Models for quasi-Sasakian and quasi-Vaisman manifolds and classification of their nilmanifolds

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Commutative Differential Graded Algebras (CDGAs)

Definition

A CDGA (A, d) is a graded vector space $A = \bigoplus_{k \in \mathbb{N}} A^k$ with

- a graded commutative product

$$A^k \times A^l \to A^{k+l}$$

$$ab = (-1)^{|a|\,|b|} ba;$$

- a degree one differential $d : A^k \to A^{k+1}, \quad d^2 = 0;$
- Leibniz rule: $d(ab) = d(a)\, b + (-1)^{|a|} a\, d(b).$

Examples of CDGAs

- Given a manifold M, the de Rham algebra

$$(\Omega(M), \wedge, d);$$

- Any graded commutative algebra A with the trivial differential $d = 0$;
- The de Rham cohomology algebra

$$(H(M), \cup, d = 0);$$

- The Chevalley-Eilenberg complex $\left(\bigwedge \mathfrak{g}^*, \wedge, d^{CE}\right)$ of a Lie algebra $\mathfrak{g}$ with the multiplication of the exterior algebra.

Cohomology of a CDGA

Given a CDGA (A, d) we can always form its cohomology

$$H^k(A) = \frac{\mathrm{Ker}\, d : A^k \to A^{k+1}}{\mathrm{Im}\, d : A^{k-1} \to A^k}.$$

It easy to check that

$$H(A) = \bigoplus_k H^k(A)$$

inherits the product of A, so we can treat $H(A)$ as a CDGA with zero differential.

Morphisms and quasi-isomorphisms

- A morphism of CDGAs is a linear map $f : A \to B$ such that
  - $f : A^k \to B^k$
  - $f(ab) = f(a)f(b)$
  - $f \circ d = d \circ f$
- A morphism of CDGAs $f : A \to B$ induces a morphism in cohomology

$$H(f) : H(A) \to H(B)$$

Definition

A *quasi-isomorphism* is a morphism of CDGAs $f : A \to B$ such that it induces an isomorphism in cohomology.

Models

- A CDGA (A, d) is a model of a CDGA (B, d) if there is a chain of quasi-isomorphisms

$$(A, d) \to (A_1, d) \leftarrow \cdots \to (A_k, d) \to \cdots \leftarrow (B, d)$$

- As a consequence one has an induced isomorphism between the cohomologies $H(A)$ and $H(B)$.

Definition

We say that a CDGA (A, d) is a model of a manifold M if it is a model of the CDGA $(\Omega(M), d)$.

Formality

Definition

We say that a manifold M is formal if the de Rham cohomology is a model of M.

- So, there is a chain of quasi-isomorphisms

$$(H(M),0) \to (A_1,d) \leftarrow \cdots \to (A_k,d) \to \cdots \leftarrow (\Omega(M),d)$$

- In this case, at least if M is formal and simply connected one can show that the real homotopy type of M is determined by the de Rham cohomology of M.

Examples of formal manifolds

- compact Lie groups
- Riemannian symmetric spaces of compact type
- compact k-connected manifolds of dimension $\leq 4k+2$ [Miller 1976]
- compact Kähler manifolds [Deligne-Griffiths-Morgan-Sullivan 1975]
- compact co-Kähler manifolds [Chinea-de Leon-Marrero 1993]

Almost contact metric manifolds

- An almost contact manifold (M, ϕ, ξ, η) is an odd-dimensional manifold M which carries a $(1,1)$-tensor field ϕ, a vector field ξ, a 1-form η, satisfying

$$\phi^2 = -I + \eta \otimes \xi \quad \textit{and} \quad \eta(\xi) = 1.$$

- Every almost contact manifold admits a compatible metric g, that is, such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all $X, Y \in \Gamma(TM)$.

Normality

- An almost contact manifold (M, ϕ, ξ, η) is said to be normal if

$$[\phi, \phi] + d\eta \otimes \xi = 0.$$

- M is normal iff the almost complex structure J on the product $M \times \mathbb{R}$ defined by setting, for any $X \in \Gamma(TM)$ and $f \in C^\infty(M \times \mathbb{R})$,

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right)$$

is integrable.

Quasi-Sasakian manifolds

- A quasi-Sasakian structure on a $(2n+1)$-dimensional manifold M is a normal almost contact metric structure (ϕ, ξ, η, g) such that $d\Phi = 0$, where Φ is defined by

$$\Phi(X, Y) = g(X, \phi Y).$$

- They were introduced by D. E. Blair to unify Sasakian geometry ($d\eta = \Phi$) and co-Kähler geometry ($d\eta = 0,\ d\Phi = 0$).
- A quasi-Sasakian manifold is said to be of rank $2p+1$ if

$$\eta \wedge (d\eta)^p \neq 0 \qquad \text{and} \qquad (d\eta)^{p+1} = 0,$$

for some $p \leq n$.

Examples of quasi-Sasakian manifolds

An example of a manifold that admits a quasi-Sasakian structure is the nilpotent Lie group

$$G = \mathrm{H}(1, l) \times \mathbb{R}^{2(n-l)},$$

where $\mathrm{H}(1, l)$ is the (generalized) Heisenberg group of dimension $2l + 1$. The Heisenberg group $\mathrm{H}(1, l)$ is the Lie subgroup of dimension $2l + 1$ in the general linear group $\mathrm{GL}_{l+2}(\mathbb{R})$ with elements of the form

$$\begin{pmatrix} 1 & P & t \\ 0 & I_l & Q \\ 0 & 0 & 1 \end{pmatrix},$$

where I_l denote the $l \times l$ identity matrix, $P, Q \in \mathbb{R}^l$ and $t \in \mathbb{R}$.

Examples of quasi-Sasakian manifolds

- If Γ is a cocompact discrete subgroup of $G = \mathrm{H}(1, l) \times \mathbb{R}^{2(n-l)}$, then the structure, being left-invariant, goes to the quotient. Thus, $\Gamma \backslash G$ is a compact quasi-Sasakian nilmanifold.
- Note that if $n \neq l$ and $l \neq 0$ then the nilmanifold $\Gamma \backslash G$ does not admit either a Sasakian or a co-Kähler structure.

Basic cohomology with respect to a given foliation

Consider a manifold M with a foliation $\mathcal{F}$. Let $T\mathcal{F} \subset TM$ be the tangent distribution to $\mathcal{F}$.
The space of basic k-forms with respect to $\mathcal{F}$ is defined as

$$\Omega_B^k(M) := \left\{ \omega \in \Omega^k(M) \,\middle|\, i_X\omega = 0,\ i_X d\omega = 0, \forall X \in \Gamma(T\mathcal{F}) \right\}.$$

The restriction of the exterior derivative d to $\Omega_B^k(M)$ sends basic forms into basic forms, so one obtains a sub-complex

$$(\Omega_B^*(M), d).$$

The basic cohomology $H_B^*(M, \mathcal{F})$ with respect to the foliation $\mathcal{F}$ is defined as the cohomology of this complex.

A model for quasi-Sasakian manifolds

Theorem

Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a compact quasi-Sasakian manifold. Then the CDGA

$$\left(H_B(M,\xi) \otimes \bigwedge \langle y \rangle, dy = [d\eta]_B\right)$$

is a model of M.

Here $\bigwedge \langle y \rangle$ is the exterior algebra generated by a free element y of degree 1 and the differential is assumed to be zero on the elements of $H_B(M,\xi)$.

- As a special case of our result one obtains the model discovered by Tievsky for Sasakian manifolds.

Almost formal CDGAs

Motivated by the model described in the above theorem, we introduce the following class of CDGAs.

Definition

We say that a CDGA (B, d) is almost formal of index l if it is quasi-isomorphic to the CDGA $(A \otimes \bigwedge \langle y \rangle, dy = z)$, where A is a connected CDGA with the zero differential and $z \in A_2$ is a closed element satisfying $z^l \neq 0$, $z^{l+1} = 0$.

A manifold M is said to be almost formal if it has an almost formal model.

Quasi-Sasakian manifold are almost formal

The previous definition and the above model suggest us to introduce the following notion for quasi-Sasakian manifolds.

Definition

Let $(M^{2n+1}, \varphi, \xi, \eta, h)$ be a quasi-Sasakian manifold. The index of M is the natural number l, $0 \leq l \leq n$, satisfying

$$[d\eta]_B^l \neq 0 \quad \text{and} \quad [d\eta]_B^{l+1} = 0.$$

From our result it follows that the model of a compact quasi-Sasakian manifold M is an almost formal CDGA with the same index of M.

Vaisman manifolds

An Hermitian manifold is a complex manifold (M, J) with a compatible Riemannian metric g, that is

$$g(JX, JY) = g(X, Y).$$

Then, the fundamental 2-form is defined by

$$\Omega(X, Y) = g(X, JY), \quad \text{for } X, Y \in \Gamma(TM),$$

If Ω is closed, then (M, J, g) is called a Kähler manifold.

Vaisman manifolds

A Hermitian manifold (M, J, g) such that the fundamental 2-form Ω satisfies

$$d\Omega = \theta \wedge \Omega.$$

for some (closed) 1-form θ, is called an LCK manifold.
Then, if θ is parallel, that is

$$\nabla\theta = 0,$$

we say that M is a Vaisman manifold.

Quasi-Vaisman manifolds

Definition

A Hermitian manifold (M, J, g) is said to be quasi-Vaisman if the fundamental 2-form Ω satisfies

$$d\Omega = \theta \wedge d\eta,$$

where θ is a closed 1-form and $\eta = -\theta \circ J$.
Moreover, the metric dual U of θ must be unitary, Killing and holomorphic (that is $\mathcal{L}_U J = 0$).

Canonical foliations on quasi-Vaisman manifolds

In a quasi-Vaisman manifold M,

- The 1-form θ and the vector field U are parallel;
- the couple $(U, V = JU)$ defines a flat foliation $\mathcal{F}$ of rank 2 which is transversely Kähler;
- the orthogonal bundle to the foliation generated by U is integrable and the leaves are quasi-Sasakian manifolds.

A quasi-Vaisman manifold is Vaisman if and only if it is LCK or equivalently

$$\theta \wedge d\eta = \theta \wedge \Omega.$$

A Model for quasi-Vaisman manifolds

Theorem

Let (M^{2n+2}, J, g) be a compact quasi-Vaisman manifold and U, V, $\mathcal{F}$ are defined as above. Then the CDGA

$$\left(H_B(M,\mathcal{F}) \otimes \bigwedge \langle x, y\rangle, dx = 0, dy = [d\eta]_B\right) \tag{1}$$

is a model of M.

Note that the model in the above theorem is in fact an almost formal CDGA. To see this we can take

$$A := H_B(M,\mathcal{F}) \otimes \bigwedge \langle x\rangle$$

and $z = [d\eta]_B$ considered as an element in A.

Quasi-Vaisman manifolds are almost formal

Now, as in the quasi-Sasakian case, we can also introduce the following definition.

Definition

The index of a quasi-Vaisman manifold (M^{2n+2}, J, g) with quasi anti-Lee 1-form η is the natural number l, $0 \leq l \leq n$, which satisfies

$$[d\eta]_B^l \neq 0 \quad \text{and} \quad [d\eta]_B^{l+1} = 0.$$

From the above theorem it follows that the model of a compact quasi-Vaisman manifold is an almost formal CDGA with the same index of M.

Examples of quasi-Vaisman manifolds

If $(N, \varphi, \xi, \eta, h)$ is a quasi-Sasakian manifold then the product $N \times \mathbb{R}$ admits a quasi-Vaisman structure (J, g), with J and g given by

$$J = \varphi + \xi \otimes dt - \frac{\partial}{\partial t} \otimes \eta, \quad g = h + dt \otimes dt,$$

So, the nilpotent Lie group

$$G = \mathrm{H}(1, l) \times \mathbb{R}^{2(n-l)+1}$$

admits a left-invariant quasi-Vaisman structure.
Thus, if Γ is a cocompact discrete subgroup then the compact nilmanifold $\Gamma \backslash G$ admits a quasi-Vaisman structure.
Note that if $n \neq l$ and $l \neq 0$ then $\Gamma \backslash G$ doesn't admit either a Vaisman or a Kähler structure.

Models of Nilmanifolds

The minimal model of a nilmanifold was found by Hasegawa using Nomizu theorem. Namely

Theorem (Hasegawa)

Let $M \cong \Gamma \backslash G$ be a compact nilmanifold. Then the Chevalley-Eilenberg complex $\left(\bigwedge \mathfrak{g}^*, d^{CE}\right)$ is a minimal model of $\Omega(M)$.

The model being minimal implies that for any other model (A, d) of $\Omega(M)$, there is a (direct) quasi-isomorphism

$$\left(\bigwedge \mathfrak{g}^*, d^{CE}\right) \longrightarrow (A, d).$$

Models of almost formal Nilmanifolds

In the case of almost formal Nilmanifolds we do have another model, by definition of almost formal manifold. Thus we have a morphism from the minimal model to the other model.
This allows us to find out what the Lie algebra $\mathfrak{g}$ can be:

Theorem

A nilmanifold $\Gamma\backslash G$ of dimension m admits an almost formal model of index l if and only if G is isomorphic to $\mathrm{H}(1,l) \times \mathbb{R}^{m-2l-1}$.

Quasi-Sasakian and quasi-Vaisman nilmanifolds

Corollary

A $2n+1$-dimensional compact nilmanifold $\Gamma\backslash G$ admits a quasi-Sasakian structure of index l if and only if

$$G \cong \mathrm{H}(1,l) \times \mathbb{R}^{2(n-l)}$$

as a Lie group.

Corollary

A $2n+2$-dimensional compact nilmanifold $\Gamma\backslash G$ admits a quasi-Vaisman structure if and only if

$$G \cong \mathrm{H}(1,l) \times \mathbb{R}^{2(n-l)+1}$$

as a Lie group.