + u^\nu u^\mu \nabla_\mu p + \nabla^\nu p = 0,$$
$$\nabla_\mu(n u^\mu) = 0.$$

- First-law of thermodynamics: entropy $S = \rho/n$

$$u^\mu \nabla_\mu S = \sigma,$$

- In presence of diffusion $T_{\mu\nu}$ is not divergence-free. Incompatibility with the twice contracted Bianchi identities $\nabla_\mu G^\mu_{\ \nu} = 0$.
- Add a matter field which interacts with the fluid particles restoring the local conservation of energy
- The new matter field plays the role of a background medium in which particles undergo diffusion
- The simplest model for this medium is a vaccum-energy described by a cosmological scalar field (varying Λ)

$$G_{\mu\nu} + \phi g_{\mu\nu} = T_{\mu\nu}$$

- The diffusion equation is

$$\nabla_\mu \phi = \sigma J_\mu$$

- When $\sigma = 0$ the model reduces to the Einstein-Euler system with cosmological constant Λ.
- A. Alho, S. Calogero, M. P. Ramos and A. J. Soares: Dynamics of Robertson-Walker spacetimes with diffusion, Annals of Physics **204** (2015)
- Stellar models in Spherical Symmetry

The Interior Region

- Comoving system of coordinates:

$$g = -e^{2\Phi(t,R)}\,dt^2 + e^{2\Psi(t,R)}\,dR^2 + r^2(t,R)\,d\Omega^2,$$
$$u = -e^{-\Phi(t,R)}\partial_t,$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\psi^2$ is the standard metric on S^2.

- Fix $\Phi(t, R_b) = 0$, so that t is the proper time of observers at rest with respect to the boundary of the star
- Fix $r(0, R) = R$, so that the comoving radius R coincides initially, i.e., at time $t = 0$, with the radius function of the group orbits.

Theorem

Let $p = 0$ and let (g, ρ, n, u, ϕ) be a spherically symmetric solution. Then ρ, n, ϕ are functions of $t \in [0, T)$ only and there exist a positive function $a : [0, T) \to (0, \infty)$ and a constant $k \in \mathbb{R}$ such that

$$g = -dt^2 + a(t)^2\left(\frac{dR^2}{1 - kR^2} + R^2 d\Omega^2\right),$$
$$u = -\partial_t.$$

- Robertson-Walker geometry! By rescaling the radial coordinate we may assume that $k \in \{-1, 0, 1\}$, $R < R_b$, where $R_b < 1$ for $k = 1$.

- From now on we consider a pressureless fluid (dust).
- The conservation of the particle number density gives $n(t) = \frac{a(0)^3 n(0)}{a(t)^3}$, while $a(t), \rho(t), \phi(t)$ satisfy

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{1}{3}(\rho + \phi) \quad , \quad \frac{\ddot{a}}{a} = -\frac{1}{6}\rho + \frac{1}{3}\phi,$$
$$\dot{\phi} = -3\frac{\beta}{a^3}, \quad , \quad \dot{\rho} = -3\rho\frac{\dot{a}}{a} - \dot{\phi},$$

where $\beta = \sigma n(0) a(0)^3/3$.

- Two classes of solutions: “Expanding” and “Collapsing”
- Collapsing (dust) solutions behave like the Friedmann-Lemaître diffusion-free solution in the limit toward the singularity.
- This implies that the spacelike singularity at $t = t_s$ is a curvature singularity,
- The spacetime is inextendible beyond the spacelike hypersurface $t = t_s$ and no outgoing light ray can emanate from the singularity.

Corollary

There exist no naked singularities in the spherical collapse of dust clouds undergoing diffusion in a cosmological scalar field.

- The diffusion forces prevents the formation naked singularities by an explicit and compelling regularizing effect: **it forces the dust interior to be spatially homogeneous**.

- Let $\chi = g^{ab}(\partial_a r)(\partial_b r)$, $a = 0, 1$:

$$\chi(t,R) = 1 - kR^2 - \dot{a}(t)^2 R^2 = (\sqrt{1-kR^2} + \dot{a}(t)R)(\sqrt{1-kR^2} - \dot{a}(t)R).$$

- Trapped region: $\chi < 0$,
- Regular region: $\chi > 0$.
- Apparent horizon: $\chi = 0$
- Introduce local mass function $m(t,R)$ via

$$\chi = 1 - \frac{2m}{r} - \frac{\phi}{3}r^2, \quad r(t,R) = a(t)R.$$

- It follows the identity

$$m(t,R) = \frac{\rho(t)}{6}r^3 = m(0,R) + \frac{R^3}{2}\beta t,$$

- The local mass is conserved only in the absence of diffusion ($\beta = 0$) and it is otherwise linearly increasing in time.
- The latter behavior is intimately connected with the irreversibility of the diffusion process: the entropy is linearly increasing in time:

$$S(t) = S(0) + \sigma t.$$

The Exterior Region

- Bondi coordinates:

$$g_{\rm ext} = -A(w,r)B(w,r)dw^2 + 2\varepsilon A(w,r)dwdr + r^2 d\Omega^2,$$

  where $\varepsilon = \pm 1$.
- $\varepsilon = 1$, w is the ingoing (advanced) null coordinate.
- $\varepsilon = -1$,w is the outgoing (retarded) null coordinate.
- The comoving boundary of the star Σ

$$\Sigma : r = r_\Sigma(w),$$

- Σ is assumed to be timelike:

$$A(w,r(w))(B(w,r(w)) - 2\varepsilon \dot{r}_\Sigma(w)) > 0, \quad \dot{r}_\Sigma = \frac{dr_\Sigma}{dw},$$

  i.e., the first fundamental form of Σ has the signature $(-,+,+)$.
- Time orientation of spacetime does not change across the boundary

$$dt/dw > 0$$

- The two metrics $g_{\rm int}$ and $g_{\rm ext}$ may be matched on Σ if and only if they satisfy the junction conditions that they induce the same first and second fundamental form on Σ,

Theorem

The metrics satisfy the junction conditions if and only if

(a) *There holds*

$$r_\Sigma(w) = a(t(w))R_b$$

(b) *The transformation of variable $t = t(w)$ satisfies*

$$\dot{t}(w) = A(w, r_\Sigma(w))(\sqrt{1 - kR_b^2} - \varepsilon\dot{a}(t(w))R_b);$$

(c) *There holds*

$$B(w, r_\Sigma(w)) = A(w, r_\Sigma(w))(1 - kR_b^2 - \dot{a}(t(w))^2 R_b^2);$$

(d) *There holds $Q(w) = 0$, where*

$$Q(w) = \Big[(B - 2\varepsilon\dot{r}_\Sigma(w))((B - \varepsilon\dot{r}_\Sigma(w))\partial_r A + \varepsilon\partial_w A) - A(2\ddot{r}_\Sigma(w) + (B - 3\varepsilon\dot{r}_\Sigma(w))\partial_r B - \varepsilon\partial_w B)\Big]_{r=r_\Sigma(w)}.$$

- Assuming $A(w,r)=1$:

Corollary

The exterior metric can be chosen to be the Vaidya type metric with variable cosmological constant given by

$$g_{\text{ext}} = -\left(1 - \frac{2M(w)}{r} - \frac{\Lambda(w)}{3} r^2\right) dw^2 + 2\varepsilon dw dr + r^2 d\Omega^2,$$

where $M(w), \Lambda(w)$ are given by

$$M(w) = m(t(w), R_b) \quad \textit{and} \quad \Lambda(w) = \phi(t(w)).$$

- The generalized Vaidya metric solves the Einstein equation with cosmological scalar field $\phi = \Lambda(w)$ and the energy-momentum tensor

$$T_{\text{ext}} = \tilde{\rho}\, dw^2, \quad \tilde{\rho} = \frac{\varepsilon}{r^2}\left(2\frac{dM}{dw} + \frac{r^3}{3}\frac{d\Lambda}{dw}\right),$$

where

$$\frac{dM}{dw} = \frac{\beta R_b^3}{2}\left(\sqrt{1 - kR_b^2} - \varepsilon \dot{a}(t(w)) R_b\right), \tag{7}$$

$$\frac{d\Lambda}{dw} = -\frac{3\beta}{a(t(w))^3}\left(\sqrt{1 - kR_b^2} - \varepsilon \dot{a}(t(w)) R_b\right). \tag{8}$$

- When $\beta = 0$, i.e., in the absence of diffusion, the functions M and Λ become constant and so $T_{\text{ext}} = 0$.

- Requiring the weak energy condition $\tilde{\rho} > 0$ to hold in the whole exterior region forces us to restrict to the outgoing Vaidya solution.

Proposition

$\tilde{\rho}(w, r_\Sigma(w)) = 0$. *Moreover the weak energy condition* $\tilde{\rho}(w, r) > 0$, *for all* $r > r_\Sigma(w)$, *holds only for the outgoing Vaidya metric* ($\varepsilon = -1$).

Proof.

By (7) and (8),

$$\tilde{\rho}(w, r) = \frac{\varepsilon\beta \frac{dt}{dw}}{r^2 a(t(w))^3}\left[r_\Sigma^3(w) - r^3\right], \quad r \geq r_\Sigma(w), \tag{9}$$

by which the result follows immediately. □

- If $\varepsilon = +1$ we have $\tilde{\rho} > 0$ for $r < r_\Sigma(w)$. Generalisation of the Einstein-Strauss void model.
- We restrict from now on to the generalized outgoing Vaidya metric ($\varepsilon = -1$),

$$g_{\text{ext}} = -\left(1 - \frac{2M(u)}{r} - \frac{\Lambda(u)}{3} r^2\right) du^2 - 2dudr + r^2 d\Omega^2. \tag{10}$$

- The apparent horizons are the hypersurfaces in the exterior where $B(u, r) = 0$.
- The interior and exterior apparent horizons intersect on the boundary.

Self-Similar solution

- The interior admits the explicit (self-similar) solution:

$$a(t) = \delta_k t \quad , \quad \phi(t) = \frac{3\beta}{2\delta_k} a(t)^{-2} \quad , \quad \rho(t) = \frac{3\beta}{\delta_k} a(t)^{-2},$$

  where δ_k is the real solution of the polynomial equation

$$\delta^3 + k\delta - \frac{3\beta}{2} = 0.$$

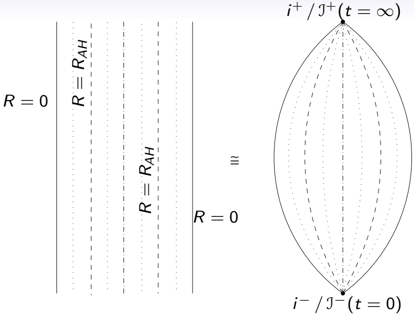
- Note that $\delta_k > 0$, for all $k = 0, \pm 1$.
- Towards the past: curvature singularity
- Towards the future: the solution is forever expanding ($\dot{a} > 0$) without acceleration ($\ddot{a} = 0$).
- Apparent horizon:

$$R_{\mathrm{AH}} = \frac{1}{\sqrt{\delta_k^2 + k}}.$$

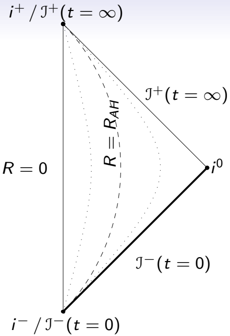
- The local mass of the interior is

$$m(t, R) = \frac{\beta}{2} R^3 t,$$

  hence $m(t, R) \to +\infty$ as $t \to +\infty$.



(a) Conformal diagram and bounded conformal diagram for $k = 1$. The solid lines correspond to the boundary $R = 0$ and the dashdotted line to the equator $R = 1$. The remaining dotted lines are curves of constant R, for $0 < R < 1$, while the dashed lines represent the apparent horizon at $R = R_{AH}$. In this case a suitable matching surface is given by the curve $R = R_b < 1$.



(b) Bounded conformal diagram $k = 0$ and $k = -1$. The dotted lines are curves of constant R and the dashed line is the apparent horizon $R = R_{AH}$. The thick solid line corresponds to a Big-Bang type (null) singularity.

Figure: Penrose diagrams for the expanding (at constant rate) interior solution. Each point represents a 2-sphere of radius $r = a(t)R$. As usual, i^- and i^+ represent past and future timelike infinity respectively, and i^0 corresponds to spacelike infinity. Also, $\mathcal{I}^-$, $\mathcal{I}^+$ denote past and future null infinity respectively.

- Let $R = R_b$ the boundary of the star. We distinguish three cases:
  (i) $R_b > R_{\mathrm{AH}}$; the interior has an apparent horizon in this case.
  (ii) $R_b = R_{\mathrm{AH}}$; the boundary of the star coincides with the apparent horizon.
  (iii) $R_b < R_{\mathrm{AH}}$; the interior has no apparent horizon.

- The matching conditions give

$$u = C_k t, \quad r_\Sigma(u) = x_\Sigma u, \quad x_\Sigma = \frac{R_b \delta_k}{C_k}, \quad C_k = \frac{1}{\sqrt{1 - kR_b^2} + \delta_k R_b} > 0,$$

- The exterior metric becomes

$$g_{\mathrm{ext}} = -\left(1 - \lambda_1 \frac{u}{r} - \lambda_2 \frac{r^2}{u^2}\right) du^2 - 2dudr + r^2 d\Omega^2, \quad r > x_\Sigma u,$$

where

$$\lambda_1 = \frac{\beta R_b^3}{C_k}, \quad \lambda_2 = \frac{\beta C_k^2}{2\delta_k^3}.$$

- Curvature singularity at $u = 0$, for its Kretschmann scalar $K = \mathrm{Riem}^2$ is given by

$$K = \frac{12\lambda_1^2 u^2}{r^6} + \frac{24\lambda_2^2}{u^4}.$$

- The apparent horizons: hypersurfaces where $B(u, r) = 0$,

$$B(u, r) = h\left(\frac{r}{u}\right), \quad h(x) = 1 - \frac{\lambda_1}{x} - \lambda_2 x^2.$$

Proposition

The following holds:

(1) *In case (i) there is no apparent horizon in the exterior region and* $B(u, r) < 0$ *for all* $r > r_\Sigma(u), u > 0$.

(2) *In case (ii) the apparent horizon in the exterior coincides with the apparent horizon of the interior, as well as with the matching surface:*

$$r_{\mathrm{AH}}(u) = r_\Sigma(u) \equiv R_{\mathrm{AH}} = R_b.$$

Moreover $B(u, r) < 0$ *for all* $r > r_\Sigma(u)$.

(3) *In case (iii) there exists* $x_{\mathrm{AH}} > x_\Sigma$ *such that the metric in the exterior has an apparent horizon at* $r = x_{\mathrm{AH}} u$. *Moreover* $B(u, r) > 0$ *for* $r_\Sigma(u) < r < x_{\mathrm{AH}} u$ *and* $B(u, r) < 0$ *for* $r > x_{\mathrm{AH}} u$.

Proof.

For the proof it suffices to notice that the function $h(x)$ attains its maximum at $x = x_\Sigma$ and $h(x_\Sigma) = 1 - (R_b/R_{\mathrm{AH}})^2$. □

- The model under discussion is self-similar, with the lines $r = xu$, $x > x_\Sigma$, on the (t, r)-plane being tangent to the homothetic vector field in the exterior. We call such curves homothetic curves.

Theorem

There exists $x_* > x_\Sigma$ such that the homothetic curve $r = xu$ is spacelike for $x > x_*$, null for $x = x_*$ and timelike for $x_\Sigma < x < x_*$. In case (iii) there holds $x_* > x_{\rm AH}$. Moreover the homothetic curve $r = x_* u$ is the first ingoing radial null geodesics that escapes to null-infinity.

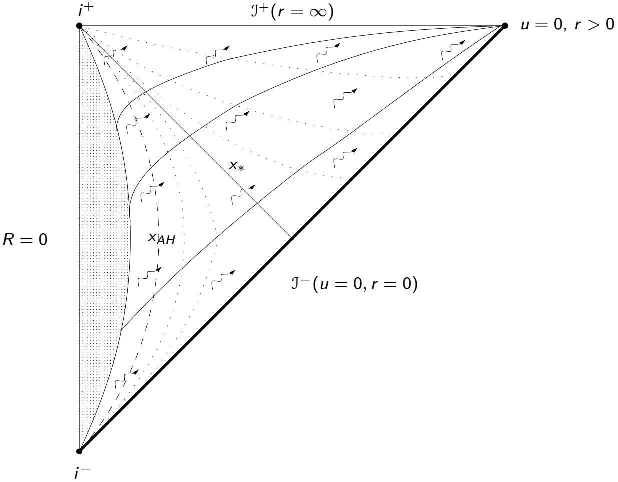


Figure: Penrose diagram for the self-similar stellar solution with diffusion in case (iii).