Conformal geodesics in spherically symmetric vacuum spacetimes with cosmological constant

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Work done in collaboration with E. Gasperín and J. A. Valiente Kroon.

Bibliography:

A. García-Parrado, E. Gasperín and J. A. Valiente Kroon *Conformal geodesics in spherically symmetric vacuum spacetimes with cosmological constant*, https://arxiv.org/abs/1704.05639

Outline

Conformal geometry in general relativity

Let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$, $(\mathcal{M}, \boldsymbol{g})$ be 4-dimensional space-times:

- $\Phi : \tilde{\mathcal{M}} \to \mathcal{M}$ is a smooth conformal map iff $\Phi^* \boldsymbol{g} = \Theta^2 \tilde{\boldsymbol{g}}$, $\Theta \in C^\infty(\tilde{M})$. If a conformal map Φ exists then $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ and $(\mathcal{M}, \boldsymbol{g})$ are conformally related.
- When $\Theta \neq 0$, the light cone of conformally related space-times can be put into one-to-one correspondence $\Longrightarrow$ Conformally related space-times have a similar causal structure.
- If Φ is an embedding then $\Phi(\tilde{\mathcal{M}}) \subset \mathcal{M}$ is diffeomorphic to $\tilde{\mathcal{M}}$ and $\Theta \neq 0$ on $\tilde{\mathcal{M}}$. Φ is then a conformal embedding and $(\mathcal{M}, g_{\mu\nu})$ is a conformal extension of $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$.

The conformal boundary of $\tilde{\mathcal{M}}$ with respect to a conformal embedding Φ is defined by

$$\partial \tilde{\mathcal{M}} \equiv \partial(\Phi(\tilde{\mathcal{M}})).$$

Conformal geometry in general relativity

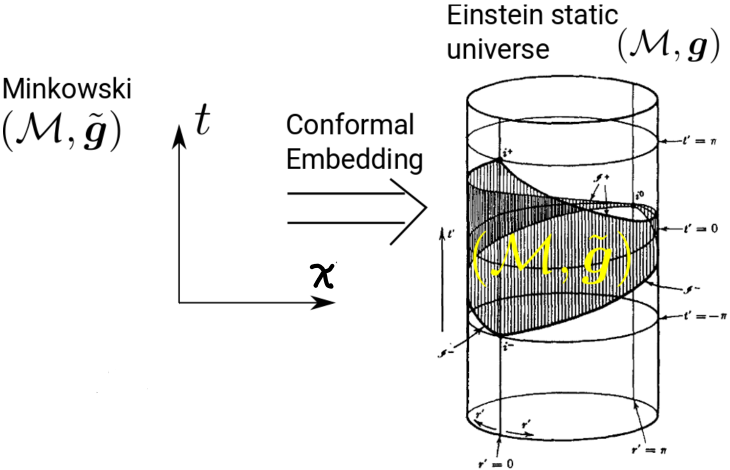


Figure : Left: conformal embedding of the 4-dimensional Minkowski space-time into the Einstein static universe. Right figure taken from S. W. Hawking and G. F. R. Ellis *The Large scale structure of space-time* Cambridge University Press, Cambridge (1973).

Conformal geometry in general relativity

- Let $(\mathcal{M}, \tilde{g}_{\mu\nu})$ be a solution of the vacuum Einstein field equations with cosmological constant λ

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}\,\tilde{g}_{\mu\nu} - \lambda\tilde{g}_{\mu\nu} = 0 \Longrightarrow \tilde{R}_{\mu\nu} = -\lambda\tilde{g}_{\mu\nu}, \qquad \tilde{R} = -4\lambda.$$

  The solution $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$ is called the physical spacetime.

- Introduce a conformal extension $(\mathcal{M}, g_{\mu\nu})$ of $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$. The space-time $(\mathcal{M}, g_{\mu\nu})$ is called the unphysical space-time.

- **The computation of a conformal extension of an Einstein space-time and the corresponding conformal boundary will enable us to gain a wealth of information about the global properties of the space-time.**

The computation of the conformal boundary

The relation between $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$ can be thought of as a conformal coordinate change $\boldsymbol{g} = \Theta^2 \tilde{\boldsymbol{g}}$ in the unphysical spacetime $(\mathcal{M}, g_{\mu\nu})$. The conformal boundary is then given by $\Theta = 0$.

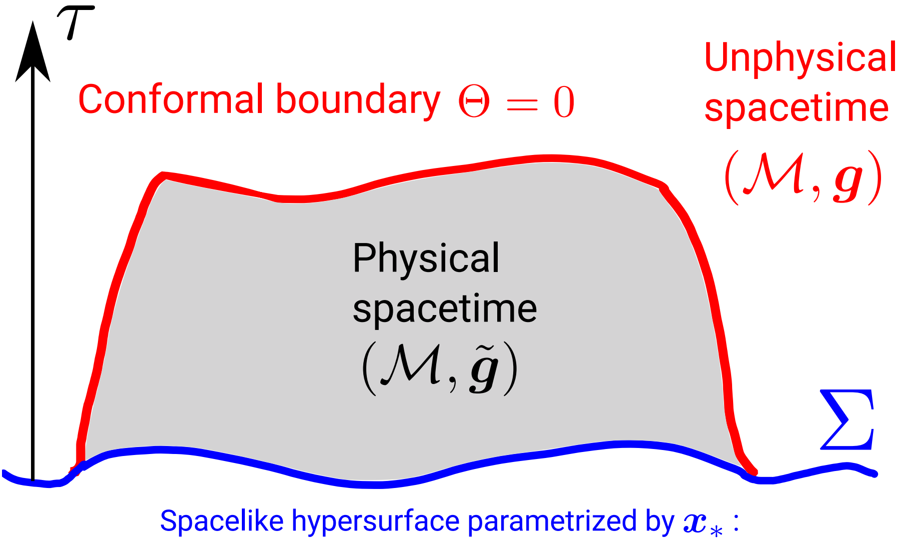
Idea

For a given physical space-time $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$, find a coordinate system $(\tau, \boldsymbol{x}_*)$ such that in the new coordinates

$$\tilde{\boldsymbol{g}}(\tau, \boldsymbol{x}_\star) = \frac{\boldsymbol{g}(\tau, \boldsymbol{x}_\star)}{\Theta(\tau, \boldsymbol{x}_\star)^2}.$$

Ideally, $\boldsymbol{x}_*$ should vary within a compact spatial domain.

- A conformal Gaussian coordinate system constructed from a congruence of conformal geodesics realize this idea.


τ
Conformal boundary $\Theta = 0$
Unphysical spacetime
$(\mathcal{M}, \boldsymbol{g})$
Physical spacetime
$(\mathcal{M}, \tilde{\boldsymbol{g}})$
Σ
Spacelike hypersurface parametrized by $\boldsymbol{x}_*$:

The role of conformal geodesics

Definition

Given an interval $I \subseteq \mathbb{R}$, let $x(\tau)$, $\tau \in I$ denote a curve in $(\mathcal{M}, \tilde{\boldsymbol{g}})$ and let $\boldsymbol{\beta}(\tau)$ denote a 1-form along $\boldsymbol{x}(\tau)$. Furthermore, let $\dot{x} \equiv \mathrm{d}\boldsymbol{x}/\mathrm{d}\tau$ denote the tangent vector field of the curve $x(\tau)$. The *conformal geodesic equations* are then given by:

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}} \dot{\boldsymbol{x}} = -2\langle \boldsymbol{\beta}, \dot{\boldsymbol{x}} \rangle \dot{\boldsymbol{x}} + \tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) \boldsymbol{\beta}^{\sharp},$$
$$\tilde{\nabla}_{\dot{\boldsymbol{x}}} \boldsymbol{\beta} = \langle \boldsymbol{\beta}, \dot{\boldsymbol{x}} \rangle \boldsymbol{\beta} - \tfrac{1}{2} \tilde{\boldsymbol{g}}^{\sharp}(\boldsymbol{\beta}, \boldsymbol{\beta}) \dot{\boldsymbol{x}}^{\flat} + \tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}}, \cdot),$$

where $\tilde{\boldsymbol{\nabla}}$ denotes the Levi-Civita connection of the physical metric $\tilde{\boldsymbol{g}}$, $\tilde{\nabla}_{\dot{\boldsymbol{x}}}$ denotes a derivative in the direction of $\dot{\boldsymbol{x}}$ and $\boldsymbol{\beta}^{\sharp}$ is the contravariant version of $\boldsymbol{\beta}$ with respect to $\tilde{\boldsymbol{g}}$.

The symbol $\tilde{\boldsymbol{L}}$ denotes the Schouten tensor of $\tilde{\boldsymbol{g}}$ defined by:

$$\tilde{L}_{\mu\nu} \equiv \frac{1}{2}\left(\tilde{R}_{\mu\nu} - \frac{1}{6} \tilde{R}\, \tilde{g}_{\mu\nu} \right).$$

The role of conformal geodesics

- Choose a parameter τ for the conformal geodesic such that

$$\boldsymbol{g}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = 1\ , \quad \text{(signature convention } (+,-,-,-)\text{).}$$

- τ is the unphysical proper time. Then a computation shows that

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\tilde{\nabla}_{\dot{\boldsymbol{x}}}\tilde{\nabla}_{\dot{\boldsymbol{x}}}\Theta = \frac{3}{\Theta}\tilde{\boldsymbol{L}}(\boldsymbol{\beta}^{\sharp}, \dot{\boldsymbol{x}}) - 3\Theta\langle\boldsymbol{\beta}, \dot{\boldsymbol{x}}\rangle\tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) + \Theta(\tilde{\nabla}_{\dot{\boldsymbol{x}}}\tilde{\boldsymbol{L}})(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}).$$

The role of conformal geodesics

- From now on assume that our space-time is an Einstein space. Then one has that $\tilde{\boldsymbol{L}} = -\frac{1}{6}\lambda\tilde{\boldsymbol{g}}$ and hence

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\tilde{\nabla}_{\dot{\boldsymbol{x}}}\tilde{\nabla}_{\dot{\boldsymbol{x}}}\Theta = 0 \ ,$$

- This equation can be integrated in terms of quantities defined at Σ (initial data quantities)

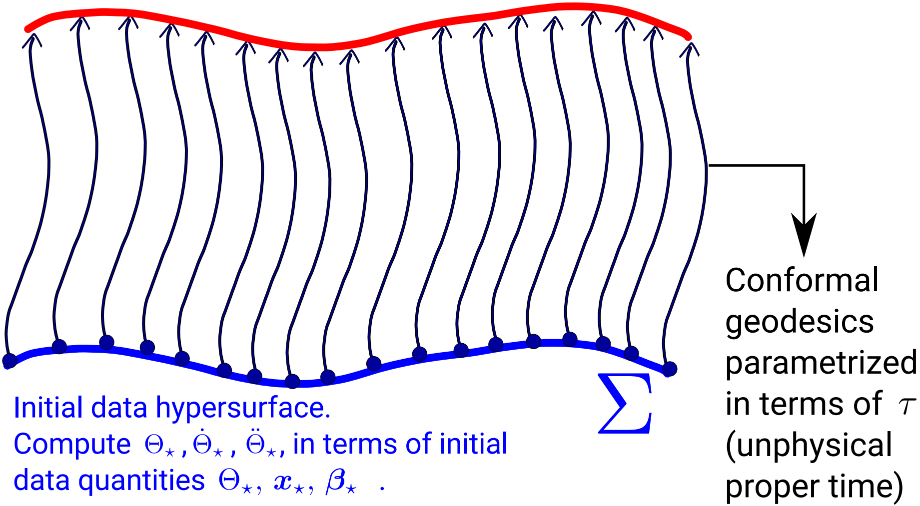
$$\Theta = \Theta_\star + \dot{\Theta}_\star(\tau - \tau_\star) + \frac{1}{2}\ddot{\Theta}_\star(\tau - \tau_\star)^2.$$

$\dot{\Theta}_\star$ and $\ddot{\Theta}_\star$ are defined in terms of the initial data quantities $\Theta_\star$ (initial conformal factor), $\boldsymbol{x}_\star$ (initial velocity) and $\boldsymbol{\beta}_\star$ (initial acceleration).

$$\dot{\Theta}_\star = \langle \boldsymbol{\beta}_\star, \dot{\boldsymbol{x}}_\star \rangle \Theta_\star, \qquad \Theta_\star\ddot{\Theta}_\star = \frac{1}{2}\tilde{\boldsymbol{g}}(\boldsymbol{\beta}_\star, \boldsymbol{\beta}_\star) - \frac{1}{6}\lambda.$$

Computation of the conformal boundary

$$\Theta = \Theta_\star + \dot{\Theta}_\star(\tau - \tau_\star) + \frac{1}{2}\ddot{\Theta}_\star(\tau - \tau_\star)^2 = 0$$



Initial data hypersurface.
Compute $\Theta_\star$, $\dot{\Theta}_\star$, $\ddot{\Theta}_\star$, in terms of initial
data quantities $\Theta_\star, \boldsymbol{x}_\star, \boldsymbol{\beta}_\star$.

The role of conformal geodesics

The conformal geodesics give us a procedure to analyze the properties of a conformal boundary without actually solving the conformal equations.

Procedure

1. *Choose a set of initial data quantities* $\Theta_\star$, $\boldsymbol{x}_\star$ *and* $\boldsymbol{\beta}_\star$.
2. *Try to show that the congruence of conformal geodesics exists for every value of the unphysical parameter* τ *and develops no caustics within the region of interest.*
3. *If the previous is true, use then the formula of* $\Theta(\tau, \boldsymbol{x}_\star)$ *to compute the location of the conformal boundary and its structure.*
4. *The set of conformal geodesics define a congruence of curves. If the congruence covers the region of interest we can use it to define a new coordinate system* $(\tau, \boldsymbol{x}_\star)$ *in the unphysical space-time* $(\mathcal{M}, \boldsymbol{g})$ *(conformal Gaussian coordinates). In these coordinates we have*

$$\boldsymbol{g}(\tau, \boldsymbol{x}_\star) = \Theta(\tau, \boldsymbol{x}_\star)^2 \tilde{\boldsymbol{g}}(\tau, \boldsymbol{x}_\star).$$

The role of conformal geodesics

In general the computation of a congruence of conformal geodesics with the right properties is a difficult problem. However, the problem can be simplified in certain cases.

- Consider the warped product physical Einstein space-time

$$\tilde{\boldsymbol{g}} = D(r)\mathbf{d}t \otimes \mathbf{d}t - \frac{\mathbf{d}r \otimes \mathbf{d}r}{D(r)} + f^2(r)k_{ij}(x^k)\mathbf{d}x^i \otimes \mathbf{d}x^j \ , \ i,j,k = 2,3.$$

- Seek solutions of the conformal geodesic equations of the form :

$$\tilde{\boldsymbol{x}}' = t'(\tilde{\tau}, r_*)\frac{\partial}{\partial t} + r'(\tilde{\tau}, r_*)\frac{\partial}{\partial r} \Longrightarrow \boldsymbol{\beta} = \beta(-r'(\tilde{\tau}, r_*)\mathbf{d}t + t'(\tilde{\tau}, r_*)\mathbf{d}r) \ ,$$
$$\tilde{\boldsymbol{g}}(\tilde{\boldsymbol{x}}', \tilde{\boldsymbol{x}}') = 1$$

where $\tilde{\tau}$ is the physical proper time, $' \equiv d/d\tilde{\tau}$ means derivative with respect to the proper physical parameter and β is a constant defined from the initial data.

The role of conformal geodesics

- Under these assumptions, the equations for the conformal geodesics are reduced to the following set

$$t'' + \frac{\partial_r D(r)}{D(r)} r' t' = \frac{\beta\, r'}{D(r)}\ ,$$

$$r'' - \frac{\partial_r D(r)}{2D(r)} r'^2 + \frac{1}{2} D(r) \partial_r D(r) t'^2 = D(r) \beta\, t'.$$

- The condition $\tilde{\boldsymbol{g}}(\tilde{\boldsymbol{x}}', \tilde{\boldsymbol{x}}') = 1$ is rendered in the form

$$D(r)\, t'^2 - \frac{r'^2}{D(r)} = 1.$$

- And the relation between the unphysical proper time and the physical proper time is

$$\tilde{\tau} = \int_{\tau_*}^{\tau} \frac{ds}{\Theta(s)}$$

The role of conformal geodesics

- The previous equations can be reduced to first order equations solvable up to quadratures

$$r' = \pm\sqrt{(\gamma+\beta r)^2 - D(r)}\ ,$$
$$\gamma \equiv -\beta r_\star + \sqrt{D_\star + r'_\star}\ , \qquad D_\star \equiv D(r_\star) > 0\ , \qquad r'_\star \equiv r'(\tilde{\tau}_\star)\ ,$$
$$t' = \frac{|\gamma+\beta r|}{|D(r)|}.$$

- We choose a congruence which is orthogonal to the initial data hypersurface. This implies the conditions

$$r'(\tilde{\tau}_\star) = 0 \Rightarrow \gamma = -\beta r_* + \sqrt{D_*}\ , \quad t'(\tilde{\tau}_\star) = \frac{1}{\sqrt{|D_*|}}.$$

- Hence we only need to choose an adequate value of $\beta = \beta(r_*)$ and Θ_*.

The warped product structure arises naturally for many Einstein spaces

- Schwarzschild solution (**H. Friedrich 2003**)

$$\tilde{\boldsymbol{g}} = \left(1 - \frac{2m}{r}\right)\mathbf{d}t \otimes \mathbf{d}t - \left(1 - \frac{2m}{r}\right)^{-1}\mathbf{d}r \otimes \mathbf{d}r - r^2\boldsymbol{\sigma}.$$

- The Reissner Nordström solution (**C. Lübbe and J.A. Valiente Kroon 2013**)

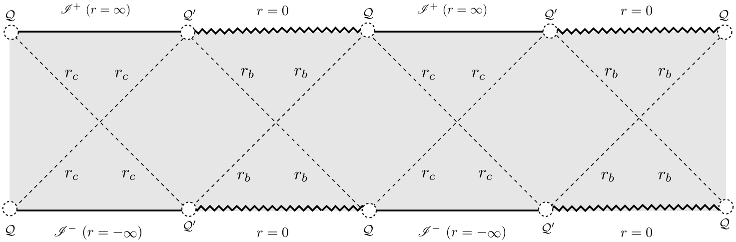
$$\tilde{\boldsymbol{g}} = \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)\mathbf{d}t \otimes \mathbf{d}t - \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1}\mathbf{d}r \otimes \mathbf{d}r - r^2\boldsymbol{\sigma}.$$

- The Kottler solution (Schwarzschild solution with cosmological constant). **(A. García-Parrado, E. Gasperín, J. A. Valiente Kroon 2017)**

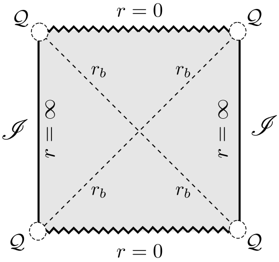
$$\tilde{\boldsymbol{g}} = \left(1 - \frac{2m}{r} - \frac{\lambda}{3}r^2\right)\mathbf{d}t \otimes \mathbf{d}t - \left(1 - \frac{2m}{r} - \frac{\lambda}{3}r^2\right)^{-1}\mathbf{d}r \otimes \mathbf{d}r - r^2\boldsymbol{\sigma}.$$

The Kottler solution

- $\lambda > 0$ (Schwarzschild-de Sitter solution, topology $\mathbb{S}^2 \times \mathbb{S}^1 \times \mathbb{R}$).



- $\lambda < 0$ (Schwarzschild-anti de Sitter solution, topology $\mathbb{S}^2 \times \mathbb{R} \times \mathbb{R}$).

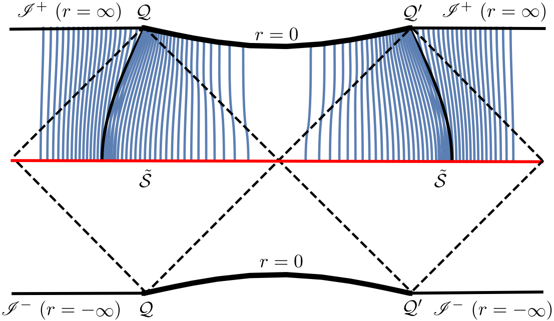


Schwarzschild de Sitter solution

- Parameters of the congruence of conformal geodesics

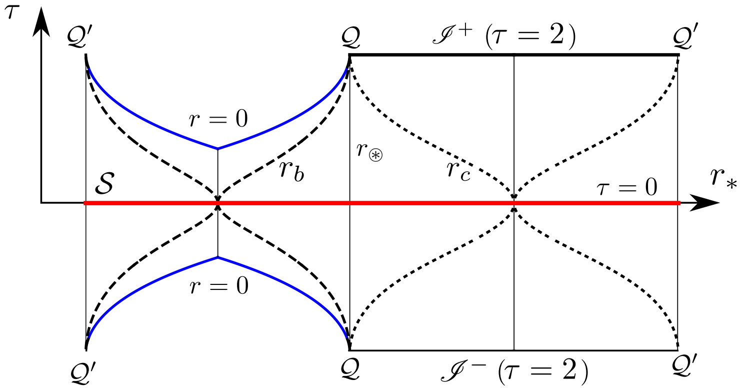
$$r_b \leq r_\star \leq r_c\ ,\ \ \boldsymbol{\beta}_\star = 0\ ,\ \ \Theta_\star = 1 \Longrightarrow \dot{\Theta}_\star = 0\ ,\ \Theta(\tau) = 1 - \frac{\tau^2}{4}.$$

- Congruence of conformal geodesics plotted on the Penrose diagram



Schwarzschild de Sitter solution

- Spacetime representation in conformal Gaussian coordinates

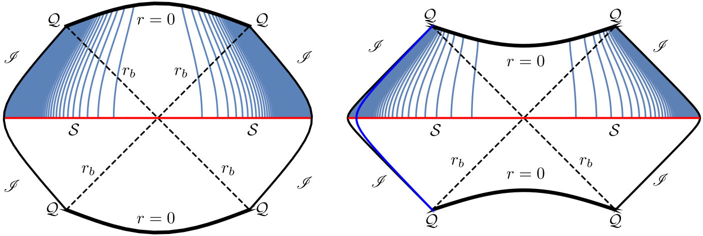


Schwarzschild anti-de Sitter solution

- Parameters of the congruence of conformal geodesics

$$\Theta_\star = \frac{1}{\sqrt{1+r_*^2}}, \qquad \boldsymbol{\beta}_\star = -\frac{r_*}{1+r_*^2}\mathbf{d}r_*$$

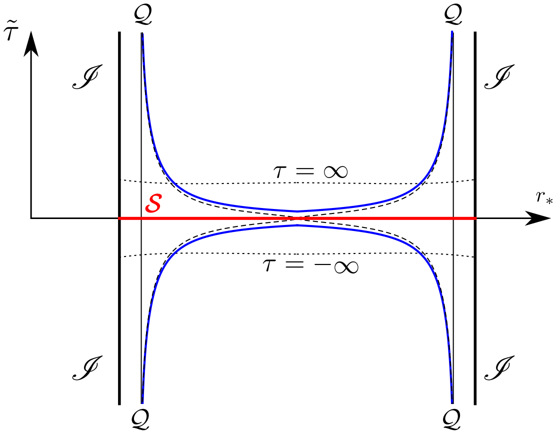
- Congruence of conformal geodesics plotted on the Penrose diagram



$M = 28.56$ $\qquad$ $M = 0.34$

Schwarzschild anti-de Sitter solution

- Spacetime representation in conformal Gaussian coordinates



Open issues

- Attempt to carry out a similar construction for the Kerr family.
- Obtain the explicit coordinate representation of the conformal metrics in conformal Gaussian coordinates. This would allow us to find the explicit coordinate representation of the unphysical spacetime.
- In the case of the Schwarzschild anti-de Sitter we have the conformal embedding of only a subset but the knowledge of the unphysical spacetime could allow us to obtain the full conformal embedding by means of an appropriate coordinate extension.